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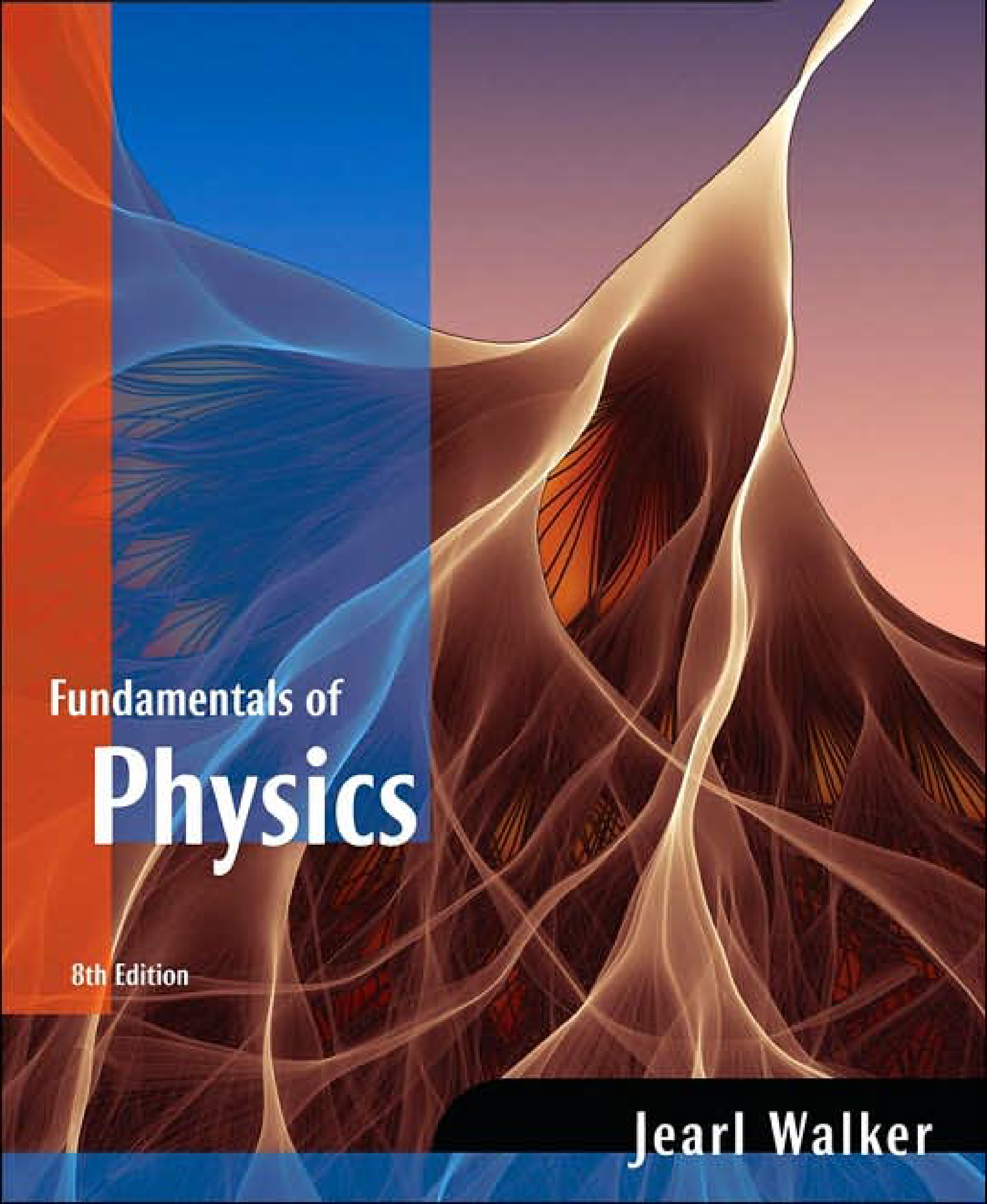

Fundamentals of

Physics

8th Edition

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PUZZLER

For thousands of years the spinning Earth provided a natural standard for our measurements of time. However, since 1972 we have added more than 20 "leap seconds" to our clocks to keep them synchronized to the Earth. Why are such adjustments needed? What does it take to be a good standard? *(Don Mason/The Stock Market* and *NASA)*

chapter

1

Physics and Measurement

Chapter Outline

Like all other sciences, physics is based on experimental observations and quantitative measurements. The main objective of physics is to find the limited number of fundamental laws that govern natural phenomena and to use them to develop theories that can predict the results of future experiments. The fundamental laws used in developing theories are expressed in the language of mathematics, the tool that provides a bridge between theory and experiment.

When a discrepancy between theory and experiment arises, new theories must be formulated to remove the discrepancy. Many times a theory is satisfactory only under limited conditions; a more general theory might be satisfactory without such limitations. For example, the laws of motion discovered by Isaac Newton (1642–1727) in the 17th century accurately describe the motion of bodies at normal speeds but do not apply to objects moving at speeds comparable with the speed of light. In contrast, the special theory of relativity developed by Albert Einstein (1879–1955) in the early 1900s gives the same results as Newton's laws at low speeds but also correctly describes motion at speeds approaching the speed of light. Hence, Einstein's is a more general theory of motion.

Classical physics, which means all of the physics developed before 1900, includes the theories, concepts, laws, and experiments in classical mechanics, thermodynamics, and electromagnetism.

Important contributions to classical physics were provided by Newton, who developed classical mechanics as a systematic theory and was one of the originators of calculus as a mathematical tool. Major developments in mechanics continued in the 18th century, but the fields of thermodynamics and electricity and magnetism were not developed until the latter part of the 19th century, principally because before that time the apparatus for controlled experiments was either too crude or unavailable.

A new era in physics, usually referred to as *modern physics,* began near the end of the 19th century. Modern physics developed mainly because of the discovery that many physical phenomena could not be explained by classical physics. The two most important developments in modern physics were the theories of relativity and quantum mechanics. Einstein's theory of relativity revolutionized the traditional concepts of space, time, and energy; quantum mechanics, which applies to both the microscopic and macroscopic worlds, was originally formulated by a number of distinguished scientists to provide descriptions of physical phenomena at the atomic level.

Scientists constantly work at improving our understanding of phenomena and fundamental laws, and new discoveries are made every day. In many research areas, a great deal of overlap exists between physics, chemistry, geology, and biology, as well as engineering. Some of the most notable developments are (1) numerous space missions and the landing of astronauts on the Moon, (2) microcircuitry and high-speed computers, and (3) sophisticated imaging techniques used in scientific research and medicine. The impact such developments and discoveries have had on our society has indeed been great, and it is very likely that future discoveries and developments will be just as exciting and challenging and of great benefit to humanity.

1.1 STANDARDS OF LENGTH, MASS, AND TIME

The laws of physics are expressed in terms of basic quantities that require a clear definition. In mechanics, the three basic quantities are length (L), mass (M), and time (T). All other quantities in mechanics can be expressed in terms of these three.

If we are to report the results of a measurement to someone who wishes to reproduce this measurement, a *standard* must be defined. It would be meaningless if a visitor from another planet were to talk to us about a length of 8 "glitches" if we do not know the meaning of the unit glitch. On the other hand, if someone familiar with our system of measurement reports that a wall is 2 meters high and our unit of length is defined to be 1 meter, we know that the height of the wall is twice our basic length unit. Likewise, if we are told that a person has a mass of 75 kilograms and our unit of mass is defined to be 1 kilogram, then that person is 75 times as massive as our basic unit.[1] Whatever is chosen as a standard must be readily accessible and possess some property that can be measured reliably—measurements taken by different people in different places must yield the same result.

In 1960, an international committee established a set of standards for length, mass, and other basic quantities. The system established is an adaptation of the metric system, and it is called the **SI system** of units. (The abbreviation SI comes from the system's French name "Système International.") In this system, the units of length, mass, and time are the meter, kilogram, and second, respectively. Other SI standards established by the committee are those for temperature (the *kelvin*), electric current (the *ampere*), luminous intensity (the *candela*), and the amount of substance (the *mole*). In our study of mechanics we shall be concerned only with the units of length, mass, and time.

Length

In A.D. 1120 the king of England decreed that the standard of length in his country would be named the *yard* and would be precisely equal to the distance from the tip of his nose to the end of his outstretched arm. Similarly, the original standard for the foot adopted by the French was the length of the royal foot of King Louis XIV. This standard prevailed until 1799, when the legal standard of length in France became the *meter,* defined as one ten-millionth the distance from the equator to the North Pole along one particular longitudinal line that passes through Paris.

Many other systems for measuring length have been developed over the years, but the advantages of the French system have caused it to prevail in almost all countries and in scientific circles everywhere. As recently as 1960, the length of the meter was defined as the distance between two lines on a specific platinum–iridium bar stored under controlled conditions in France. This standard was abandoned for several reasons, a principal one being that the limited accuracy with which the separation between the lines on the bar can be determined does not meet the current requirements of science and technology. In the 1960s and 1970s, the meter was defined as 1 650 763.73 wavelengths of orange-red light emitted from a krypton-86 lamp. However, in October 1983, the **meter (m) was redefined as the distance traveled by light in vacuum during a time of 1/299 792 458 second.** In effect, this latest definition establishes that the speed of light in vacuum is precisely 299 792 458 m per second.

Table 1.1 lists approximate values of some measured lengths.

[1] The need for assigning numerical values to various measured physical quantities was expressed by Lord Kelvin (William Thomson) as follows: "I often say that when you can measure what you are speaking about, and express it in numbers, you should know something about it, but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind. It may be the beginning of knowledge but you have scarcely in your thoughts advanced to the state of science."

TABLE 1.1 Approximate Values of Some Measured Lengths

	Length (m)
Distance from the Earth to most remote known quasar	1.4×10^{26}
Distance from the Earth to most remote known normal galaxies	9×10^{25}
Distance from the Earth to nearest large galaxy (M 31, the Andromeda galaxy)	2×10^{22}
Distance from the Sun to nearest star (Proxima Centauri)	4×10^{16}
One lightyear	9.46×10^{15}
Mean orbit radius of the Earth about the Sun	1.50×10^{11}
Mean distance from the Earth to the Moon	3.84×10^{8}
Distance from the equator to the North Pole	1.00×10^{7}
Mean radius of the Earth	6.37×10^{6}
Typical altitude (above the surface) of a satellite orbiting the Earth	2×10^{5}
Length of a football field	9.1×10^{1}
Length of a housefly	5×10^{-3}
Size of smallest dust particles	$\sim 10^{-4}$
Size of cells of most living organisms	$\sim 10^{-5}$
Diameter of a hydrogen atom	$\sim 10^{-10}$
Diameter of an atomic nucleus	$\sim 10^{-14}$
Diameter of a proton	$\sim 10^{-15}$

Mass

The basic SI unit of mass, **the kilogram (kg), is defined as the mass of a specific platinum–iridium alloy cylinder kept at the International Bureau of Weights and Measures at Sèvres, France.** This mass standard was established in 1887 and has not been changed since that time because platinum–iridium is an unusually stable alloy (Fig. 1.1a). A duplicate of the Sèvres cylinder is kept at the National Institute of Standards and Technology (NIST) in Gaithersburg, Maryland.

Table 1.2 lists approximate values of the masses of various objects.

web

Visit the Bureau at **www.bipm.fr** or the National Institute of Standards at **www.NIST.gov**

TABLE 1.2 Masses of Various Bodies (Approximate Values)

Body	Mass (kg)
Visible Universe	$\sim 10^{52}$
Milky Way galaxy	7×10^{41}
Sun	1.99×10^{30}
Earth	5.98×10^{24}
Moon	7.36×10^{22}
Horse	$\sim 10^{3}$
Human	$\sim 10^{2}$
Frog	$\sim 10^{-1}$
Mosquito	$\sim 10^{-5}$
Bacterium	$\sim 10^{-15}$
Hydrogen atom	1.67×10^{-27}
Electron	9.11×10^{-31}

Time

Before 1960, the standard of time was defined in terms of the *mean solar day* for the year 1900.[2] The *mean solar second* was originally defined as $(\frac{1}{60})(\frac{1}{60})(\frac{1}{24})$ of a mean solar day. The rotation of the Earth is now known to vary slightly with time, however, and therefore this motion is not a good one to use for defining a standard.

In 1967, consequently, the second was redefined to take advantage of the high precision obtainable in a device known as an *atomic clock* (Fig. 1.1b). In this device, the frequencies associated with certain atomic transitions can be measured to a precision of one part in 10^{12}. This is equivalent to an uncertainty of less than one second every 30 000 years. Thus, in 1967 the SI unit of time, the *second,* was redefined using the characteristic frequency of a particular kind of cesium atom as the "reference clock." The basic SI unit of time, **the second (s), is defined as 9 192 631 770 times the period of vibration of radiation from the cesium-133 atom.**[3] To keep these atomic clocks—and therefore all common clocks and

[2] One solar day is the time interval between successive appearances of the Sun at the highest point it reaches in the sky each day.

[3] *Period* is defined as the time interval needed for one complete vibration.

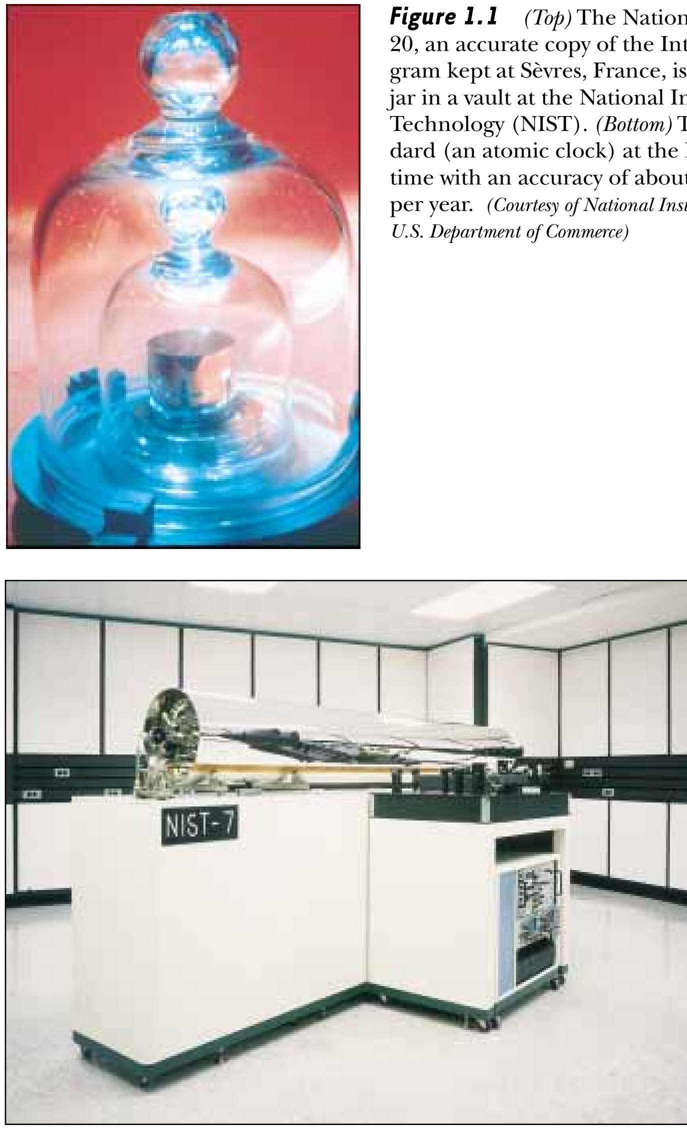


Figure 1.1 *(Top)* The National Standard Kilogram No. 20, an accurate copy of the International Standard Kilogram kept at Sèvres, France, is housed under a double bell jar in a vault at the National Institute of Standards and Technology (NIST). *(Bottom)* The primary frequency standard (an atomic clock) at the NIST. This device keeps time with an accuracy of about 3 millionths of a second per year. *(Courtesy of National Institute of Standards and Technology, U.S. Department of Commerce)*

watches that are set to them—synchronized, it has sometimes been necessary to add leap seconds to our clocks. This is not a new idea. In 46 B.C. Julius Caesar began the practice of adding extra days to the calendar during leap years so that the seasons occurred at about the same date each year.

Since Einstein's discovery of the linkage between space and time, precise measurement of time intervals requires that we know both the state of motion of the clock used to measure the interval and, in some cases, the location of the clock as well. Otherwise, for example, global positioning system satellites might be unable to pinpoint your location with sufficient accuracy, should you need rescuing.

Approximate values of time intervals are presented in Table 1.3.

In addition to SI, another system of units, the *British engineering system* (sometimes called the *conventional system*), is still used in the United States despite acceptance of SI by the rest of the world. In this system, the units of length, mass, and

TABLE 1.3 Approximate Values of Some Time Intervals

	Interval (s)
Age of the Universe	5×10^{17}
Age of the Earth	1.3×10^{17}
Average age of a college student	6.3×10^{8}
One year	3.16×10^{7}
One day (time for one rotation of the Earth about its axis)	8.64×10^{4}
Time between normal heartbeats	8×10^{-1}
Period of audible sound waves	$\sim 10^{-3}$
Period of typical radio waves	$\sim 10^{-6}$
Period of vibration of an atom in a solid	$\sim 10^{-13}$
Period of visible light waves	$\sim 10^{-15}$
Duration of a nuclear collision	$\sim 10^{-22}$
Time for light to cross a proton	$\sim 10^{-24}$

time are the foot (ft), slug, and second, respectively. In this text we shall use SI units because they are almost universally accepted in science and industry. We shall make some limited use of British engineering units in the study of classical mechanics.

In addition to the basic SI units of meter, kilogram, and second, we can also use other units, such as millimeters and nanoseconds, where the prefixes *milli-* and *nano-* denote various powers of ten. Some of the most frequently used prefixes for the various powers of ten and their abbreviations are listed in Table 1.4. For

TABLE 1.4 Prefixes for SI Units

Power	Prefix	Abbreviation
10^{-24}	yocto	y
10^{-21}	zepto	z
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deka	da
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	Z
10^{24}	yotta	Y

example, 10^{-3} m is equivalent to 1 millimeter (mm), and 10^3 m corresponds to 1 kilometer (km). Likewise, 1 kg is 10^3 grams (g), and 1 megavolt (MV) is 10^6 volts (V).

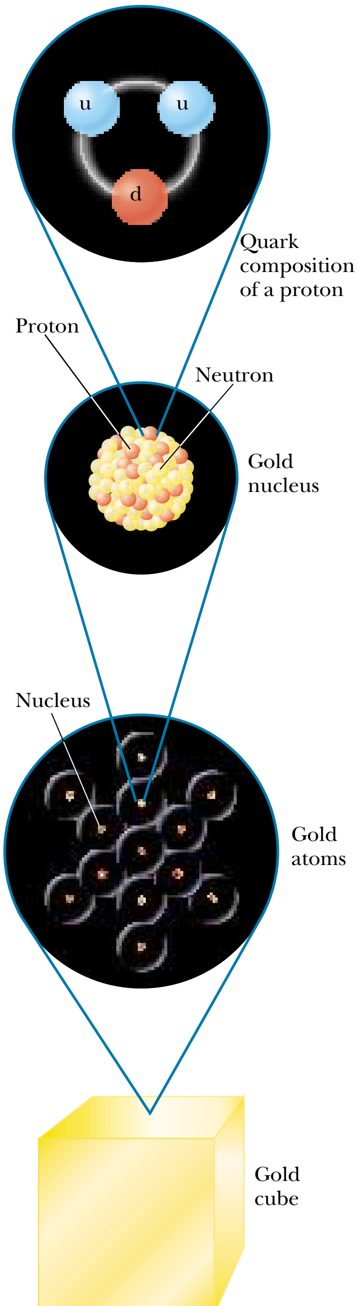


Figure 1.2 Levels of organization in matter. Ordinary matter consists of atoms, and at the center of each atom is a compact nucleus consisting of protons and neutrons. Protons and neutrons are composed of quarks. The quark composition of a proton is shown.

1.2 THE BUILDING BLOCKS OF MATTER

A 1-kg cube of solid gold has a length of 3.73 cm on a side. Is this cube nothing but wall-to-wall gold, with no empty space? If the cube is cut in half, the two pieces still retain their chemical identity as solid gold. But what if the pieces are cut again and again, indefinitely? Will the smaller and smaller pieces always be gold? Questions such as these can be traced back to early Greek philosophers. Two of them—Leucippus and his student Democritus—could not accept the idea that such cuttings could go on forever. They speculated that the process ultimately must end when it produces a particle that can no longer be cut. In Greek, *atomos* means "not sliceable." From this comes our English word *atom.*

Let us review briefly what is known about the structure of matter. All ordinary matter consists of atoms, and each atom is made up of electrons surrounding a central nucleus. Following the discovery of the nucleus in 1911, the question arose: Does it have structure? That is, is the nucleus a single particle or a collection of particles? The exact composition of the nucleus is not known completely even today, but by the early 1930s a model evolved that helped us understand how the nucleus behaves. Specifically, scientists determined that occupying the nucleus are two basic entities, protons and neutrons. The *proton* carries a positive charge, and a specific element is identified by the number of protons in its nucleus. This number is called the **atomic number** of the element. For instance, the nucleus of a hydrogen atom contains one proton (and so the atomic number of hydrogen is 1), the nucleus of a helium atom contains two protons (atomic number 2), and the nucleus of a uranium atom contains 92 protons (atomic number 92). In addition to atomic number, there is a second number characterizing atoms—**mass number,** defined as the number of protons plus neutrons in a nucleus. As we shall see, the atomic number of an element never varies (i.e., the number of protons does not vary) but the mass number can vary (i.e., the number of neutrons varies). Two or more atoms of the same element having different mass numbers are **isotopes** of one another.

The existence of neutrons was verified conclusively in 1932. A *neutron* has no charge and a mass that is about equal to that of a proton. One of its primary purposes is to act as a "glue" that holds the nucleus together. If neutrons were not present in the nucleus, the repulsive force between the positively charged particles would cause the nucleus to come apart.

But is this where the breaking down stops? Protons, neutrons, and a host of other exotic particles are now known to be composed of six different varieties of particles called **quarks,** which have been given the names of *up, down, strange, charm, bottom,* and *top.* The up, charm, and top quarks have charges of $+\frac{2}{3}$ that of the proton, whereas the down, strange, and bottom quarks have charges of $-\frac{1}{3}$ that of the proton. The proton consists of two up quarks and one down quark (Fig. 1.2), which you can easily show leads to the correct charge for the proton. Likewise, the neutron consists of two down quarks and one up quark, giving a net charge of zero.

1.3 DENSITY

A property of any substance is its **density** ρ (Greek letter rho), defined as the amount of mass contained in a unit volume, which we usually express as *mass per unit volume:*

A table of the letters in the Greek alphabet is provided on the back endsheet of this textbook.

$$\rho \equiv \frac{m}{V} \tag{1.1}$$

For example, aluminum has a density of 2.70 g/cm^3, and lead has a density of 11.3 g/cm^3. Therefore, a piece of aluminum of volume 10.0 cm^3 has a mass of 27.0 g, whereas an equivalent volume of lead has a mass of 113 g. A list of densities for various substances is given Table 1.5.

The difference in density between aluminum and lead is due, in part, to their different *atomic masses.* The **atomic mass** of an element is the average mass of one atom in a sample of the element that contains all the element's isotopes, where the relative amounts of isotopes are the same as the relative amounts found in nature. The unit for atomic mass is the *atomic mass unit* (u), where 1 u = $1.660\ 540\ 2 \times 10^{-27}$ kg. The atomic mass of lead is 207 u, and that of aluminum is 27.0 u. However, the ratio of atomic masses, 207 u/27.0 u = 7.67, does not correspond to the ratio of densities, $(11.3\ g/cm^3)/(2.70\ g/cm^3) = 4.19$. The discrepancy is due to the difference in atomic separations and atomic arrangements in the crystal structure of these two substances.

The mass of a nucleus is measured relative to the mass of the nucleus of the carbon-12 isotope, often written as ^{12}C. (This isotope of carbon has six protons and six neutrons. Other carbon isotopes have six protons but different numbers of neutrons.) Practically all of the mass of an atom is contained within the nucleus. Because the atomic mass of ^{12}C is defined to be exactly 12 u, the proton and neutron each have a mass of about 1 u.

One mole (mol) of a substance is that amount of the substance that contains as many particles (atoms, molecules, or other particles) as there are atoms in 12 g of the carbon-12 isotope. One mole of substance A contains the same number of particles as there are in 1 mol of any other substance B. For example, 1 mol of aluminum contains the same number of atoms as 1 mol of lead.

TABLE 1.5 Densities of Various Substances

Substance	Density ρ (10^3 kg/m^3)
Gold	19.3
Uranium	18.7
Lead	11.3
Copper	8.92
Iron	7.86
Aluminum	2.70
Magnesium	1.75
Water	1.00
Air	0.0012

Experiments have shown that this number, known as Avogadro's number, N_A, is

$$N_A = 6.022\,137 \times 10^{23} \text{ particles/mol}$$

Avogadro's number is defined so that 1 mol of carbon-12 atoms has a mass of exactly 12 g. In general, the mass in 1 mol of any element is the element's atomic mass expressed in grams. For example, 1 mol of iron (atomic mass = 55.85 u) has a mass of 55.85 g (we say its *molar mass* is 55.85 g/mol), and 1 mol of lead (atomic mass = 207 u) has a mass of 207 g (its molar mass is 207 g/mol). Because there are 6.02×10^{23} particles in 1 mol of *any* element, the mass per atom for a given element is

$$m_{\text{atom}} = \frac{\text{molar mass}}{N_A} \tag{1.2}$$

For example, the mass of an iron atom is

$$m_{\text{Fe}} = \frac{55.85 \text{ g/mol}}{6.02 \times 10^{23} \text{ atoms/mol}} = 9.28 \times 10^{-23} \text{ g/atom}$$

EXAMPLE 1.1 How Many Atoms in the Cube?

A solid cube of aluminum (density 2.7 g/cm^3) has a volume of 0.20 cm^3. How many aluminum atoms are contained in the cube?

Solution Since density equals mass per unit volume, the mass m of the cube is

$$m = \rho V = (2.7 \text{ g/cm}^3)(0.20 \text{ cm}^3) = 0.54 \text{ g}$$

To find the number of atoms N in this mass of aluminum, we can set up a proportion using the fact that one mole of aluminum (27 g) contains 6.02×10^{23} atoms:

$$\frac{N_A}{27 \text{ g}} = \frac{N}{0.54 \text{ g}}$$

$$\frac{6.02 \times 10^{23} \text{ atoms}}{27 \text{ g}} = \frac{N}{0.54 \text{ g}}$$

$$N = \frac{(0.54 \text{ g})(6.02 \times 10^{23} \text{ atoms})}{27 \text{ g}} = 1.2 \times 10^{22} \text{ atoms}$$

1.4 DIMENSIONAL ANALYSIS

The word *dimension* has a special meaning in physics. It usually denotes the physical nature of a quantity. Whether a distance is measured in the length unit feet or the length unit meters, it is still a distance. We say the dimension—the physical nature—of distance is *length.*

The symbols we use in this book to specify length, mass, and time are L, M, and T, respectively. We shall often use brackets [ ] to denote the dimensions of a physical quantity. For example, the symbol we use for speed in this book is v, and in our notation the dimensions of speed are written $[v] = \text{L/T}$. As another example, the dimensions of area, for which we use the symbol A, are $[A] = \text{L}^2$. The dimensions of area, volume, speed, and acceleration are listed in Table 1.6.

In solving problems in physics, there is a useful and powerful procedure called *dimensional analysis.* This procedure, which should always be used, will help minimize the need for rote memorization of equations. Dimensional analysis makes use of the fact that **dimensions can be treated as algebraic quantities.** That is, quantities can be added or subtracted only if they have the same dimensions. Furthermore, the terms on both sides of an equation must have the same dimensions.

TABLE 1.6 Dimensions and Common Units of Area, Volume, Speed, and Acceleration

System	Area (L^2)	Volume (L^3)	Speed (L/T)	Acceleration (L/T^2)
SI	m^2	m^3	m/s	m/s^2
British engineering	ft^2	ft^3	ft/s	ft/s^2

By following these simple rules, you can use dimensional analysis to help determine whether an expression has the correct form. The relationship can be correct only if the dimensions are the same on both sides of the equation.

To illustrate this procedure, suppose you wish to derive a formula for the distance x traveled by a car in a time t if the car starts from rest and moves with constant acceleration a. In Chapter 2, we shall find that the correct expression is $x = \frac{1}{2}at^2$. Let us use dimensional analysis to check the validity of this expression. The quantity x on the left side has the dimension of length. For the equation to be dimensionally correct, the quantity on the right side must also have the dimension of length. We can perform a dimensional check by substituting the dimensions for acceleration, L/T^2, and time, T, into the equation. That is, the dimensional form of the equation $x = \frac{1}{2}at^2$ is

$$L = \frac{L}{\cancel{T^2}} \cdot \cancel{T^2} = L$$

The units of time squared cancel as shown, leaving the unit of length.

A more general procedure using dimensional analysis is to set up an expression of the form

$$x \propto a^n t^m$$

where n and m are exponents that must be determined and the symbol $\propto$ indicates a proportionality. This relationship is correct only if the dimensions of both sides are the same. Because the dimension of the left side is length, the dimension of the right side must also be length. That is,

$$[a^n t^m] = L = LT^0$$

Because the dimensions of acceleration are L/T^2 and the dimension of time is T, we have

$$\left(\frac{L}{T^2}\right)^n T^m = L^1$$

$$L^n T^{m-2n} = L^1$$

Because the exponents of L and T must be the same on both sides, the dimensional equation is balanced under the conditions $m - 2n = 0$, $n = 1$, and $m = 2$. Returning to our original expression $x \propto a^n t^m$, we conclude that $x \propto at^2$. This result differs by a factor of 2 from the correct expression, which is $x = \frac{1}{2}at^2$. Because the factor $\frac{1}{2}$ is dimensionless, there is no way of determining it using dimensional analysis.

Quick Quiz 1.1

True or False: Dimensional analysis can give you the numerical value of constants of proportionality that may appear in an algebraic expression.

EXAMPLE 1.2 Analysis of an Equation

Show that the expression $v = at$ is dimensionally correct, where v represents speed, a acceleration, and t a time interval.

Solution For the speed term, we have from Table 1.6

$$[v] = \frac{\mathrm{L}}{\mathrm{T}}$$

The same table gives us $\mathrm{L/T^2}$ for the dimensions of acceleration, and so the dimensions of at are

$$[at] = \left(\frac{\mathrm{L}}{\mathrm{T}^{\not 2}}\right)(\not{\mathrm{T}}) = \frac{\mathrm{L}}{\mathrm{T}}$$

Therefore, the expression is dimensionally correct. (If the expression were given as $v = at^2$, it would be dimensionally *incorrect.* Try it and see!)

EXAMPLE 1.3 Analysis of a Power Law

Suppose we are told that the acceleration a of a particle moving with uniform speed v in a circle of radius r is proportional to some power of r, say r^n, and some power of v, say v^m. How can we determine the values of n and m?

Solution Let us take a to be

$$a = kr^n v^m$$

where k is a dimensionless constant of proportionality. Knowing the dimensions of a, r, and v, we see that the dimensional equation must be

$$\mathrm{L/T^2} = \mathrm{L}^n(\mathrm{L/T})^m = \mathrm{L}^{n+m}/\mathrm{T}^m$$

This dimensional equation is balanced under the conditions

$$n + m = 1 \quad \text{and} \quad m = 2$$

Therefore $n = -1$, and we can write the acceleration expression as

$$a = kr^{-1}v^2 = k\frac{v^2}{r}$$

When we discuss uniform circular motion later, we shall see that $k = 1$ if a consistent set of units is used. The constant k would not equal 1 if, for example, v were in km/h and you wanted a in $\mathrm{m/s^2}$.

QuickLab

Estimate the weight (in pounds) of two large bottles of soda pop. Note that 1 L of water has a mass of about 1 kg. Use the fact that an object weighing 2.2 lb has a mass of 1 kg. Find some bathroom scales and check your estimate.

1.5 CONVERSION OF UNITS

Sometimes it is necessary to convert units from one system to another. Conversion factors between the SI units and conventional units of length are as follows:

$$1 \text{ mi} = 1\,609 \text{ m} = 1.609 \text{ km} \qquad 1 \text{ ft} = 0.304\,8 \text{ m} = 30.48 \text{ cm}$$

$$1 \text{ m} = 39.37 \text{ in.} = 3.281 \text{ ft} \qquad 1 \text{ in.} \equiv 0.025\,4 \text{ m} = 2.54 \text{ cm (exactly)}$$

A more complete list of conversion factors can be found in Appendix A.

Units can be treated as algebraic quantities that can cancel each other. For example, suppose we wish to convert 15.0 in. to centimeters. Because 1 in. is defined as exactly 2.54 cm, we find that

$$15.0 \text{ in.} = (15.0 \not{\text{in.}})(2.54 \text{ cm}/\not{\text{in.}}) = 38.1 \text{ cm}$$

This works because multiplying by $\left(\frac{2.54 \text{ cm}}{1 \text{ in.}}\right)$ is the same as multiplying by 1, because the numerator and denominator describe identical things.

(Left) This road sign near Raleigh, North Carolina, shows distances in miles and kilometers. How accurate are the conversions? *(Billy E. Barnes/Stock Boston).*

(Right) This vehicle's speedometer gives speed readings in miles per hour and in kilometers per hour. Try confirming the conversion between the two sets of units for a few readings of the dial. *(Paul Silverman/Fundamental Photographs)*

EXAMPLE 1.4 The Density of a Cube

The mass of a solid cube is 856 g, and each edge has a length of 5.35 cm. Determine the density ρ of the cube in basic SI units.

Solution Because 1 g = 10^{-3} kg and 1 cm = 10^{-2} m, the mass m and volume V in basic SI units are

$$m = 856\ \cancel{\text{g}} \times 10^{-3}\ \text{kg}/\cancel{\text{g}} = 0.856\ \text{kg}$$

$$V = L^3 = (5.35\ \cancel{\text{cm}} \times 10^{-2}\ \text{m}/\cancel{\text{cm}})^3$$
$$= (5.35)^3 \times 10^{-6}\ \text{m}^3 = 1.53 \times 10^{-4}\ \text{m}^3$$

Therefore,

$$\rho = \frac{m}{V} = \frac{0.856\ \text{kg}}{1.53 \times 10^{-4}\ \text{m}^3} = 5.59 \times 10^3\ \text{kg/m}^3$$

1.6 ESTIMATES AND ORDER-OF-MAGNITUDE CALCULATIONS

It is often useful to compute an approximate answer to a physical problem even where little information is available. Such an approximate answer can then be used to determine whether a more accurate calculation is necessary. Approximations are usually based on certain assumptions, which must be modified if greater accuracy is needed. Thus, we shall sometimes refer to the order of magnitude of a certain quantity as the power of ten of the number that describes that quantity. If, for example, we say that a quantity increases in value by three orders of magnitude, this means that its value is increased by a factor of $10^3 = 1000$. Also, if a quantity is given as 3×10^3, we say that the order of magnitude of that quantity is 10^3 (or in symbolic form, $3 \times 10^3 \sim 10^3$). Likewise, the quantity $8 \times 10^7 \sim 10^8$.

The spirit of order-of-magnitude calculations, sometimes referred to as "guesstimates" or "ball-park figures," is given in the following quotation: "Make an estimate before every calculation, try a simple physical argument . . . before every derivation, guess the answer to every puzzle. Courage: no one else needs to know what the guess is."[4] Inaccuracies caused by guessing too low for one number are often canceled out by other guesses that are too high. You will find that with practice your guesstimates get better and better. Estimation problems can be fun to work as you freely drop digits, venture reasonable approximations for unknown numbers, make simplifying assumptions, and turn the question around into something you can answer in your head.

EXAMPLE 1.5 Breaths in a Lifetime

Estimate the number of breaths taken during an average life span.

Solution We shall start by guessing that the typical life span is about 70 years. The only other estimate we must make in this example is the average number of breaths that a person takes in 1 min. This number varies, depending on whether the person is exercising, sleeping, angry, serene, and so forth. To the nearest order of magnitude, we shall choose 10 breaths per minute as our estimate of the average. (This is certainly closer to the true value than 1 breath per minute or 100 breaths per minute.) The number of minutes in a year is approximately

$$1 \cancel{\text{yr}} \times 400 \frac{\cancel{\text{days}}}{\cancel{\text{yr}}} \times 25 \frac{\cancel{\text{h}}}{\cancel{\text{day}}} \times 60 \frac{\text{min}}{\cancel{\text{h}}} = 6 \times 10^5 \text{ min}$$

Notice how much simpler it is to multiply 400×25 than it is to work with the more accurate 365×24. These approximate values for the number of days in a year and the number of hours in a day are close enough for our purposes. Thus, in 70 years there will be $(70 \text{ yr})(6 \times 10^5 \text{ min/yr}) = 4 \times 10^7$ min. At a rate of 10 breaths/min, an individual would take 4×10^8 breaths in a lifetime.

EXAMPLE 1.6 It's a Long Way to San Jose

Estimate the number of steps a person would take walking from New York to Los Angeles.

Solution Without looking up the distance between these two cities, you might remember from a geography class that they are about 3 000 mi apart. The next approximation we must make is the length of one step. Of course, this length depends on the person doing the walking, but we can estimate that each step covers about 2 ft. With our estimated step size, we can determine the number of steps in 1 mi. Because this is a rough calculation, we round 5 280 ft/mi to 5 000 ft/mi. (What percentage error does this introduce?) This conversion factor gives us

$$\frac{5\,000 \cancel{\text{ft}}/\text{mi}}{2 \cancel{\text{ft}}/\text{step}} = 2\,500 \text{ steps/mi}$$

Now we switch to scientific notation so that we can do the calculation mentally:

$$(3 \times 10^3 \cancel{\text{mi}})(2.5 \times 10^3 \text{ steps}/\cancel{\text{mi}}) = 7.5 \times 10^6 \text{ steps}$$

$$\sim 10^7 \text{ steps}$$

So if we intend to walk across the United States, it will take us on the order of ten million steps. This estimate is almost certainly too small because we have not accounted for curving roads and going up and down hills and mountains. Nonetheless, it is probably within an order of magnitude of the correct answer.

EXAMPLE 1.7 How Much Gas Do We Use?

Estimate the number of gallons of gasoline used each year by all the cars in the United States.

Solution There are about 270 million people in the United States, and so we estimate that the number of cars in the country is 100 million (guessing that there are between two and three people per car). We also estimate that the average distance each car travels per year is 10 000 mi. If we assume a gasoline consumption of 20 mi/gal or 0.05 gal/mi, then each car uses about 500 gal/yr. Multiplying this by the total number of cars in the United States gives an estimated total consumption of 5×10^{10} gal $\sim 10^{11}$ gal.

[4] E. Taylor and J. A. Wheeler, *Spacetime Physics,* San Francisco, W. H. Freeman & Company, Publishers, 1966, p. 60.

1.7 SIGNIFICANT FIGURES

When physical quantities are measured, the measured values are known only to within the limits of the experimental uncertainty. The value of this uncertainty can depend on various factors, such as the quality of the apparatus, the skill of the experimenter, and the number of measurements performed.

Suppose that we are asked to measure the area of a computer disk label using a meter stick as a measuring instrument. Let us assume that the accuracy to which we can measure with this stick is ± 0.1 cm. If the length of the label is measured to be 5.5 cm, we can claim only that its length lies somewhere between 5.4 cm and 5.6 cm. In this case, we say that the measured value has two significant figures. Likewise, if the label's width is measured to be 6.4 cm, the actual value lies between 6.3 cm and 6.5 cm. Note that the significant figures include the first estimated digit. Thus we could write the measured values as (5.5 ± 0.1) cm and (6.4 ± 0.1) cm.

Now suppose we want to find the area of the label by multiplying the two measured values. If we were to claim the area is $(5.5 \text{ cm})(6.4 \text{ cm}) = 35.2 \text{ cm}^2$, our answer would be unjustifiable because it contains three significant figures, which is greater than the number of significant figures in either of the measured lengths. A good rule of thumb to use in determining the number of significant figures that can be claimed is as follows:

> When multiplying several quantities, the number of significant figures in the final answer is the same as the number of significant figures in the *least* accurate of the quantities being multiplied, where "least accurate" means "having the lowest number of significant figures." The same rule applies to division.

Applying this rule to the multiplication example above, we see that the answer for the area can have only two significant figures because our measured lengths have only two significant figures. Thus, all we can claim is that the area is 35 cm^2, realizing that the value can range between $(5.4 \text{ cm})(6.3 \text{ cm}) = 34 \text{ cm}^2$ and $(5.6 \text{ cm})(6.5 \text{ cm}) = 36 \text{ cm}^2$.

Zeros may or may not be significant figures. Those used to position the decimal point in such numbers as 0.03 and 0.007 5 are not significant. Thus, there are one and two significant figures, respectively, in these two values. When the zeros come after other digits, however, there is the possibility of misinterpretation. For example, suppose the mass of an object is given as 1 500 g. This value is ambiguous because we do not know whether the last two zeros are being used to locate the decimal point or whether they represent significant figures in the measurement. To remove this ambiguity, it is common to use scientific notation to indicate the number of significant figures. In this case, we would express the mass as 1.5×10^3 g if there are two significant figures in the measured value, 1.50×10^3 g if there are three significant figures, and 1.500×10^3 g if there are four. The same rule holds when the number is less than 1, so that 2.3×10^{-4} has two significant figures (and so could be written 0.000 23) and 2.30×10^{-4} has three significant figures (also written 0.000 230). In general, **a significant figure is a reliably known digit** (other than a zero used to locate the decimal point).

For addition and subtraction, you must consider the number of decimal places when you are determining how many significant figures to report.

QuickLab

Determine the thickness of a page from this book. (Note that numbers that have no measurement errors—like the count of a number of pages—do not affect the significant figures in a calculation.) In terms of significant figures, why is it better to measure the thickness of as many pages as possible and then divide by the number of sheets?

When numbers are added or subtracted, the number of decimal places in the result should equal the smallest number of decimal places of any term in the sum.

For example, if we wish to compute 123 + 5.35, the answer given to the correct number of significant figures is 128 and not 128.35. If we compute the sum 1.000 1 + 0.000 3 = 1.000 4, the result has five significant figures, even though one of the terms in the sum, 0.000 3, has only one significant figure. Likewise, if we perform the subtraction 1.002 − 0.998 = 0.004, the result has only one significant figure even though one term has four significant figures and the other has three. In this book, **most of the numerical examples and end-of-chapter problems will yield answers having three significant figures.** When carrying out estimates we shall typically work with a single significant figure.

Quick Quiz 1.2

Suppose you measure the position of a chair with a meter stick and record that the center of the seat is 1.043 860 564 2 m from a wall. What would a reader conclude from this recorded measurement?

EXAMPLE 1.8 The Area of a Rectangle

A rectangular plate has a length of (21.3 ± 0.2) cm and a width of (9.80 ± 0.1) cm. Find the area of the plate and the uncertainty in the calculated area.

Solution

$$\text{Area} = \ell w = (21.3 \pm 0.2 \text{ cm}) \times (9.80 \pm 0.1 \text{ cm})$$

$$\approx (21.3 \times 9.80 \pm 21.3 \times 0.1 \pm 0.2 \times 9.80) \text{ cm}^2$$

$$\approx (209 \pm 4) \text{ cm}^2$$

Because the input data were given to only three significant figures, we cannot claim any more in our result. Do you see why we did not need to multiply the uncertainties 0.2 cm and 0.1 cm?

EXAMPLE 1.9 Installing a Carpet

A carpet is to be installed in a room whose length is measured to be 12.71 m and whose width is measured to be 3.46 m. Find the area of the room.

Solution If you multiply 12.71 m by 3.46 m on your calculator, you will get an answer of 43.976 6 m^2. How many of these numbers should you claim? Our rule of thumb for multiplication tells us that you can claim only the number of significant figures in the least accurate of the quantities being measured. In this example, we have only three significant figures in our least accurate measurement, so we should express our final answer as 44.0 m^2.

Note that in reducing 43.976 6 to three significant figures for our answer, we used a general rule for rounding off numbers that states that the last digit retained (the 9 in this example) is increased by 1 if the first digit dropped (here, the 7) is 5 or greater. (A technique for avoiding error accumulation is to delay rounding of numbers in a long calculation until you have the final result. Wait until you are ready to copy the answer from your calculator before rounding to the correct number of significant figures.)

SUMMARY

The three fundamental physical quantities of mechanics are length, mass, and time, which in the SI system have the units meters (m), kilograms (kg), and seconds (s), respectively. Prefixes indicating various powers of ten are used with these three basic units. The **density** of a substance is defined as its *mass per unit volume.* Different substances have different densities mainly because of differences in their atomic masses and atomic arrangements.

The number of particles in one mole of any element or compound, called **Avogadro's number,** N_A, is 6.02×10^{23}.

The method of *dimensional analysis* is very powerful in solving physics problems. Dimensions can be treated as algebraic quantities. By making estimates and making order-of-magnitude calculations, you should be able to approximate the answer to a problem when there is not enough information available to completely specify an exact solution.

When you compute a result from several measured numbers, each of which has a certain accuracy, you should give the result with the correct number of significant figures.

QUESTIONS

1. In this chapter we described how the Earth's daily rotation on its axis was once used to define the standard unit of time. What other types of natural phenomena could serve as alternative time standards?

2. Suppose that the three fundamental standards of the metric system were length, density, and time rather than length, mass, and time. The standard of density in this system is to be defined as that of water. What considerations about water would you need to address to make sure that the standard of density is as accurate as possible?

3. A hand is defined as 4 in.; a foot is defined as 12 in. Why should the hand be any less acceptable as a unit than the foot, which we use all the time?

4. Express the following quantities using the prefixes given in Table 1.4: (a) 3×10^{-4} m (b) 5×10^{-5} s (c) 72×10^{2} g.

5. Suppose that two quantities A and B have different dimensions. Determine which of the following arithmetic operations *could* be physically meaningful: (a) $A + B$ (b) A/B (c) $B - A$ (d) AB.

6. What level of accuracy is implied in an order-of-magnitude calculation?

7. Do an order-of-magnitude calculation for an everyday situation you might encounter. For example, how far do you walk or drive each day?

8. Estimate your age in seconds.

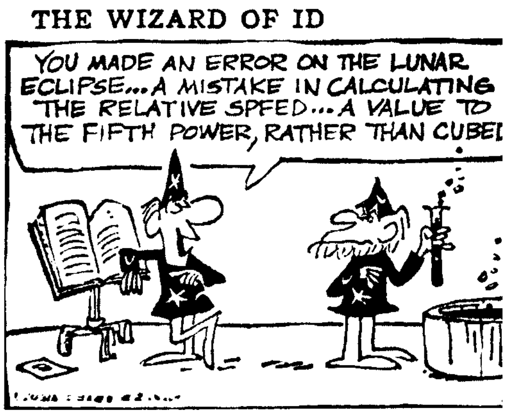
9. Estimate the mass of this textbook in kilograms. If a scale is available, check your estimate.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 1.3 Density

1. The standard kilogram is a platinum–iridium cylinder 39.0 mm in height and 39.0 mm in diameter. What is the density of the material?
2. The mass of the planet Saturn (Fig. P1.2) is 5.64×10^{26} kg, and its radius is 6.00×10^{7} m. Calculate its density.
3. How many grams of copper are required to make a hollow spherical shell having an inner radius of 5.70 cm and an outer radius of 5.75 cm? The density of copper is 8.92 g/cm^3.
4. What mass of a material with density ρ is required to make a hollow spherical shell having inner radius r_1 and outer radius r_2?
5. Iron has molar mass 55.8 g/mol. (a) Find the volume of 1 mol of iron. (b) Use the value found in (a) to determine the volume of one iron atom. (c) Calculate the cube root of the atomic volume, to have an estimate for the distance between atoms in the solid. (d) Repeat the calculations for uranium, finding its molar mass in the periodic table of the elements in Appendix C.



By permission of John Hart and Field Enterprises, Inc.

Figure P1.2 A view of Saturn from *Voyager 2.* *(Courtesy of NASA)*

6. Two spheres are cut from a certain uniform rock. One has radius 4.50 cm. The mass of the other is five times greater. Find its radius.

WEB **7.** Calculate the mass of an atom of (a) helium, (b) iron, and (c) lead. Give your answers in atomic mass units and in grams. The molar masses are 4.00, 55.9, and 207 g/mol, respectively, for the atoms given.

8. On your wedding day your lover gives you a gold ring of mass 3.80 g. Fifty years later its mass is 3.35 g. As an average, how many atoms were abraded from the ring during each second of your marriage? The molar mass of gold is 197 g/mol.

9. A small cube of iron is observed under a microscope. The edge of the cube is 5.00×10^{-6} cm long. Find (a) the mass of the cube and (b) the number of iron atoms in the cube. The molar mass of iron is 55.9 g/mol, and its density is 7.86 g/cm^3.

10. A structural I-beam is made of steel. A view of its cross-section and its dimensions are shown in Figure P1.10.

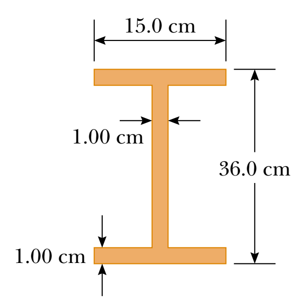


Figure P1.10

(a) What is the mass of a section 1.50 m long? (b) How many atoms are there in this section? The density of steel is 7.56×10^3 kg/m^3.

11. A child at the beach digs a hole in the sand and, using a pail, fills it with water having a mass of 1.20 kg. The molar mass of water is 18.0 g/mol. (a) Find the number of water molecules in this pail of water. (b) Suppose the quantity of water on the Earth is 1.32×10^{21} kg and remains constant. How many of the water molecules in this pail of water were likely to have been in an equal quantity of water that once filled a particular claw print left by a dinosaur?

Section 1.4 Dimensional Analysis

12. The radius r of a circle inscribed in any triangle whose sides are a, b, and c is given by

$$r = [(s - a)(s - b)(s - c)/s]^{1/2}$$

where s is an abbreviation for $(a + b + c)/2$. Check this formula for dimensional consistency.

13. The displacement of a particle moving under uniform acceleration is some function of the elapsed time and the acceleration. Suppose we write this displacement $s = ka^m t^n$, where k is a dimensionless constant. Show by dimensional analysis that this expression is satisfied if $m = 1$ and $n = 2$. Can this analysis give the value of k?

14. The period T of a simple pendulum is measured in time units and is described by

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

where ℓ is the length of the pendulum and g is the free-fall acceleration in units of length divided by the square of time. Show that this equation is dimensionally correct.

15. Which of the equations below are dimensionally correct?
(a) $v = v_0 + ax$
(b) $y = (2\text{ m})\cos(kx)$, where $k = 2\text{ m}^{-1}$

16. Newton's law of universal gravitation is represented by

$$F = \frac{GMm}{r^2}$$

Here F is the gravitational force, M and m are masses, and r is a length. Force has the SI units kg·m/s^2. What are the SI units of the proportionality constant G?

WEB **17.** The consumption of natural gas by a company satisfies the empirical equation $V = 1.50t + 0.008\,00t^2$, where V is the volume in millions of cubic feet and t the time in months. Express this equation in units of cubic feet and seconds. Put the proper units on the coefficients. Assume a month is 30.0 days.

Section 1.5 Conversion of Units

18. Suppose your hair grows at the rate 1/32 in. per day. Find the rate at which it grows in nanometers per second. Since the distance between atoms in a molecule is on the order of 0.1 nm, your answer suggests how rapidly layers of atoms are assembled in this protein synthesis.

19. A rectangular building lot is 100 ft by 150 ft. Determine the area of this lot in m^2.

20. An auditorium measures 40.0 m × 20.0 m × 12.0 m. The density of air is 1.20 kg/m^3. What are (a) the volume of the room in cubic feet and (b) the weight of air in the room in pounds?

21. Assume that it takes 7.00 min to fill a 30.0-gal gasoline tank. (a) Calculate the rate at which the tank is filled in gallons per second. (b) Calculate the rate at which the tank is filled in cubic meters per second. (c) Determine the time, in hours, required to fill a 1-cubic-meter volume at the same rate. (1 U.S. gal = 231 $in.^3$)

22. A creature moves at a speed of 5.00 furlongs per fortnight (not a very common unit of speed). Given that 1 furlong = 220 yards and 1 fortnight = 14 days, determine the speed of the creature in meters per second. What kind of creature do you think it might be?

23. A section of land has an area of 1 mi^2 and contains 640 acres. Determine the number of square meters in 1 acre.

24. A quart container of ice cream is to be made in the form of a cube. What should be the length of each edge in centimeters? (Use the conversion 1 gal = 3.786 L.)

25. A solid piece of lead has a mass of 23.94 g and a volume of 2.10 cm^3. From these data, calculate the density of lead in SI units (kg/m^3).

26. An astronomical unit (AU) is defined as the average distance between the Earth and the Sun. (a) How many astronomical units are there in one lightyear? (b) Determine the distance from the Earth to the Andromeda galaxy in astronomical units.

27. The mass of the Sun is 1.99×10^{30} kg, and the mass of an atom of hydrogen, of which the Sun is mostly composed, is 1.67×10^{-27} kg. How many atoms are there in the Sun?

28. (a) Find a conversion factor to convert from miles per hour to kilometers per hour. (b) In the past, a federal law mandated that highway speed limits would be 55 mi/h. Use the conversion factor of part (a) to find this speed in kilometers per hour. (c) The maximum highway speed is now 65 mi/h in some places. In kilometers per hour, how much of an increase is this over the 55-mi/h limit?

29. At the time of this book's printing, the U. S. national debt is about \$6 trillion. (a) If payments were made at the rate of \$1 000/s, how many years would it take to pay off a \$6-trillion debt, assuming no interest were charged? (b) A dollar bill is about 15.5 cm long. If six trillion dollar bills were laid end to end around the Earth's equator, how many times would they encircle the Earth? Take the radius of the Earth at the equator to be 6 378 km. (*Note:* Before doing any of these calculations, try to guess at the answers. You may be very surprised.)

30. (a) How many seconds are there in a year? (b) If one micrometeorite (a sphere with a diameter of 1.00×10^{-6} m) strikes each square meter of the Moon each second, how many years will it take to cover the Moon to a depth of 1.00 m? (*Hint:* Consider a cubic box on the Moon 1.00 m on a side, and find how long it will take to fill the box.)

WEB **31.** One gallon of paint (volume = 3.78×10^{-3} m^3) covers an area of 25.0 m^2. What is the thickness of the paint on the wall?

32. A pyramid has a height of 481 ft, and its base covers an area of 13.0 acres (Fig. P1.32). If the volume of a pyramid is given by the expression $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height, find the volume of this pyramid in cubic meters. (1 acre = 43 560 ft^2)

Figure P1.32 Problems 32 and 33.

33. The pyramid described in Problem 32 contains approximately two million stone blocks that average 2.50 tons each. Find the weight of this pyramid in pounds.

34. Assuming that 70% of the Earth's surface is covered with water at an average depth of 2.3 mi, estimate the mass of the water on the Earth in kilograms.

35. The amount of water in reservoirs is often measured in acre-feet. One acre-foot is a volume that covers an area of 1 acre to a depth of 1 ft. An acre is an area of 43 560 ft^2. Find the volume in SI units of a reservoir containing 25.0 acre-ft of water.

36. A hydrogen atom has a diameter of approximately 1.06×10^{-10} m, as defined by the diameter of the spherical electron cloud around the nucleus. The hydrogen nucleus has a diameter of approximately 2.40×10^{-15} m. (a) For a scale model, represent the diameter of the hydrogen atom by the length of an American football field (100 yards = 300 ft), and determine the diameter of the nucleus in millimeters. (b) The atom is how many times larger in volume than its nucleus?

37. The diameter of our disk-shaped galaxy, the Milky Way, is about 1.0×10^5 lightyears. The distance to Messier 31—which is Andromeda, the spiral galaxy nearest to the Milky Way—is about 2.0 million lightyears. If a scale model represents the Milky Way and Andromeda galaxies as dinner plates 25 cm in diameter, determine the distance between the two plates.

38. The mean radius of the Earth is 6.37×10^6 m, and that of the Moon is 1.74×10^8 cm. From these data calculate (a) the ratio of the Earth's surface area to that of the Moon and (b) the ratio of the Earth's volume to that of the Moon. Recall that the surface area of a sphere is $4\pi r^2$ and that the volume of a sphere is $\frac{4}{3}\pi r^3$.

WEB **39.** One cubic meter (1.00 m^3) of aluminum has a mass of 2.70×10^3 kg, and 1.00 m^3 of iron has a mass of 7.86×10^3 kg. Find the radius of a solid aluminum sphere that balances a solid iron sphere of radius 2.00 cm on an equal-arm balance.

40. Let ρ_{Al} represent the density of aluminum and ρ_{Fe} that of iron. Find the radius of a solid aluminum sphere that balances a solid iron sphere of radius r_{Fe} on an equal-arm balance.

Section 1.6 Estimates and Order-of-Magnitude Calculations

WEB **41.** Estimate the number of Ping-Pong balls that would fit into an average-size room (without being crushed). In your solution state the quantities you measure or estimate and the values you take for them.

42. McDonald's sells about 250 million packages of French fries per year. If these fries were placed end to end, estimate how far they would reach.

43. An automobile tire is rated to last for 50 000 miles. Estimate the number of revolutions the tire will make in its lifetime.

44. Approximately how many raindrops fall on a 1.0-acre lot during a 1.0-in. rainfall?

45. Grass grows densely everywhere on a quarter-acre plot of land. What is the order of magnitude of the number of blades of grass on this plot of land? Explain your reasoning. (1 acre = 43 560 ft^2.)

46. Suppose that someone offers to give you $1 billion if you can finish counting it out using only one-dollar bills. Should you accept this offer? Assume you can count one bill every second, and be sure to note that you need about 8 hours a day for sleeping and eating and that right now you are probably at least 18 years old.

47. Compute the order of magnitude of the mass of a bathtub half full of water and of the mass of a bathtub half full of pennies. In your solution, list the quantities you take as data and the value you measure or estimate for each.

48. Soft drinks are commonly sold in aluminum containers. Estimate the number of such containers thrown away or recycled each year by U.S. consumers. Approximately how many tons of aluminum does this represent?

49. To an order of magnitude, how many piano tuners are there in New York City? The physicist Enrico Fermi was famous for asking questions like this on oral Ph.D. qualifying examinations and for his own facility in making order-of-magnitude calculations.

Section 1.7 Significant Figures

50. Determine the number of significant figures in the following measured values: (a) 23 cm (b) 3.589 s (c) 4.67×10^3 m/s (d) 0.003 2 m.

51. The radius of a circle is measured to be 10.5 ± 0.2 m. Calculate the (a) area and (b) circumference of the circle and give the uncertainty in each value.

52. Carry out the following arithmetic operations: (a) the sum of the measured values 756, 37.2, 0.83, and 2.5; (b) the product $0.003\,2 \times 356.3$; (c) the product $5.620 \times \pi$.

53. The radius of a solid sphere is measured to be (6.50 ± 0.20) cm, and its mass is measured to be (1.85 ± 0.02) kg. Determine the density of the sphere in kilograms per cubic meter and the uncertainty in the density.

54. How many significant figures are in the following numbers: (a) 78.9 ± 0.2, (b) 3.788×10^9, (c) 2.46×10^{-6}, and (d) 0.005 3?

55. A farmer measures the distance around a rectangular field. The length of the long sides of the rectangle is found to be 38.44 m, and the length of the short sides is found to be 19.5 m. What is the total distance around the field?

56. A sidewalk is to be constructed around a swimming pool that measures (10.0 ± 0.1) m by (17.0 ± 0.1) m. If the sidewalk is to measure (1.00 ± 0.01) m wide by (9.0 ± 0.1) cm thick, what volume of concrete is needed, and what is the approximate uncertainty of this volume?

ADDITIONAL PROBLEMS

57. In a situation where data are known to three significant digits, we write 6.379 m = 6.38 m and 6.374 m = 6.37 m. When a number ends in 5, we arbitrarily choose to write 6.375 m = 6.38 m. We could equally well write 6.375 m = 6.37 m, "rounding down" instead of "rounding up," since we would change the number 6.375 by equal increments in both cases. Now consider an order-of-magnitude estimate, in which we consider factors rather than increments. We write 500 m $\sim 10^3$ m because 500 differs from 100 by a factor of 5 whereas it differs from 1000 by only a factor of 2. We write 437 m $\sim 10^3$ m and 305 m $\sim 10^2$ m. What distance differs from 100 m and from 1000 m by equal factors, so that we could equally well choose to represent its order of magnitude either as $\sim 10^2$ m or as $\sim 10^3$ m?

58. When a droplet of oil spreads out on a smooth water surface, the resulting "oil slick" is approximately one molecule thick. An oil droplet of mass 9.00×10^{-7} kg and density 918 kg/m^3 spreads out into a circle of radius 41.8 cm on the water surface. What is the diameter of an oil molecule?

59. The basic function of the carburetor of an automobile is to "atomize" the gasoline and mix it with air to promote rapid combustion. As an example, assume that 30.0 cm^3 of gasoline is atomized into N spherical droplets, each with a radius of 2.00×10^{-5} m. What is the total surface area of these N spherical droplets?

60. In physics it is important to use mathematical approximations. Demonstrate for yourself that for small angles ($< 20°$)

$$\tan \alpha \approx \sin \alpha \approx \alpha = \pi\alpha'/180°$$

where α is in radians and α' is in degrees. Use a calculator to find the largest angle for which $\tan \alpha$ may be approximated by $\sin \alpha$ if the error is to be less than 10.0%.

61. A high fountain of water is located at the center of a circular pool as in Figure P1.61. Not wishing to get his feet wet, a student walks around the pool and measures its circumference to be 15.0 m. Next, the student stands at the edge of the pool and uses a protractor to gauge the angle of elevation of the top of the fountain to be 55.0°. How high is the fountain?

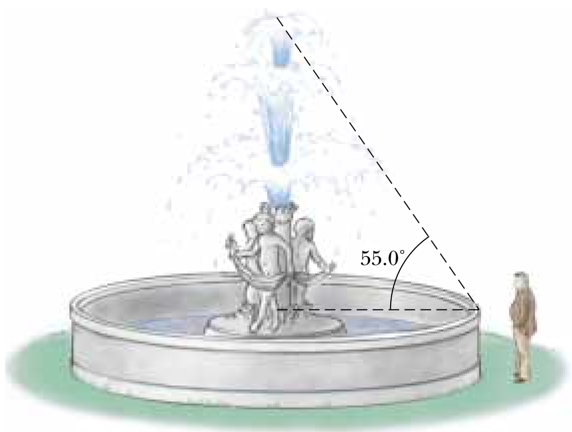


Figure P1.61

62. Assume that an object covers an area A and has a uniform height h. If its cross-sectional area is uniform over its height, then its volume is given by $V = Ah$. (a) Show that $V = Ah$ is dimensionally correct. (b) Show that the volumes of a cylinder and of a rectangular box can be written in the form $V = Ah$, identifying A in each case. (Note that A, sometimes called the "footprint" of the object, can have any shape and that the height can be replaced by average thickness in general.)

63. A useful fact is that there are about $\pi \times 10^7$ s in one year. Find the percentage error in this approximation, where "percentage error" is defined as

$$\frac{|\text{Assumed value} - \text{true value}|}{\text{True value}} \times 100\%$$

64. A crystalline solid consists of atoms stacked up in a repeating lattice structure. Consider a crystal as shown in Figure P1.64a. The atoms reside at the corners of cubes of side $L = 0.200$ nm. One piece of evidence for the regular arrangement of atoms comes from the flat surfaces along which a crystal separates, or "cleaves," when it is broken. Suppose this crystal cleaves along a face diagonal, as shown in Figure P1.64b. Calculate the spacing d between two adjacent atomic planes that separate when the crystal cleaves.

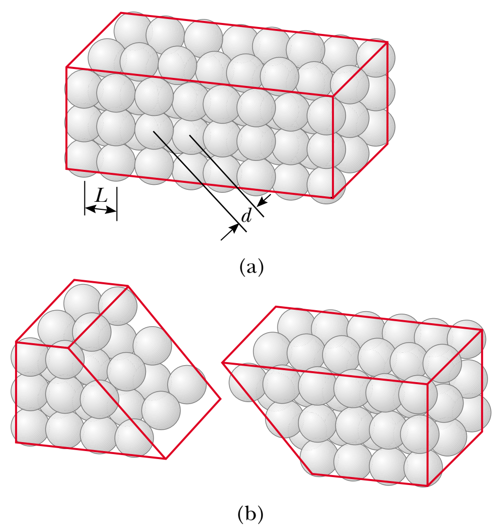


Figure P1.64

65. A child loves to watch as you fill a transparent plastic bottle with shampoo. Every horizontal cross-section of the bottle is a circle, but the diameters of the circles all have different values, so that the bottle is much wider in some places than in others. You pour in bright green shampoo with constant volume flow rate 16.5 cm^3/s. At what rate is its level in the bottle rising (a) at a point where the diameter of the bottle is 6.30 cm and (b) at a point where the diameter is 1.35 cm?

66. As a child, the educator and national leader Booker T. Washington was given a spoonful (about 12.0 cm^3) of molasses as a treat. He pretended that the quantity increased when he spread it out to cover uniformly all of a tin plate (with a diameter of about 23.0 cm). How thick a layer did it make?

67. Assume there are 100 million passenger cars in the United States and that the average fuel consumption is 20 mi/gal of gasoline. If the average distance traveled by each car is 10 000 mi/yr, how much gasoline would be saved per year if average fuel consumption could be increased to 25 mi/gal?

68. One cubic centimeter of water has a mass of 1.00×10^{-3} kg. (a) Determine the mass of 1.00 m^3 of water. (b) Assuming biological substances are 98% water, estimate the mass of a cell that has a diameter of 1.0 μm, a human kidney, and a fly. Assume that a kidney is roughly a sphere with a radius of 4.0 cm and that a fly is roughly a cylinder 4.0 mm long and 2.0 mm in diameter.

69. The distance from the Sun to the nearest star is 4×10^{16} m. The Milky Way galaxy is roughly a disk of diameter $\sim 10^{21}$ m and thickness $\sim 10^{19}$ m. Find the order of magnitude of the number of stars in the Milky Way. Assume the 4×10^{16}-m distance between the Sun and the nearest star is typical.

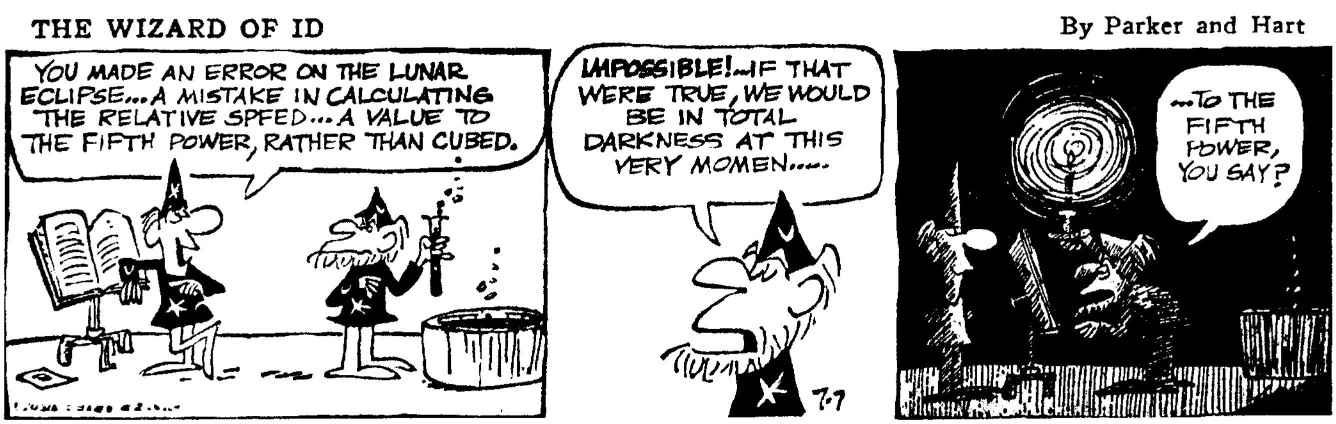
70. The data in the following table represent measurements of the masses and dimensions of solid cylinders of aluminum, copper, brass, tin, and iron. Use these data to calculate the densities of these substances. Compare your results for aluminum, copper, and iron with those given in Table 1.5.

Substance	**Mass (g)**	**Diameter (cm)**	**Length (cm)**
Aluminum	51.5	2.52	3.75
Copper	56.3	1.23	5.06
Brass	94.4	1.54	5.69
Tin	69.1	1.75	3.74
Iron	216.1	1.89	9.77

ANSWERS TO QUICK QUIZZES

1.1 False. Dimensional analysis gives the units of the proportionality constant but provides no information about its numerical value. For example, experiments show that doubling the radius of a solid sphere increases its mass 8-fold, and tripling the radius increases the mass 27-fold. Therefore, its mass is proportional to the cube of its radius. Because $m \propto r^3$ we can write $m = kr^3$. Dimensional analysis shows that the proportionality constant k must have units kg/m^3, but to determine its numerical value requires either experimental data or geometrical reasoning.

1.2 Reporting all these digits implies you have determined the location of the center of the chair's seat to the nearest $\pm 0.000\,000\,000\,1$ m. This roughly corresponds to being able to count the atoms in your meter stick because each of them is about that size! It would probably be better to record the measurement as 1.044 m: this indicates that you know the position to the nearest millimeter, assuming the meter stick has millimeter markings on its scale.



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PUZZLER

In a moment the arresting cable will be pulled taut, and the 140-mi/h landing of this F/A-18 Hornet on the aircraft carrier USS *Nimitz* will be brought to a sudden conclusion. The pilot cuts power to the engine, and the plane is stopped in less than 2 s. If the cable had not been successfully engaged, the pilot would have had to take off quickly before reaching the end of the flight deck. Can the motion of the plane be described quantitatively in a way that is useful to ship and aircraft designers and to pilots learning to land on a "postage stamp?" *(Courtesy of the USS* Nimitz*/U.S. Navy)*

chapter 2

Motion in One Dimension

Chapter Outline

As a first step in studying classical mechanics, we describe motion in terms of space and time while ignoring the agents that caused that motion. This portion of classical mechanics is called *kinematics.* (The word *kinematics* has the same root as *cinema.* Can you see why?) In this chapter we consider only motion in one dimension. We first define displacement, velocity, and acceleration. Then, using these concepts, we study the motion of objects traveling in one dimension with a constant acceleration.

From everyday experience we recognize that motion represents a continuous change in the position of an object. In physics we are concerned with three types of motion: translational, rotational, and vibrational. A car moving down a highway is an example of translational motion, the Earth's spin on its axis is an example of rotational motion, and the back-and-forth movement of a pendulum is an example of vibrational motion. In this and the next few chapters, we are concerned only with translational motion. (Later in the book we shall discuss rotational and vibrational motions.)

In our study of translational motion, we describe the moving object as a *particle* regardless of its size. In general, **a particle is a point-like mass having infinitesimal size.** For example, if we wish to describe the motion of the Earth around the Sun, we can treat the Earth as a particle and obtain reasonably accurate data about its orbit. This approximation is justified because the radius of the Earth's orbit is large compared with the dimensions of the Earth and the Sun. As an example on a much smaller scale, it is possible to explain the pressure exerted by a gas on the walls of a container by treating the gas molecules as particles.

2.1 DISPLACEMENT, VELOCITY, AND SPEED

TABLE 2.1
Position of the Car at Various Times

Position	t(s)	x(m)
Ⓐ	0	30
Ⓑ	10	52
Ⓒ	20	38
Ⓓ	30	0
Ⓔ	40	−37
Ⓕ	50	−53

The motion of a particle is completely known if the particle's position in space is known at all times. Consider a car moving back and forth along the x axis, as shown in Figure 2.1a. When we begin collecting position data, the car is 30 m to the right of a road sign. (Let us assume that all data in this example are known to two significant figures. To convey this information, we should report the initial position as 3.0×10^1 m. We have written this value in this simpler form to make the discussion easier to follow.) We start our clock and once every 10 s note the car's location relative to the sign. As you can see from Table 2.1, the car is moving to the right (which we have defined as the positive direction) during the first 10 s of motion, from position Ⓐ to position Ⓑ. The position values now begin to decrease, however, because the car is backing up from position Ⓑ through position Ⓕ. In fact, at Ⓓ, 30 s after we start measuring, the car is alongside the sign we are using as our origin of coordinates. It continues moving to the left and is more than 50 m to the left of the sign when we stop recording information after our sixth data point. A graph of this information is presented in Figure 2.1b. Such a plot is called a *position–time graph.*

If a particle is moving, we can easily determine its change in position. **The displacement of a particle is defined as its change in position.** As it moves from an initial position x_i to a final position x_f, its displacement is given by $x_f - x_i$. We use the Greek letter delta (Δ) to denote the *change* in a quantity. Therefore, we write the displacement, or change in position, of the particle as

$$\Delta x \equiv x_f - x_i \quad \textbf{(2.1)}$$

From this definition we see that Δx is positive if x_f is greater than x_i and negative if x_f is less than x_i.

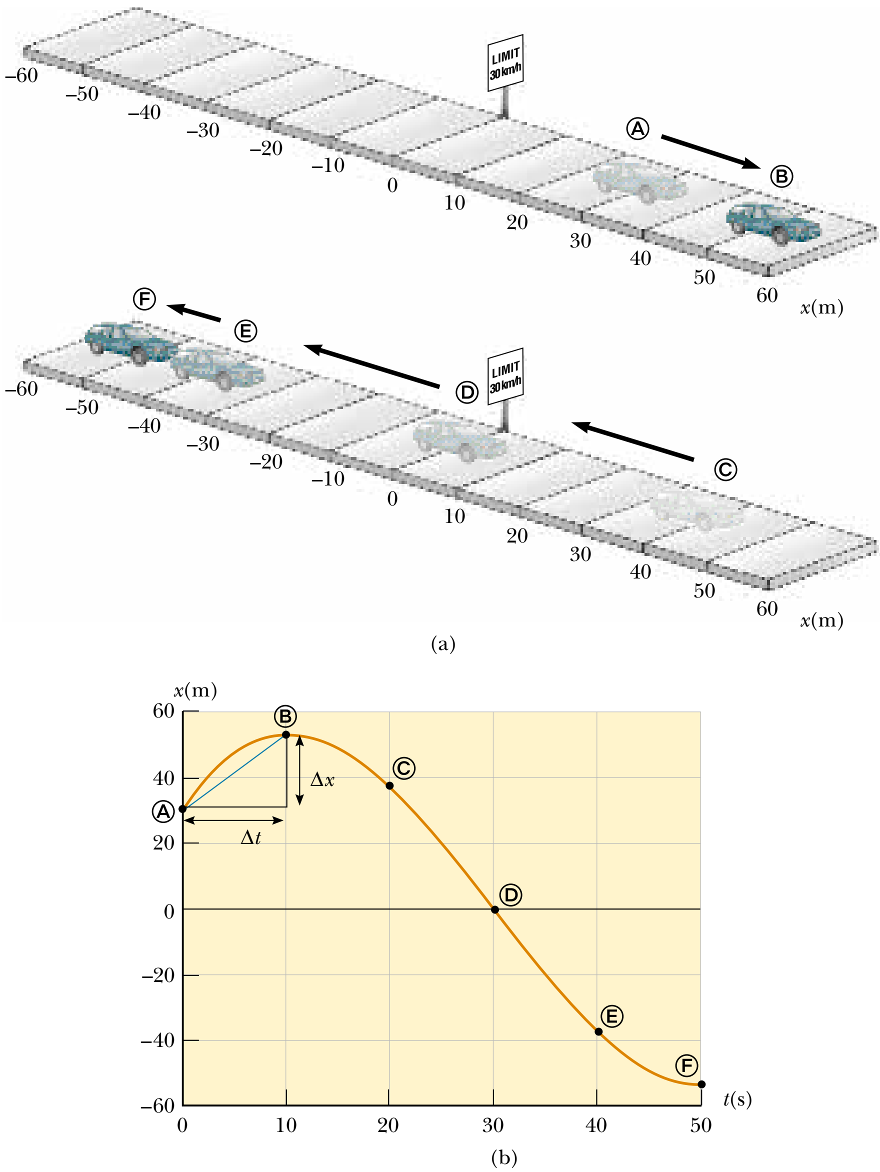


Figure 2.1 (a) A car moves back and forth along a straight line taken to be the *x* axis. Because we are interested only in the car's translational motion, we can treat it as a particle. (b) Position–time graph for the motion of the "particle."

A very easy mistake to make is not to recognize the difference between displacement and distance traveled (Fig. 2.2). A baseball player hitting a home run travels a distance of 360 ft in the trip around the bases. However, the player's displacement is zero because his final and initial positions are identical.

Displacement is an example of a vector quantity. Many other physical quantities, including velocity and acceleration, also are vectors. In general, **a vector is a physical quantity that requires the specification of both direction and magnitude.** By contrast, **a scalar is a quantity that has magnitude and no direction.** In this chapter, we use plus and minus signs to indicate vector direction. We can do this because the chapter deals with one-dimensional motion only; this means that any object we study can be moving only along a straight line. For example, for horizontal motion, let us arbitrarily specify to the right as being the positive direction. It follows that any object always moving to the right undergoes a

Figure 2.2 Bird's-eye view of a baseball diamond. A batter who hits a home run travels 360 ft as he rounds the bases, but his displacement for the round trip is zero. *(Mark C. Burnett/Photo Researchers, Inc.)*

positive displacement $+\Delta x$, and any object moving to the left undergoes a negative displacement $-\Delta x$. We shall treat vectors in greater detail in Chapter 3.

There is one very important point that has not yet been mentioned. Note that the graph in Figure 2.1b does not consist of just six data points but is actually a smooth curve. The graph contains information about the entire 50-s interval during which we watched the car move. It is much easier to see changes in position from the graph than from a verbal description or even a table of numbers. For example, it is clear that the car was covering more ground during the middle of the 50-s interval than at the end. Between positions Ⓒ and Ⓓ, the car traveled almost 40 m, but during the last 10 s, between positions Ⓔ and Ⓕ, it moved less than half that far. A common way of comparing these different motions is to divide the displacement Δx that occurs between two clock readings by the length of that particular time interval Δt. This turns out to be a very useful ratio, one that we shall use many times. For convenience, the ratio has been given a special name—*average velocity*. **The average velocity $\bar{v}_x$ of a particle is defined as the particle's displacement Δx divided by the time interval Δt during which that displacement occurred:**

Average velocity

$$\bar{v}_x \equiv \frac{\Delta x}{\Delta t} \tag{2.2}$$

3.2 where the subscript x indicates motion along the x axis. From this definition we see that average velocity has dimensions of length divided by time (L/T)—meters per second in SI units.

Although the distance traveled for any motion is always positive, the average velocity of a particle moving in one dimension can be positive or negative, depending on the sign of the displacement. (The time interval Δt is always positive.) If the coordinate of the particle increases in time (that is, if $x_f > x_i$), then Δx is positive and $\bar{v}_x = \Delta x/\Delta t$ is positive. This case corresponds to motion in the positive x direction. If the coordinate decreases in time (that is, if $x_f < x_i$), then Δx is negative and hence $\bar{v}_x$ is negative. This case corresponds to motion in the negative x direction.

We can interpret average velocity geometrically by drawing a straight line between any two points on the position–time graph in Figure 2.1b. This line forms the hypotenuse of a right triangle of height Δx and base Δt. The slope of this line is the ratio $\Delta x/\Delta t$. For example, the line between positions Ⓐ and Ⓑ has a slope equal to the average velocity of the car between those two times, (52 m − 30 m)/(10 s − 0) = 2.2 m/s.

In everyday usage, the terms *speed* and *velocity* are interchangeable. In physics, however, there is a clear distinction between these two quantities. Consider a marathon runner who runs more than 40 km, yet ends up at his starting point. His average velocity is zero! Nonetheless, we need to be able to quantify how fast he was running. A slightly different ratio accomplishes this for us. **The average speed of a particle, a scalar quantity, is defined as the total distance traveled divided by the total time it takes to travel that distance:**

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}}$$

Average speed

The SI unit of average speed is the same as the unit of average velocity: meters per second. However, unlike average velocity, average speed has no direction and hence carries no algebraic sign.

Knowledge of the average speed of a particle tells us nothing about the details of the trip. For example, suppose it takes you 8.0 h to travel 280 km in your car. The average speed for your trip is 35 km/h. However, you most likely traveled at various speeds during the trip, and the average speed of 35 km/h could result from an infinite number of possible speed values.

EXAMPLE 2.1 Calculating the Variables of Motion

Find the displacement, average velocity, and average speed of the car in Figure 2.1a between positions Ⓐ and Ⓕ.

Solution The units of displacement must be meters, and the numerical result should be of the same order of magnitude as the given position data (which means probably not 10 or 100 times bigger or smaller). From the position–time graph given in Figure 2.1b, note that $x_A = 30$ m at $t_A = 0$ s and that $x_F = -53$ m at $t_F = 50$ s. Using these values along with the definition of displacement, Equation 2.1, we find that

$$\Delta x = x_F - x_A = -53\text{ m} - 30\text{ m} = \boxed{-83\text{ m}}$$

This result means that the car ends up 83 m in the negative direction (to the left, in this case) from where it started. This number has the correct units and is of the same order of magnitude as the supplied data. A quick look at Figure 2.1a indicates that this is the correct answer.

It is difficult to estimate the average velocity without completing the calculation, but we expect the units to be meters per second. Because the car ends up to the left of where we started taking data, we know the average velocity must be negative. From Equation 2.2,

$$\bar{v}_x = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x_F - x_A}{t_F - t_A}$$

$$= \frac{-53\text{ m} - 30\text{ m}}{50\text{ s} - 0\text{ s}} = \frac{-83\text{ m}}{50\text{ s}} = \boxed{-1.7\text{ m/s}}$$

We find the car's average speed for this trip by adding the distances traveled and dividing by the total time:

$$\text{Average speed} = \frac{22\text{ m} + 52\text{ m} + 53\text{ m}}{50\text{ s}} = \boxed{2.5\text{ m/s}}$$

2.2 INSTANTANEOUS VELOCITY AND SPEED

Often we need to know the velocity of a particle at a particular instant in time, rather than over a finite time interval. For example, even though you might want to calculate your average velocity during a long automobile trip, you would be especially interested in knowing your velocity at the *instant* you noticed the police

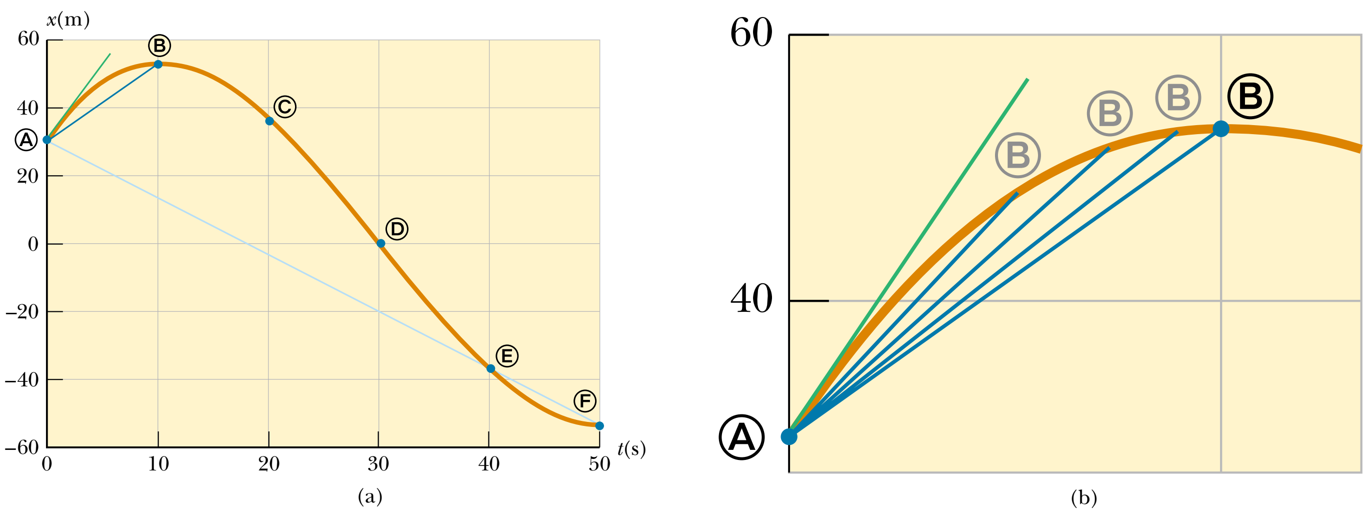


Figure 2.3 (a) Graph representing the motion of the car in Figure 2.1. (b) An enlargement of the upper left-hand corner of the graph shows how the blue line between positions Ⓐ and Ⓑ approaches the green tangent line as point Ⓑ gets closer to point Ⓐ.

car parked alongside the road in front of you. In other words, you would like to be able to specify your velocity just as precisely as you can specify your position by noting what is happening at a specific clock reading—that is, at some specific instant. It may not be immediately obvious how to do this. What does it mean to talk about how fast something is moving if we "freeze time" and talk only about an individual instant? This is a subtle point not thoroughly understood until the late 1600s. At that time, with the invention of calculus, scientists began to understand how to describe an object's motion at any moment in time.

To see how this is done, consider Figure 2.3a. We have already discussed the average velocity for the interval during which the car moved from position Ⓐ to position Ⓑ (given by the slope of the dark blue line) and for the interval during which it moved from Ⓐ to Ⓕ (represented by the slope of the light blue line). Which of these two lines do you think is a closer approximation of the initial velocity of the car? The car starts out by moving to the right, which we defined to be the positive direction. Therefore, being positive, the value of the average velocity during the Ⓐ to Ⓑ interval is probably closer to the initial value than is the value of the average velocity during the Ⓐ to Ⓕ interval, which we determined to be negative in Example 2.1. Now imagine that we start with the dark blue line and slide point Ⓑ to the left along the curve, toward point Ⓐ, as in Figure 2.3b. The line between the points becomes steeper and steeper, and as the two points get extremely close together, the line becomes a tangent line to the curve, indicated by the green line on the graph. The slope of this tangent line represents the velocity of the car at the moment we started taking data, at point Ⓐ. What we have done is determine the *instantaneous velocity* at that moment. In other words, the **instantaneous velocity v_x equals the limiting value of the ratio $\Delta x/\Delta t$ as Δt approaches zero:**[1]

Definition of instantaneous velocity

3.3

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad \textbf{(2.3)}$$

[1] Note that the displacement Δx also approaches zero as Δt approaches zero. As Δx and Δt become smaller and smaller, the ratio $\Delta x/\Delta t$ approaches a value equal to the slope of the line tangent to the x-versus-t curve.

In calculus notation, this limit is called the *derivative* of x with respect to t, written dx/dt:

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{2.4}$$

The instantaneous velocity can be positive, negative, or zero. When the slope of the position–time graph is positive, such as at any time during the first 10 s in Figure 2.3, v_x is positive. After point Ⓑ, v_x is negative because the slope is negative. At the peak, the slope and the instantaneous velocity are zero.

From here on, we use the word *velocity* to designate instantaneous velocity. When it is *average velocity* we are interested in, we always use the adjective *average*.

The instantaneous speed of a particle is defined as the magnitude of its velocity. As with average speed, instantaneous speed has no direction associated with it and hence carries no algebraic sign. For example, if one particle has a velocity of +25 m/s along a given line and another particle has a velocity of −25 m/s along the same line, both have a speed[2] of 25 m/s.

EXAMPLE 2.2 Average and Instantaneous Velocity

A particle moves along the x axis. Its x coordinate varies with time according to the expression $x = -4t + 2t^2$, where x is in meters and t is in seconds.[3] The position–time graph for this motion is shown in Figure 2.4. Note that the particle moves in the negative x direction for the first second of motion, is at rest at the moment $t = 1$ s, and moves in the positive x direction for $t > 1$ s. (a) Determine the displacement of the particle in the time intervals $t = 0$ to $t = 1$ s and $t = 1$ s to $t = 3$ s.

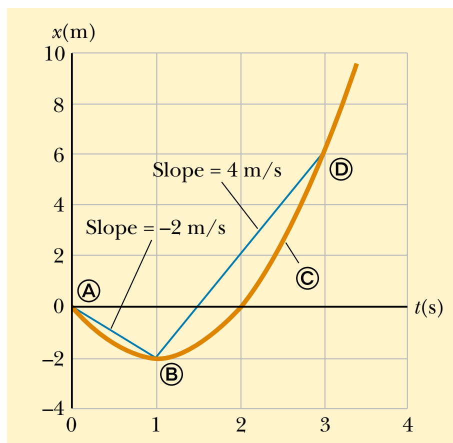


Figure 2.4 Position–time graph for a particle having an x coordinate that varies in time according to the expression $x = -4t + 2t^2$.

Solution During the first time interval, we have a negative slope and hence a negative velocity. Thus, we know that the displacement between Ⓐ and Ⓑ must be a negative number having units of meters. Similarly, we expect the displacement between Ⓑ and Ⓓ to be positive.

In the first time interval, we set $t_i = t_A = 0$ and $t_f = t_B = 1$ s. Using Equation 2.1, with $x = -4t + 2t^2$, we obtain for the first displacement

$$\begin{aligned}\Delta x_{A\to B} &= x_f - x_i = x_B - x_A \\ &= [-4(1) + 2(1)^2] - [-4(0) + 2(0)^2] \\ &= \boxed{-2\text{ m}}\end{aligned}$$

To calculate the displacement during the second time interval, we set $t_i = t_B = 1$ s and $t_f = t_D = 3$ s:

$$\begin{aligned}\Delta x_{B\to D} &= x_f - x_i = x_D - x_B \\ &= [-4(3) + 2(3)^2] - [-4(1) + 2(1)^2] \\ &= \boxed{+8\text{ m}}\end{aligned}$$

These displacements can also be read directly from the position–time graph.

[2] As with velocity, we drop the adjective for instantaneous speed: "Speed" means instantaneous speed.

[3] Simply to make it easier to read, we write the empirical equation as $x = -4t + 2t^2$ rather than as $x = (-4.00\text{ m/s})t + (2.00\text{ m/s}^2)t^{2.00}$. When an equation summarizes measurements, consider its coefficients to have as many significant digits as other data quoted in a problem. Consider its coefficients to have the units required for dimensional consistency. When we start our clocks at $t = 0$ s, we usually do not mean to limit the precision to a single digit. Consider any zero value in this book to have as many significant figures as you need.

(b) Calculate the average velocity during these two time intervals.

Solution In the first time interval, $\Delta t = t_f - t_i = t_B - t_A = 1$ s. Therefore, using Equation 2.2 and the displacement calculated in (a), we find that

$$\overline{v}_{x(A\rightarrow B)} = \frac{\Delta x_{A\rightarrow B}}{\Delta t} = \frac{-2\text{ m}}{1\text{ s}} = \boxed{-2\text{ m/s}}$$

In the second time interval, $\Delta t = 2$ s; therefore,

$$\overline{v}_{x(B\rightarrow D)} = \frac{\Delta x_{B\rightarrow D}}{\Delta t} = \frac{8\text{ m}}{2\text{ s}} = \boxed{+4\text{ m/s}}$$

These values agree with the slopes of the lines joining these points in Figure 2.4.

(c) Find the instantaneous velocity of the particle at $t = 2.5$ s.

Solution Certainly we can guess that this instantaneous velocity must be of the same order of magnitude as our previous results, that is, around 4 m/s. Examining the graph, we see that the slope of the tangent at position Ⓒ is greater than the slope of the blue line connecting points Ⓑ and Ⓓ. Thus, we expect the answer to be greater than 4 m/s. By measuring the slope of the position–time graph at $t = 2.5$ s, we find that

$$v_x = \boxed{+6\text{ m/s}}$$

2.3 ACCELERATION

In the last example, we worked with a situation in which the velocity of a particle changed while the particle was moving. This is an extremely common occurrence. (How constant is your velocity as you ride a city bus?) It is easy to quantify changes in velocity as a function of time in exactly the same way we quantify changes in position as a function of time. When the velocity of a particle changes with time, the particle is said to be *accelerating.* For example, the velocity of a car increases when you step on the gas and decreases when you apply the brakes. However, we need a better definition of acceleration than this.

Suppose a particle moving along the x axis has a velocity v_{xi} at time t_i and a velocity v_{xf} at time t_f, as in Figure 2.5a.

The average acceleration of the particle is defined as the *change* in velocity Δv_x divided by the time interval Δt during which that change occurred:

Average acceleration

$$\overline{a}_x \equiv \frac{\Delta v_x}{\Delta t} = \frac{v_{xf} - v_{xi}}{t_f - t_i} \tag{2.5}$$

As with velocity, when the motion being analyzed is one-dimensional, we can use positive and negative signs to indicate the direction of the acceleration. Because the dimensions of velocity are L/T and the dimension of time is T, accelera-

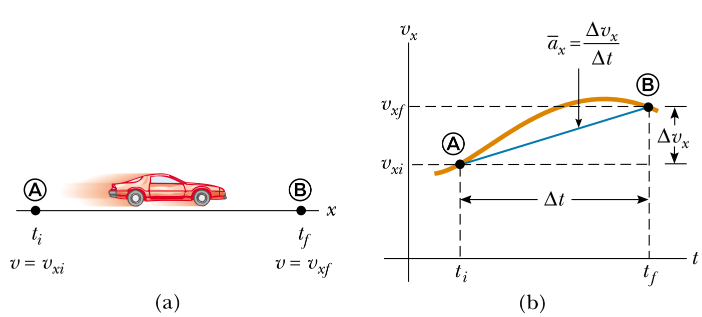


Figure 2.5 (a) A "particle" moving along the x axis from Ⓐ to Ⓑ has velocity v_{xi} at $t = t_i$ and velocity v_{xf} at $t = t_f$. (b) Velocity–time graph for the particle moving in a straight line. The slope of the blue straight line connecting Ⓐ and Ⓑ is the average acceleration in the time interval $\Delta t = t_f - t_i$.

tion has dimensions of length divided by time squared, or L/T^2. The SI unit of acceleration is meters per second squared (m/s^2). It might be easier to interpret these units if you think of them as meters per second per second. For example, suppose an object has an acceleration of 2 m/s^2. You should form a mental image of the object having a velocity that is along a straight line and is increasing by 2 m/s during every 1-s interval. If the object starts from rest, you should be able to picture it moving at a velocity of + 2 m/s after 1 s, at + 4 m/s after 2 s, and so on.

In some situations, the value of the average acceleration may be different over different time intervals. It is therefore useful to define the *instantaneous acceleration* as the limit of the average acceleration as Δt approaches zero. This concept is analogous to the definition of instantaneous velocity discussed in the previous section. If we imagine that point Ⓑ is brought closer and closer to point Ⓐ in Figure 2.5a and take the limit of $\Delta v_x/\Delta t$ as Δt approaches zero, we obtain the instantaneous acceleration:

$$a_x \equiv \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \quad \textbf{(2.6)}$$

Instantaneous acceleration

That is, **the instantaneous acceleration equals the derivative of the velocity with respect to time,** which by definition is the slope of the velocity–time graph (Fig. 2.5b). Thus, we see that just as the velocity of a moving particle is the slope of the particle's x-t graph, the acceleration of a particle is the slope of the particle's v_x-t graph. One can interpret the derivative of the velocity with respect to time as the time rate of change of velocity. If a_x is positive, then the acceleration is in the positive x direction; if a_x is negative, then the acceleration is in the negative x direction.

From now on we shall use the term *acceleration* to mean instantaneous acceleration. When we mean average acceleration, we shall always use the adjective *average.*

Because $v_x = dx/dt$, the acceleration can also be written

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \quad \textbf{(2.7)}$$

That is, in one-dimensional motion, the acceleration equals the *second derivative* of x with respect to time.

Figure 2.6 illustrates how an acceleration–time graph is related to a velocity–time graph. The acceleration at any time is the slope of the velocity–time graph at that time. Positive values of acceleration correspond to those points in Figure 2.6a where the velocity is increasing in the positive x direction. The acceler-

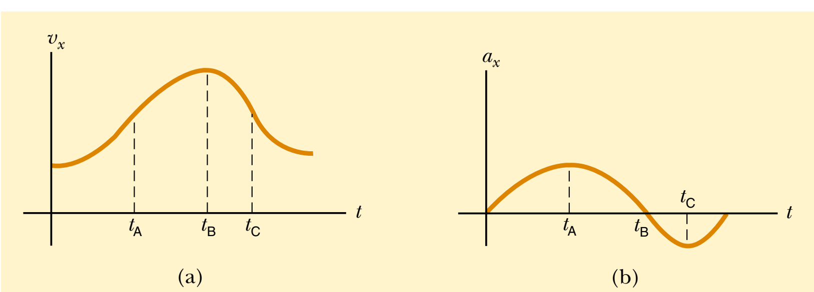


Figure 2.6 Instantaneous acceleration can be obtained from the v_x-t graph. (a) The velocity–time graph for some motion. (b) The acceleration–time graph for the same motion. The acceleration given by the a_x-t graph for any value of t equals the slope of the line tangent to the v_x-t graph at the same value of t.

ation reaches a maximum at time t_A, when the slope of the velocity–time graph is a maximum. The acceleration then goes to zero at time t_B, when the velocity is a maximum (that is, when the slope of the v_x-t graph is zero). The acceleration is negative when the velocity is decreasing in the positive x direction, and it reaches its most negative value at time t_C.

CONCEPTUAL EXAMPLE 2.3 Graphical Relationships Between x, v_x, and a_x

The position of an object moving along the x axis varies with time as in Figure 2.7a. Graph the velocity versus time and the acceleration versus time for the object.

Solution The velocity at any instant is the slope of the tangent to the x-t graph at that instant. Between $t = 0$ and $t = t_A$, the slope of the x-t graph increases uniformly, and so the velocity increases linearly, as shown in Figure 2.7b. Between t_A and t_B, the slope of the x-t graph is constant, and so the velocity remains constant. At t_D, the slope of the x-t graph is zero, so the velocity is zero at that instant. Between t_D and t_E, the slope of the x-t graph and thus the velocity are negative and decrease uniformly in this interval. In the interval t_E to t_F, the slope of the x-t graph is still negative, and at t_F it goes to zero. Finally, after t_F, the slope of the x-t graph is zero, meaning that the object is at rest for $t > t_F$.

The acceleration at any instant is the slope of the tangent to the v_x-t graph at that instant. The graph of acceleration versus time for this object is shown in Figure 2.7c. The acceleration is constant and positive between 0 and t_A, where the slope of the v_x-t graph is positive. It is zero between t_A and t_B and for $t > t_F$ because the slope of the v_x-t graph is zero at these times. It is negative between t_B and t_E because the slope of the v_x-t graph is negative during this interval.

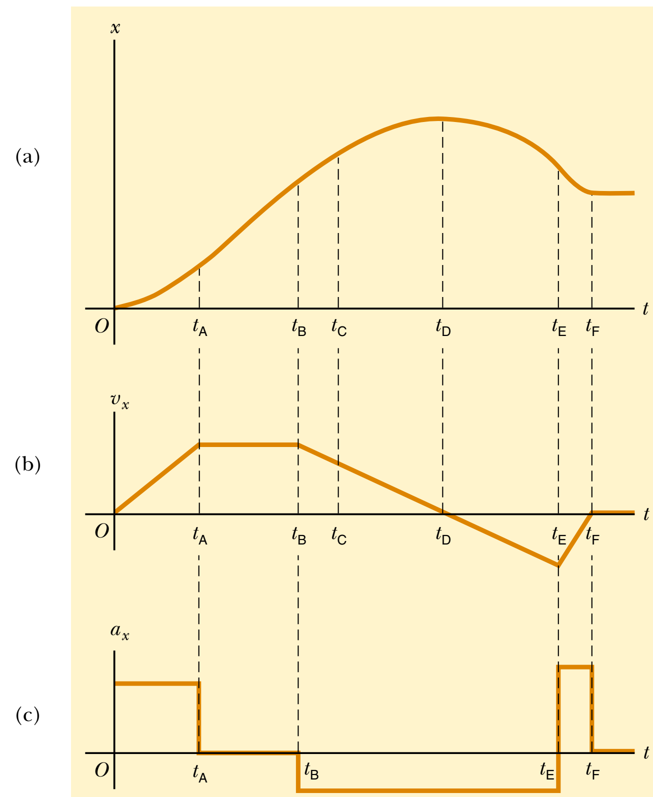


Figure 2.7 (a) Position–time graph for an object moving along the x axis. (b) The velocity–time graph for the object is obtained by measuring the slope of the position–time graph at each instant. (c) The acceleration–time graph for the object is obtained by measuring the slope of the velocity–time graph at each instant.

Quick Quiz 2.1

Make a velocity–time graph for the car in Figure 2.1a and use your graph to determine whether the car ever exceeds the speed limit posted on the road sign (30 km/h).

EXAMPLE 2.4 Average and Instantaneous Acceleration

The velocity of a particle moving along the x axis varies in time according to the expression $v_x = (40 - 5t^2)$ m/s, where t is in seconds. (a) Find the average acceleration in the time interval $t = 0$ to $t = 2.0$ s.

Solution Figure 2.8 is a v_x-t graph that was created from the velocity versus time expression given in the problem statement. Because the slope of the entire v_x-t curve is negative, we expect the acceleration to be negative.

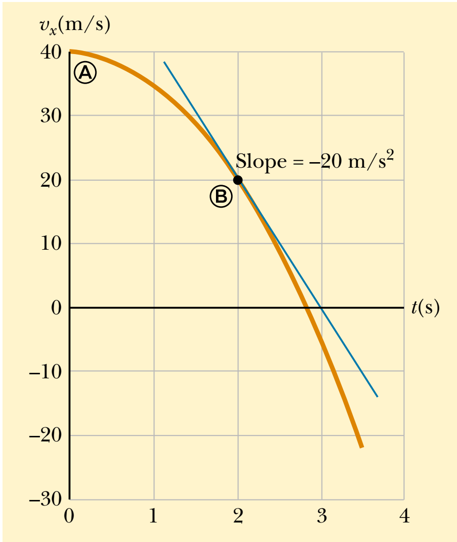


Figure 2.8 The velocity–time graph for a particle moving along the x axis according to the expression $v_x = (40 - 5t^2)$ m/s. The acceleration at $t = 2$ s is equal to the slope of the blue tangent line at that time.

We find the velocities at $t_i = t_A = 0$ and $t_f = t_B = 2.0$ s by substituting these values of t into the expression for the velocity:

$$v_{xA} = (40 - 5t_A^2)\text{ m/s} = [40 - 5(0)^2]\text{ m/s} = +40\text{ m/s}$$

$$v_{xB} = (40 - 5t_B^2)\text{ m/s} = [40 - 5(2.0)^2]\text{ m/s} = +20\text{ m/s}$$

Therefore, the average acceleration in the specified time interval $\Delta t = t_B - t_A = 2.0$ s is

$$\bar{a}_x = \frac{v_{xf} - v_{xi}}{t_f - t_i} = \frac{v_{xB} - v_{xA}}{t_B - t_A} = \frac{(20 - 40)\text{ m/s}}{(2.0 - 0)\text{ s}}$$

$$= \boxed{-10\text{ m/s}^2}$$

The negative sign is consistent with our expectations—namely, that the average acceleration, which is represented by the slope of the line (not shown) joining the initial and final points on the velocity–time graph, is negative.

(b) Determine the acceleration at $t = 2.0$ s.

Solution The velocity at any time t is $v_{xi} = (40 - 5t^2)$ m/s, and the velocity at any later time $t + \Delta t$ is

$$v_{xf} = 40 - 5(t + \Delta t)^2 = 40 - 5t^2 - 10t\,\Delta t - 5(\Delta t)^2$$

Therefore, the change in velocity over the time interval Δt is

$$\Delta v_x = v_{xf} - v_{xi} = [-10t\,\Delta t - 5(\Delta t)^2]\text{ m/s}$$

Dividing this expression by Δt and taking the limit of the result as Δt approaches zero gives the acceleration at *any* time t:

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \lim_{\Delta t \to 0} (-10t - 5\Delta t) = -10t\text{ m/s}^2$$

Therefore, at $t = 2.0$ s,

$$a_x = (-10)(2.0)\text{ m/s}^2 = \boxed{-20\text{ m/s}^2}$$

What we have done by comparing the average acceleration during the interval between Ⓐ and Ⓑ (-10 m/s^2) with the instantaneous value at Ⓑ (-20 m/s^2) is compare the slope of the line (not shown) joining Ⓐ and Ⓑ with the slope of the tangent at Ⓑ.

Note that the acceleration is not constant in this example. Situations involving constant acceleration are treated in Section 2.5.

So far we have evaluated the derivatives of a function by starting with the definition of the function and then taking the limit of a specific ratio. Those of you familiar with calculus should recognize that there are specific rules for taking derivatives. These rules, which are listed in Appendix B.6, enable us to evaluate derivatives quickly. For instance, one rule tells us that the derivative of any constant is zero. As another example, suppose x is proportional to some power of t, such as in the expression

$$x = At^n$$

where A and n are constants. (This is a very common functional form.) The derivative of x with respect to t is

$$\frac{dx}{dt} = nAt^{n-1}$$

Applying this rule to Example 2.4, in which $v_x = 40 - 5t^2$, we find that $a_x = dv_x/dt = -10t$.

2.4 MOTION DIAGRAMS

The concepts of velocity and acceleration are often confused with each other, but in fact they are quite different quantities. It is instructive to use motion diagrams to describe the velocity and acceleration while an object is in motion. In order not to confuse these two vector quantities, for which both magnitude and direction are important, we use red for velocity vectors and violet for acceleration vectors, as shown in Figure 2.9. The vectors are sketched at several instants during the motion of the object, and the time intervals between adjacent positions are assumed to be equal. This illustration represents three sets of strobe photographs of a car moving from left to right along a straight roadway. The time intervals between flashes are equal in each diagram.

In Figure 2.9a, the images of the car are equally spaced, showing us that the car moves the same distance in each time interval. Thus, the car moves with *constant positive velocity* and has *zero acceleration.*

In Figure 2.9b, the images become farther apart as time progresses. In this case, the velocity vector increases in time because the car's displacement between adjacent positions increases in time. The car is moving with a *positive velocity* and a *positive acceleration.*

In Figure 2.9c, we can tell that the car slows as it moves to the right because its displacement between adjacent images decreases with time. In this case, the car moves to the right with a constant negative acceleration. The velocity vector decreases in time and eventually reaches zero. From this diagram we see that the acceleration and velocity vectors are *not* in the same direction. The car is moving with a *positive velocity* but with a *negative acceleration.*

You should be able to construct motion diagrams for a car that moves initially to the left with a constant positive or negative acceleration.

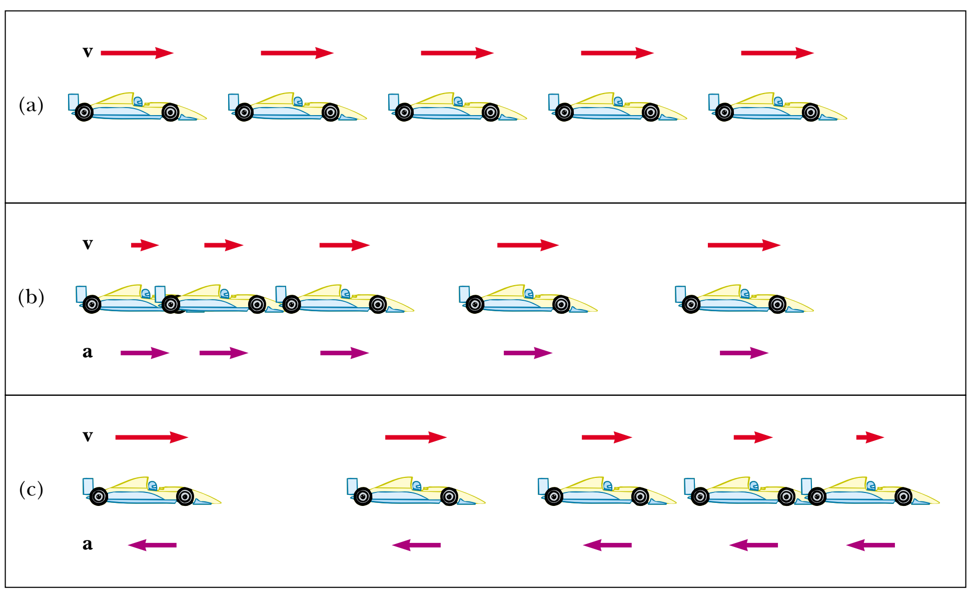


Figure 2.9 (a) Motion diagram for a car moving at constant velocity (zero acceleration). (b) Motion diagram for a car whose constant acceleration is in the direction of its velocity. The velocity vector at each instant is indicated by a red arrow, and the constant acceleration by a violet arrow. (c) Motion diagram for a car whose constant acceleration is in the direction *opposite* the velocity at each instant.

Quick Quiz 2.2

(a) If a car is traveling eastward, can its acceleration be westward? (b) If a car is slowing down, can its acceleration be positive?

2.5 ONE-DIMENSIONAL MOTION WITH CONSTANT ACCELERATION

If the acceleration of a particle varies in time, its motion can be complex and difficult to analyze. However, a very common and simple type of one-dimensional motion is that in which the acceleration is constant. When this is the case, the average acceleration over any time interval equals the instantaneous acceleration at any instant within the interval, and the velocity changes at the same rate throughout the motion.

If we replace $\overline{a}_x$ by a_x in Equation 2.5 and take $t_i = 0$ and t_f to be any later time t, we find that

$$a_x = \frac{v_{xf} - v_{xi}}{t}$$

or

$$v_{xf} = v_{xi} + a_x t \quad \text{(for constant } a_x\text{)} \tag{2.8}$$

Velocity as a function of time

This powerful expression enables us to determine an object's velocity at *any* time t if we know the object's initial velocity and its (constant) acceleration. A velocity–time graph for this constant-acceleration motion is shown in Figure 2.10a. The graph is a straight line, the (constant) slope of which is the acceleration a_x; this is consistent with the fact that $a_x = dv_x/dt$ is a constant. Note that the slope is positive; this indicates a positive acceleration. If the acceleration were negative, then the slope of the line in Figure 2.10a would be negative.

When the acceleration is constant, the graph of acceleration versus time (Fig. 2.10b) is a straight line having a slope of zero.

Quick Quiz 2.3

Describe the meaning of each term in Equation 2.8.

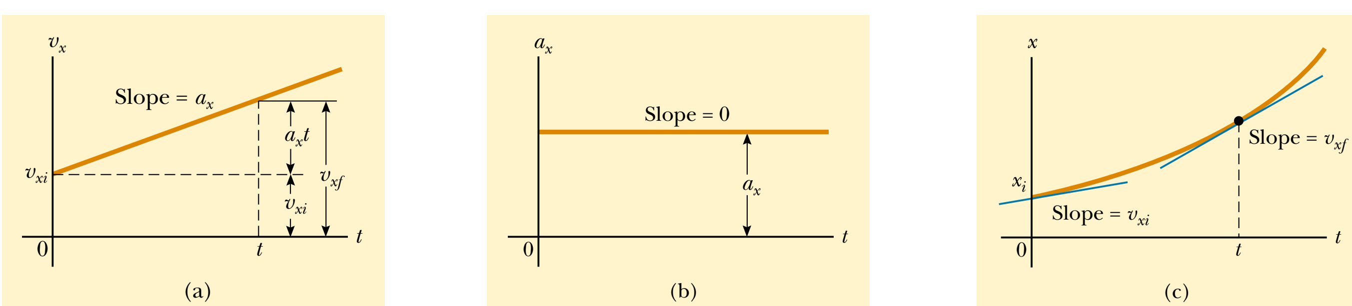


Figure 2.10 An object moving along the x axis with constant acceleration a_x. (a) The velocity–time graph. (b) The acceleration–time graph. (c) The position–time graph.

Because velocity at constant acceleration varies linearly in time according to Equation 2.8, we can express the average velocity in any time interval as the arithmetic mean of the initial velocity v_{xi} and the final velocity v_{xf}:

$$\bar{v}_x = \frac{v_{xi} + v_{xf}}{2} \qquad \text{(for constant } a_x\text{)} \tag{2.9}$$

Note that this expression for average velocity applies *only* in situations in which the acceleration is constant.

We can now use Equations 2.1, 2.2, and 2.9 to obtain the displacement of any object as a function of time. Recalling that Δx in Equation 2.2 represents $x_f - x_i$, and now using t in place of Δt (because we take $t_i = 0$), we can say

Displacement as a function of velocity and time

$$x_f - x_i = \bar{v}_x t = \tfrac{1}{2}(v_{xi} + v_{xf})\,t \qquad \text{(for constant } a_x\text{)} \tag{2.10}$$

We can obtain another useful expression for displacement at constant acceleration by substituting Equation 2.8 into Equation 2.10:

$$x_f - x_i = \tfrac{1}{2}(v_{xi} + v_{xi} + a_x t)\,t$$

$$x_f - x_i = v_{xi}t + \tfrac{1}{2}a_x t^2 \tag{2.11}$$

The position–time graph for motion at constant (positive) acceleration shown in Figure 2.10c is obtained from Equation 2.11. Note that the curve is a parabola. The slope of the tangent line to this curve at $t = t_i = 0$ equals the initial velocity v_{xi}, and the slope of the tangent line at any later time t equals the velocity at that time, v_{xf}.

We can check the validity of Equation 2.11 by moving the x_i term to the right-hand side of the equation and differentiating the equation with respect to time:

$$v_{xf} = \frac{dx_f}{dt} = \frac{d}{dt}\left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right) = v_{xi} + a_x t$$

Finally, we can obtain an expression for the final velocity that does not contain a time interval by substituting the value of t from Equation 2.8 into Equation 2.10:

$$x_f - x_i = \frac{1}{2}(v_{xi} + v_{xf})\left(\frac{v_{xf} - v_{xi}}{a_x}\right) = \frac{v_{xf}^2 - v_{xi}^2}{2a_x}$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i) \qquad \text{(for constant } a_x\text{)} \tag{2.12}$$

For motion at *zero* acceleration, we see from Equations 2.8 and 2.11 that

$$\left.\begin{aligned} v_{xf} &= v_{xi} = v_x \\ x_f - x_i &= v_x t \end{aligned}\right\} \quad \text{when } a_x = 0$$

That is, when acceleration is zero, velocity is constant and displacement changes linearly with time.

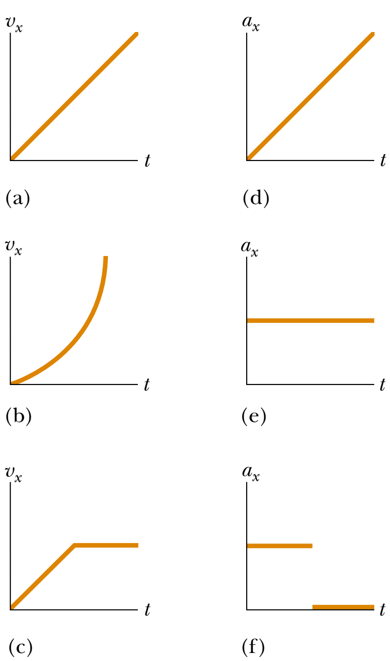


Figure 2.11 Parts (a), (b), and (c) are v_x-t graphs of objects in one-dimensional motion. The possible accelerations of each object as a function of time are shown in scrambled order in (d), (e), and (f).

Quick Quiz 2.4

In Figure 2.11, match each v_x-t graph with the a_x-t graph that best describes the motion.

Equations 2.8 through 2.12 are **kinematic expressions that may be used to solve any problem involving one-dimensional motion at constant accelera-**

TABLE 2.2 Kinematic Equations for Motion in a Straight Line Under Constant Acceleration

Equation	Information Given by Equation
$v_{xf} = v_{xi} + a_x t$	Velocity as a function of time
$x_f - x_i = \frac{1}{2}(v_{xi} + v_{xf})t$	Displacement as a function of velocity and time
$x_f - x_i = v_{xi}t + \frac{1}{2}a_x t^2$	Displacement as a function of time
$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$	Velocity as a function of displacement

Note: Motion is along the *x* axis.

tion. Keep in mind that these relationships were derived from the definitions of velocity and acceleration, together with some simple algebraic manipulations and the requirement that the acceleration be constant.

The four kinematic equations used most often are listed in Table 2.2 for convenience. The choice of which equation you use in a given situation depends on what you know beforehand. Sometimes it is necessary to use two of these equations to solve for two unknowns. For example, suppose initial velocity v_{xi} and acceleration a_x are given. You can then find (1) the velocity after an interval t has elapsed, using $v_{xf} = v_{xi} + a_x t$, and (2) the displacement after an interval t has elapsed, using $x_f - x_i = v_{xi}t + \frac{1}{2}a_x t^2$. You should recognize that the quantities that vary during the motion are velocity, displacement, and time.

You will get a great deal of practice in the use of these equations by solving a number of exercises and problems. Many times you will discover that more than one method can be used to obtain a solution. Remember that these equations of kinematics *cannot* be used in a situation in which the acceleration varies with time. They can be used only when the acceleration is constant.

CONCEPTUAL EXAMPLE 2.5 The Velocity of Different Objects

Consider the following one-dimensional motions: (a) A ball thrown directly upward rises to a highest point and falls back into the thrower's hand. (b) A race car starts from rest and speeds up to 100 m/s. (c) A spacecraft drifts through space at constant velocity. Are there any points in the motion of these objects at which the instantaneous velocity is the same as the average velocity over the entire motion? If so, identify the point(s).

Solution (a) The average velocity for the thrown ball is zero because the ball returns to the starting point; thus its displacement is zero. (Remember that average velocity is defined as $\Delta x/\Delta t$.) There is one point at which the instantaneous velocity is zero—at the top of the motion.

(b) The car's average velocity cannot be evaluated unambiguously with the information given, but it must be some value between 0 and 100 m/s. Because the car will have every instantaneous velocity between 0 and 100 m/s at some time during the interval, there must be some instant at which the instantaneous velocity is equal to the average velocity.

(c) Because the spacecraft's instantaneous velocity is constant, its instantaneous velocity at *any* time and its average velocity over *any* time interval are the same.

EXAMPLE 2.6 Entering the Traffic Flow

(a) Estimate your average acceleration as you drive up the entrance ramp to an interstate highway.

Solution This problem involves more than our usual amount of estimating! We are trying to come up with a value of a_x, but that value is hard to guess directly. The other three variables involved in kinematics are position, velocity, and time. Velocity is probably the easiest one to approximate. Let us assume a final velocity of 100 km/h, so that you can merge with traffic. We multiply this value by 1 000 to convert kilometers to meters and then divide by 3 600 to convert hours to seconds. These two calculations together are roughly equivalent to dividing by 3. In fact, let us just say that the final velocity is $v_{xf} \approx 30$ m/s. (Remember, you can get away with this type of approximation and with dropping digits when performing mental calculations. If you were starting with British units, you could approximate 1 mi/h as roughly 0.5 m/s and continue from there.)

Now we assume that you started up the ramp at about one-third your final velocity, so that $v_{xi} \approx 10$ m/s. Finally, we assume that it takes about 10 s to get from v_{xi} to v_{xf}, basing this guess on our previous experience in automobiles. We can then find the acceleration, using Equation 2.8:

$$a_x = \frac{v_{xf} - v_{xi}}{t} \approx \frac{30 \text{ m/s} - 10 \text{ m/s}}{10 \text{ s}} = \boxed{2 \text{ m/s}^2}$$

Granted, we made many approximations along the way, but this type of mental effort can be surprisingly useful and often yields results that are not too different from those derived from careful measurements.

(b) How far did you go during the first half of the time interval during which you accelerated?

Solution We can calculate the distance traveled during the first 5 s from Equation 2.11:

$$x_f - x_i = v_{xi}t + \tfrac{1}{2}a_x t^2 \approx (10 \text{ m/s})(5 \text{ s}) + \tfrac{1}{2}(2 \text{ m/s}^2)(5 \text{ s})^2$$
$$= 50 \text{ m} + 25 \text{ m} = \boxed{75 \text{ m}}$$

This result indicates that if you had not accelerated, your initial velocity of 10 m/s would have resulted in a 50-m movement up the ramp during the first 5 s. The additional 25 m is the result of your increasing velocity during that interval.

Do not be afraid to attempt making educated guesses and doing some fairly drastic number rounding to simplify mental calculations. Physicists engage in this type of thought analysis all the time.

EXAMPLE 2.7 Carrier Landing

A jet lands on an aircraft carrier at 140 mi/h ($\approx$63 m/s). (a) What is its acceleration if it stops in 2.0 s?

Solution We define our x axis as the direction of motion of the jet. A careful reading of the problem reveals that in addition to being given the initial speed of 63 m/s, we also know that the final speed is zero. We also note that we are not given the displacement of the jet while it is slowing down. Equation 2.8 is the only equation in Table 2.2 that does not involve displacement, and so we use it to find the acceleration:

$$a_x = \frac{v_{xf} - v_{xi}}{t} \approx \frac{0 - 63 \text{ m/s}}{2.0 \text{ s}} = \boxed{-31 \text{ m/s}^2}$$

(b) What is the displacement of the plane while it is stopping?

Solution We can now use any of the other three equations in Table 2.2 to solve for the displacement. Let us choose Equation 2.10:

$$x_f - x_i = \tfrac{1}{2}(v_{xi} + v_{xf})t = \tfrac{1}{2}(63 \text{ m/s} + 0)(2.0 \text{ s}) = \boxed{63 \text{ m}}$$

If the plane travels much farther than this, it might fall into the ocean. Although the idea of using arresting cables to enable planes to land safely on ships originated at about the time of the First World War, the cables are still a vital part of the operation of modern aircraft carriers.

EXAMPLE 2.8 Watch Out for the Speed Limit!

A car traveling at a constant speed of 45.0 m/s passes a trooper hidden behind a billboard. One second after the speeding car passes the billboard, the trooper sets out from the billboard to catch it, accelerating at a constant rate of 3.00 m/s^2. How long does it take her to overtake the car?

Solution A careful reading lets us categorize this as a constant-acceleration problem. We know that after the 1-s delay in starting, it will take the trooper 15 additional seconds to accelerate up to 45.0 m/s. Of course, she then has to continue to pick up speed (at a rate of 3.00 m/s per second) to catch up to the car. While all this is going on, the car continues to move. We should therefore expect our result to be well over 15 s. A sketch (Fig. 2.12) helps clarify the sequence of events.

First, we write expressions for the position of each vehicle as a function of time. It is convenient to choose the position of the billboard as the origin and to set $t_B \equiv 0$ as the time the trooper begins moving. At that instant, the car has already traveled a distance of 45.0 m because it has traveled at a constant speed of $v_x = 45.0$ m/s for 1 s. Thus, the initial position of the speeding car is $x_B = 45.0$ m.

Because the car moves with constant speed, its accelera-

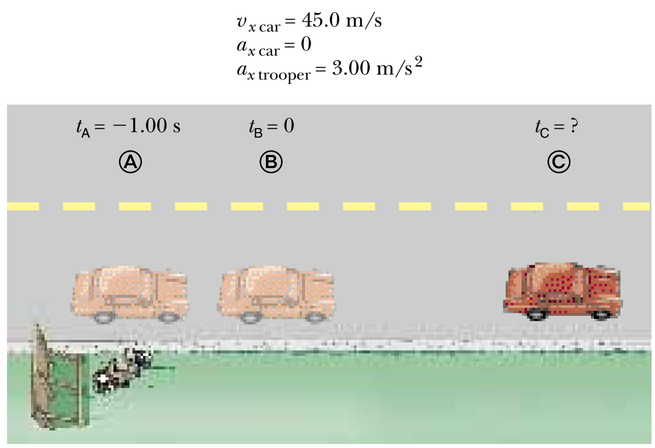


Figure 2.12 A speeding car passes a hidden police officer.

tion is zero, and applying Equation 2.11 (with $a_x = 0$) gives for the car's position at any time t:

$$x_{car} = x_B + v_{x\,car}t = 45.0 \text{ m} + (45.0 \text{ m/s})t$$

A quick check shows that at $t = 0$, this expression gives the car's correct initial position when the trooper begins to move: $x_{car} = x_B = 45.0$ m. Looking at limiting cases to see whether they yield expected values is a very useful way to make sure that you are obtaining reasonable results.

The trooper starts from rest at $t = 0$ and accelerates at 3.00 m/s^2 away from the origin. Hence, her position after any time interval t can be found from Equation 2.11:

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_xt^2$$

$$x_{trooper} = 0 + 0t + \tfrac{1}{2}a_xt^2 = \tfrac{1}{2}(3.00 \text{ m/s}^2)t^2$$

The trooper overtakes the car at the instant her position matches that of the car, which is position Ⓒ:

$$x_{trooper} = x_{car}$$

$$\tfrac{1}{2}(3.00 \text{ m/s}^2)t^2 = 45.0 \text{ m} + (45.0 \text{ m/s})t$$

This gives the quadratic equation

$$1.50t^2 - 45.0t - 45.0 = 0$$

The positive solution of this equation is $t =$ 31.0 s .

(For help in solving quadratic equations, see Appendix B.2.) Note that in this 31.0-s time interval, the trooper travels a distance of about 1440 m. [This distance can be calculated from the car's constant speed: (45.0 m/s)(31 + 1) s = 1 440 m.]

Exercise This problem can be solved graphically. On the same graph, plot position versus time for each vehicle, and from the intersection of the two curves determine the time at which the trooper overtakes the car.

2.6 FREELY FALLING OBJECTS

It is now well known that, in the absence of air resistance, all objects dropped near the Earth's surface fall toward the Earth with the same constant acceleration under the influence of the Earth's gravity. It was not until about 1600 that this conclusion was accepted. Before that time, the teachings of the great philosopher Aristotle (384–322 B.C.) had held that heavier objects fall faster than lighter ones.

It was the Italian Galileo Galilei (1564–1642) who originated our present-day ideas concerning falling objects. There is a legend that he demonstrated the law of falling objects by observing that two different weights dropped simultaneously from the Leaning Tower of Pisa hit the ground at approximately the same time. Although there is some doubt that he carried out this particular experiment, it is well established that Galileo performed many experiments on objects moving on inclined planes. In his experiments he rolled balls down a slight incline and measured the distances they covered in successive time intervals. The purpose of the incline was to reduce the acceleration; with the acceleration reduced, Galileo was able to make accurate measurements of the time intervals. By gradually increasing the slope of the incline, he was finally able to draw conclusions about freely falling objects because a freely falling ball is equivalent to a ball moving down a vertical incline.

Astronaut David Scott released a hammer and a feather simultaneously, and they fell in unison to the lunar surface. *(Courtesy of NASA)*

QuickLab

Use a pencil to poke a hole in the bottom of a paper or polystyrene cup. Cover the hole with your finger and fill the cup with water. Hold the cup up in front of you and release it. Does water come out of the hole while the cup is falling? Why or why not?

You might want to try the following experiment. Simultaneously drop a coin and a crumpled-up piece of paper from the same height. If the effects of air resistance are negligible, both will have the same motion and will hit the floor at the same time. In the idealized case, in which air resistance is absent, such motion is referred to as *free fall.* If this same experiment could be conducted in a vacuum, in which air resistance is truly negligible, the paper and coin would fall with the same acceleration even when the paper is not crumpled. On August 2, 1971, such a demonstration was conducted on the Moon by astronaut David Scott. He simultaneously released a hammer and a feather, and in unison they fell to the lunar surface. This demonstration surely would have pleased Galileo!

Definition of free fall

When we use the expression *freely falling object,* we do not necessarily refer to an object dropped from rest. **A freely falling object is any object moving freely under the influence of gravity alone, regardless of its initial motion. Objects thrown upward or downward and those released from rest are all falling freely once they are released. Any freely falling object experiences an acceleration directed *downward*, regardless of its initial motion.**

Free-fall acceleration $g = 9.80\ \text{m/s}^2$

We shall denote the magnitude of the *free-fall acceleration* by the symbol g. The value of g near the Earth's surface decreases with increasing altitude. Furthermore, slight variations in g occur with changes in latitude. It is common to define "up" as the $+y$ direction and to use y as the position variable in the kinematic equations. At the Earth's surface, the value of g is approximately $9.80\ \text{m/s}^2$. Unless stated otherwise, we shall use this value for g when performing calculations. For making quick estimates, use $g = 10\ \text{m/s}^2$.

If we neglect air resistance and assume that the free-fall acceleration does not vary with altitude over short vertical distances, then the motion of a freely falling object moving vertically is equivalent to motion in one dimension under constant acceleration. Therefore, the equations developed in Section 2.5 for objects moving with constant acceleration can be applied. The only modification that we need to make in these equations for freely falling objects is to note that the motion is in the vertical direction (the y direction) rather than in the horizontal (x) direction and that the acceleration is downward and has a magnitude of $9.80\ \text{m/s}^2$. Thus, we always take $a_y = -g = -9.80\ \text{m/s}^2$, where the minus sign means that the acceleration of a freely falling object is downward. In Chapter 14 we shall study how to deal with variations in g with altitude.

CONCEPTUAL EXAMPLE 2.9 The Daring Sky Divers

A sky diver jumps out of a hovering helicopter. A few seconds later, another sky diver jumps out, and they both fall along the same vertical line. Ignore air resistance, so that both sky divers fall with the same acceleration. Does the difference in their speeds stay the same throughout the fall? Does the vertical distance between them stay the same throughout the fall? If the two divers were connected by a long bungee cord, would the tension in the cord increase, lessen, or stay the same during the fall?

Solution At any given instant, the speeds of the divers are different because one had a head start. In any time interval Δt after this instant, however, the two divers increase their speeds by the same amount because they have the same acceleration. Thus, the difference in their speeds remains the same throughout the fall.

The first jumper always has a greater speed than the second. Thus, in a given time interval, the first diver covers a greater distance than the second. Thus, the separation distance between them increases.

Once the distance between the divers reaches the length of the bungee cord, the tension in the cord begins to increase. As the tension increases, the distance between the divers becomes greater and greater.

EXAMPLE 2.10 Describing the Motion of a Tossed Ball

A ball is tossed straight up at 25 m/s. Estimate its velocity at 1-s intervals.

Solution Let us choose the upward direction to be positive. Regardless of whether the ball is moving upward or downward, its vertical velocity changes by approximately -10 m/s for every second it remains in the air. It starts out at 25 m/s. After 1 s has elapsed, it is still moving upward but at 15 m/s because its acceleration is downward (downward acceleration causes its velocity to decrease). After another second, its upward velocity has dropped to 5 m/s. Now comes the tricky part—after another half second, its velocity is zero. The ball has gone as high as it will go. After the last half of this 1-s interval, the ball is moving at -5 m/s. (The minus sign tells us that the ball is now moving in the negative direction, that is, *downward.* Its velocity has changed from $+5$ m/s to -5 m/s during that 1-s interval. The change in velocity is still $-5 - [+5] = -10$ m/s in that second.) It continues downward, and after another 1 s has elapsed, it is falling at a velocity of -15 m/s. Finally, after another 1 s, it has reached its original starting point and is moving downward at -25 m/s. If the ball had been tossed vertically off a cliff so that it could continue downward, its velocity would continue to change by about -10 m/s every second.

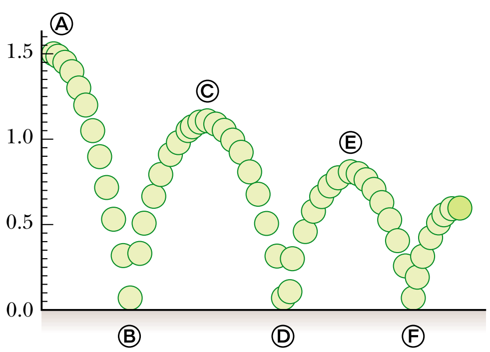
CONCEPTUAL EXAMPLE 2.11 Follow the Bouncing Ball

A tennis ball is dropped from shoulder height (about 1.5 m) and bounces three times before it is caught. Sketch graphs of its position, velocity, and acceleration as functions of time, with the $+y$ direction defined as upward.

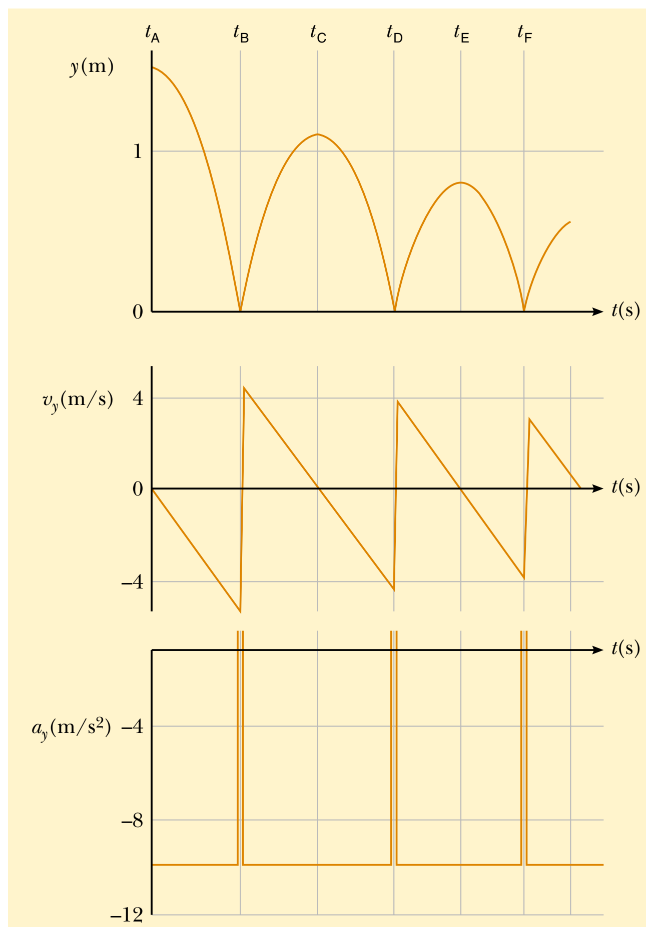
Solution For our sketch let us stretch things out horizontally so that we can see what is going on. (Even if the ball were moving horizontally, this motion would not affect its vertical motion.)

From Figure 2.13 we see that the ball is in contact with the floor at points Ⓑ, Ⓓ, and Ⓕ. Because the velocity of the ball changes from negative to positive three times during these bounces, the slope of the position–time graph must change in the same way. Note that the time interval between bounces decreases. Why is that?

During the rest of the ball's motion, the slope of the velocity–time graph should be -9.80 m/s^2. The acceleration–time graph is a horizontal line at these times because the acceleration does not change when the ball is in free fall. When the ball is in contact with the floor, the velocity changes substantially during a very short time interval, and so the acceleration must be quite great. This corresponds to the very steep upward lines on the velocity–time graph and to the spikes on the acceleration–time graph.



(a)



(b)

Figure 2.13 (a) A ball is dropped from a height of 1.5 m and bounces from the floor. (The horizontal motion is not considered here because it does not affect the vertical motion.) (b) Graphs of position, velocity, and acceleration versus time.

Quick Quiz 2.5

Which values represent the ball's velocity and acceleration at points Ⓐ, Ⓒ, and Ⓔ in Figure 2.13?

(a) $v_y = 0$, $a_y = 0$
(b) $v_y = 0$, $a_y = 9.80 \text{ m/s}^2$
(c) $v_y = 0$, $a_y = -9.80 \text{ m/s}^2$
(d) $v_y = -9.80 \text{ m/s}$, $a_y = 0$

Example 2.12 Not a Bad Throw for a Rookie!

A stone thrown from the top of a building is given an initial velocity of 20.0 m/s straight upward. The building is 50.0 m high, and the stone just misses the edge of the roof on its way down, as shown in Figure 2.14. Using $t_A = 0$ as the time the stone leaves the thrower's hand at position Ⓐ, determine (a) the time at which the stone reaches its maximum height, (b) the maximum height, (c) the time at which the stone returns to the height from which it was thrown, (d) the velocity of the stone at this instant, and (e) the velocity and position of the stone at $t = 5.00$ s.

Solution (a) As the stone travels from Ⓐ to Ⓑ, its velocity must change by 20 m/s because it stops at Ⓑ. Because gravity causes vertical velocities to change by about 10 m/s for every second of free fall, it should take the stone about 2 s to go from Ⓐ to Ⓑ in our drawing. (In a problem like this, a sketch definitely helps you organize your thoughts.) To calculate the time t_B at which the stone reaches maximum height, we use Equation 2.8, $v_{yB} = v_{yA} + a_y t$, noting that $v_{yB} = 0$ and setting the start of our clock readings at $t_A \equiv 0$:

$$20.0 \text{ m/s} + (-9.80 \text{ m/s}^2)t = 0$$

$$t = t_B = \frac{20.0 \text{ m/s}}{9.80 \text{ m/s}^2} = \boxed{2.04 \text{ s}}$$

Our estimate was pretty close.

(b) Because the average velocity for this first interval is 10 m/s (the average of 20 m/s and 0 m/s) and because it travels for about 2 s, we expect the stone to travel about 20 m. By substituting our time interval into Equation 2.11, we can find the maximum height as measured from the position of the thrower, where we set $y_i = y_A = 0$:

$$y_{max} = y_B = v_{yA}\, t + \tfrac{1}{2}a_y t^2$$

$$y_B = (20.0 \text{ m/s})(2.04 \text{ s}) + \tfrac{1}{2}(-9.80 \text{ m/s}^2)(2.04 \text{ s})^2$$

$$= \boxed{20.4 \text{ m}}$$

Our free-fall estimates are very accurate.

(c) There is no reason to believe that the stone's motion from Ⓑ to Ⓒ is anything other than the reverse of its motion

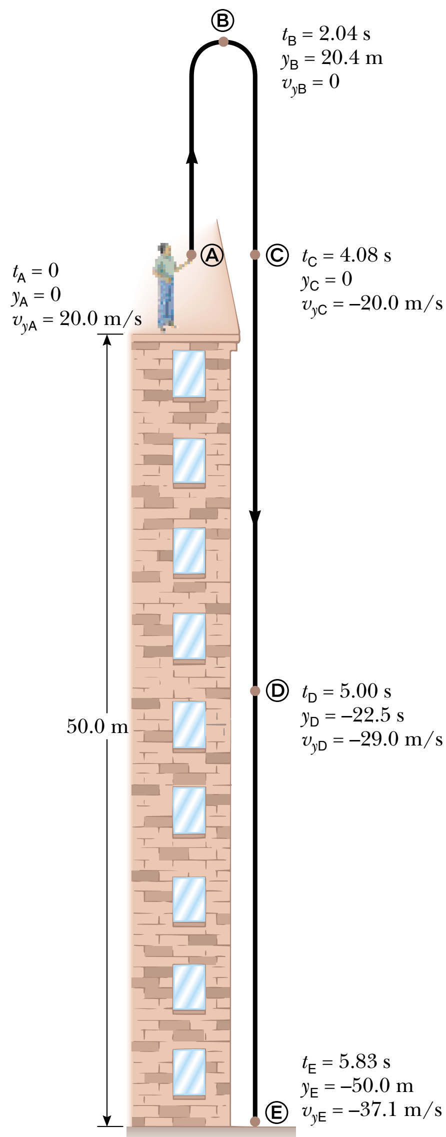


Figure 2.14 Position and velocity versus time for a freely falling stone thrown initially upward with a velocity $v_{yi} = 20.0$ m/s.

from Ⓐ to Ⓑ. Thus, the time needed for it to go from Ⓐ to Ⓒ should be twice the time needed for it to go from Ⓐ to Ⓑ. When the stone is back at the height from which it was thrown (position Ⓒ), the y coordinate is again zero. Using Equation 2.11, with $y_f = y_C = 0$ and $y_i = y_A = 0$, we obtain

$$y_C - y_A = v_{yA}\, t + \tfrac{1}{2}a_y t^2$$
$$0 = 20.0t - 4.90t^2$$

This is a quadratic equation and so has two solutions for $t = t_C$. The equation can be factored to give

$$t(20.0 - 4.90t) = 0$$

One solution is $t = 0$, corresponding to the time the stone starts its motion. The other solution is $t = 4.08$ s, which is the solution we are after. Notice that it is double the value we calculated for t_B.

(d) Again, we expect everything at Ⓒ to be the same as it is at Ⓐ, except that the velocity is now in the opposite direction. The value for t found in (c) can be inserted into Equation 2.8 to give

$$v_{yC} = v_{yA} + a_y t = 20.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(4.08 \text{ s})$$
$$= -20.0 \text{ m/s}$$

The velocity of the stone when it arrives back at its original height is equal in magnitude to its initial velocity but opposite in direction. This indicates that the motion is symmetric.

(e) For this part we consider what happens as the stone falls from position Ⓑ, where it had zero vertical velocity, to position Ⓓ. Because the elapsed time for this part of the motion is about 3 s, we estimate that the acceleration due to gravity will have changed the speed by about 30 m/s. We can calculate this from Equation 2.8, where we take $t = t_D - t_B$:

$$v_{yD} = v_{yB} + a_y t = 0 \text{ m/s} + (-9.80 \text{ m/s}^2)(5.00 \text{ s} - 2.04 \text{ s})$$
$$= -29.0 \text{ m/s}$$

We could just as easily have made our calculation between positions Ⓐ and Ⓓ by making sure we use the correct time interval, $t = t_D - t_A = 5.00$ s:

$$v_{yD} = v_{yA} + a_y t = 20.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(5.00 \text{ s})$$
$$= -29.0 \text{ m/s}$$

To demonstrate the power of our kinematic equations, we can use Equation 2.11 to find the position of the stone at $t_D = 5.00$ s by considering the change in position between a different pair of positions, Ⓒ and Ⓓ. In this case, the time is $t_D - t_C$:

$$y_D = y_C + v_{yC}t + \tfrac{1}{2}a_y t^2$$
$$= 0 \text{ m} + (-20.0 \text{ m/s})(5.00 \text{ s} - 4.08 \text{ s})$$
$$+ \tfrac{1}{2}(-9.80 \text{ m/s}^2)(5.00 \text{ s} - 4.08 \text{ s})^2$$
$$= -22.5 \text{ m}$$

Exercise Find (a) the velocity of the stone just before it hits the ground at Ⓔ and (b) the total time the stone is in the air.

Answer (a) -37.1 m/s (b) 5.83 s

Optional Section

2.7 KINEMATIC EQUATIONS DERIVED FROM CALCULUS

This is an optional section that assumes the reader is familiar with the techniques of integral calculus. If you have not yet studied integration in your calculus course, you should skip this section or cover it after you become familiar with integration.

The velocity of a particle moving in a straight line can be obtained if its position as a function of time is known. Mathematically, the velocity equals the derivative of the position coordinate with respect to time. It is also possible to find the displacement of a particle if its velocity is known as a function of time. In calculus, the procedure used to perform this task is referred to either as *integration* or as finding the *antiderivative*. Graphically, it is equivalent to finding the area under a curve.

Suppose the v_x-t graph for a particle moving along the x axis is as shown in Figure 2.15. Let us divide the time interval $t_f - t_i$ into many small intervals, each of duration Δt_n. From the definition of average velocity we see that the displacement during any small interval, such as the one shaded in Figure 2.15, is given by $\Delta x_n = \overline{v}_{xn}\,\Delta t_n$, where $\overline{v}_{xn}$ is the average velocity in that interval. Therefore, the displacement during this small interval is simply the area of the shaded rectangle.

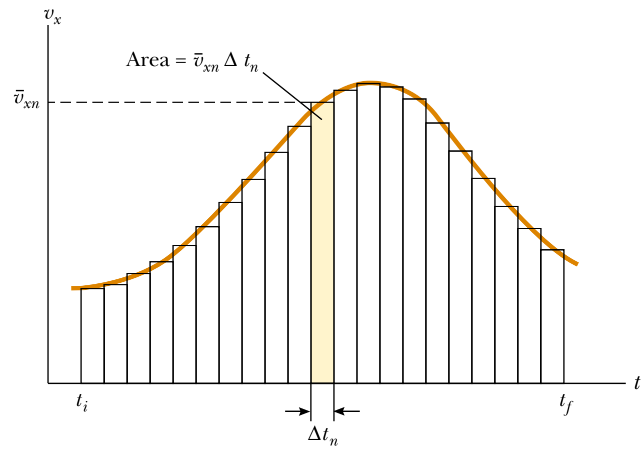


Figure 2.15 Velocity versus time for a particle moving along the x axis. The area of the shaded rectangle is equal to the displacement Δx in the time interval Δt_n, while the total area under the curve is the total displacement of the particle.

The total displacement for the interval $t_f - t_i$ is the sum of the areas of all the rectangles:

$$\Delta x = \sum_n \bar{v}_{xn}\,\Delta t_n$$

where the symbol Σ (upper case Greek sigma) signifies a sum over all terms. In this case, the sum is taken over all the rectangles from t_i to t_f. Now, as the intervals are made smaller and smaller, the number of terms in the sum increases and the sum approaches a value equal to the area under the velocity–time graph. Therefore, in the limit $n \rightarrow \infty$, or $\Delta t_n \rightarrow 0$, the displacement is

$$\Delta x = \lim_{\Delta t_n \to 0} \sum_n v_{xn}\,\Delta t_n \tag{2.13}$$

or

Displacement = area under the v_x-t graph

Note that we have replaced the average velocity $\bar{v}_{xn}$ with the instantaneous velocity v_{xn} in the sum. As you can see from Figure 2.15, this approximation is clearly valid in the limit of very small intervals. We conclude that if we know the v_x-t graph for motion along a straight line, we can obtain the displacement during any time interval by measuring the area under the curve corresponding to that time interval.

The limit of the sum shown in Equation 2.13 is called a **definite integral** and is written

Definite integral

$$\lim_{\Delta t_n \to 0} \sum_n v_{xn}\Delta t_n = \int_{t_i}^{t_f} v_x(t)\ dt \tag{2.14}$$

where $v_x(t)$ denotes the velocity at any time t. If the explicit functional form of $v_x(t)$ is known and the limits are given, then the integral can be evaluated.

Sometimes the v_x-t graph for a moving particle has a shape much simpler than that shown in Figure 2.15. For example, suppose a particle moves at a constant ve-

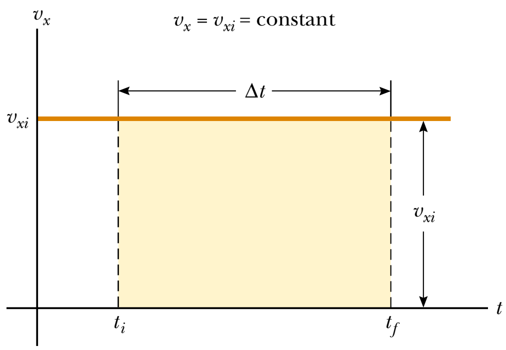


Figure 2.16 The velocity–time curve for a particle moving with constant velocity v_{xi}. The displacement of the particle during the time interval $t_f - t_i$ is equal to the area of the shaded rectangle.

locity v_{xi}. In this case, the v_x-t graph is a horizontal line, as shown in Figure 2.16, and its displacement during the time interval Δt is simply the area of the shaded rectangle:

$$\Delta x = v_{xi}\Delta t \qquad \text{(when } v_{xf} = v_{xi} = \text{constant)}$$

As another example, consider a particle moving with a velocity that is proportional to t, as shown in Figure 2.17. Taking $v_x = a_x t$, where a_x is the constant of proportionality (the acceleration), we find that the displacement of the particle during the time interval $t = 0$ to $t = t_{\text{A}}$ is equal to the area of the shaded triangle in Figure 2.17:

$$\Delta x = \tfrac{1}{2}(t_{\text{A}})(a_x t_{\text{A}}) = \tfrac{1}{2}a_x t_{\text{A}}{}^2$$

Kinematic Equations

We now use the defining equations for acceleration and velocity to derive two of our kinematic equations, Equations 2.8 and 2.11.

The defining equation for acceleration (Eq. 2.6),

$$a_x = \frac{dv_x}{dt}$$

may be written as $dv_x = a_x dt$ or, in terms of an integral (or antiderivative), as

$$v_x = \int a_x\, dt + C_1$$

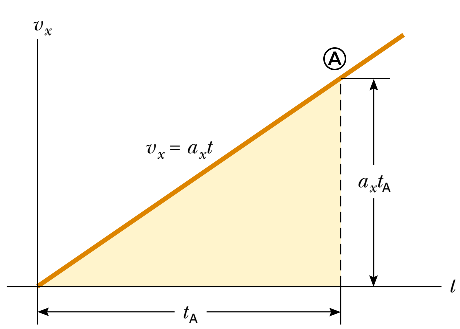


Figure 2.17 The velocity–time curve for a particle moving with a velocity that is proportional to the time.

where C_1 is a constant of integration. For the special case in which the acceleration is constant, the a_x can be removed from the integral to give

$$v_x = a_x \int dt + C_1 = a_x t + C_1 \tag{2.15}$$

The value of C_1 depends on the initial conditions of the motion. If we take $v_x = v_{xi}$ when $t = 0$ and substitute these values into the last equation, we have

$$v_{xi} = a_x(0) + C_1$$

$$C_1 = v_{xi}$$

Calling $v_x = v_{xf}$ the velocity after the time interval t has passed and substituting this and the value just found for C_1 into Equation 2.15, we obtain kinematic Equation 2.8:

$$v_{xf} = v_{xi} + a_x t \qquad \text{(for constant } a_x\text{)}$$

Now let us consider the defining equation for velocity (Eq. 2.4):

$$v_x = \frac{dx}{dt}$$

We can write this as $dx = v_x dt$ or in integral form as

$$x = \int v_x \, dt + C_2$$

where C_2 is another constant of integration. Because $v_x = v_{xf} = v_{xi} + a_x t$, this expression becomes

$$x = \int (v_{xi} + a_x t)\, dt + C_2$$

$$x = \int v_{xi} \, dt + a_x \int t \, dt + C_2$$

$$x = v_{xi} t + \tfrac{1}{2} a_x t^2 + C_2$$

To find C_2, we make use of the initial condition that $x = x_i$ when $t = 0$. This gives $C_2 = x_i$. Therefore, after substituting x_f for x, we have

$$x_f = x_i + v_{xi} t + \tfrac{1}{2} a_x t^2 \qquad \text{(for constant } a_x\text{)}$$

Once we move x_i to the left side of the equation, we have kinematic Equation 2.11. Recall that $x_f - x_i$ is equal to the displacement of the object, where x_i is its initial position.

Besides what you might expect to learn about physics concepts, a very valuable skill you should hope to take away from your physics course is the ability to solve complicated problems. The way physicists approach complex situations and break them down into manageable pieces is extremely useful. We have developed a memory aid to help you easily recall the steps required for successful problem solving. When working on problems, the secret is to keep your GOAL in mind!

GOAL Problem-Solving Steps

Gather information

The first thing to do when approaching a problem is to understand the situation. Carefully read the problem statement, looking for key phrases like "at rest" or "freely falls." What information is given? Exactly what is the question asking? Don't forget to gather information from your own experiences and common sense. What should a reasonable answer look like? You wouldn't expect to calculate the speed of an automobile to be 5×10^6 m/s. Do you know what units to expect? Are there any limiting cases you can consider? What happens when an angle approaches 0° or 90° or when a mass becomes huge or goes to zero? Also make sure you carefully study any drawings that accompany the problem.

Organize your approach

Once you have a really good idea of what the problem is about, you need to think about what to do next. Have you seen this type of question before? Being able to classify a problem can make it much easier to lay out a plan to solve it. You should almost always make a quick drawing of the situation. Label important events with circled letters. Indicate any known values, perhaps in a table or directly on your sketch.

Analyze the problem

Because you have already categorized the problem, it should not be too difficult to select relevant equations that apply to this type of situation. Use algebra (and calculus, if necessary) to solve for the unknown variable in terms of what is given. Substitute in the appropriate numbers, calculate the result, and round it to the proper number of significant figures.

Learn from your efforts

This is the most important part. Examine your numerical answer. Does it meet your expectations from the first step? What about the algebraic form of the result—before you plugged in numbers? Does it make sense? (Try looking at the variables in it to see whether the answer would change in a physically meaningful way if they were drastically increased or decreased or even became zero.) Think about how this problem compares with others you have done. How was it similar? In what critical ways did it differ? Why was this problem assigned? You should have learned something by doing it. Can you figure out what?

When solving complex problems, you may need to identify a series of subproblems and apply the GOAL process to each. For very simple problems, you probably don't need GOAL at all. But when you are looking at a problem and you don't know what to do next, remember what the letters in GOAL stand for and use that as a guide.

SUMMARY

After a particle moves along the x axis from some initial position x_i to some final position x_f, its **displacement** is

$$\Delta x \equiv x_f - x_i \tag{2.1}$$

The **average velocity** of a particle during some time interval is the displacement Δx divided by the time interval Δt during which that displacement occurred:

$$\bar{v}_x \equiv \frac{\Delta x}{\Delta t} \tag{2.2}$$

The **average speed** of a particle is equal to the ratio of the total distance it travels to the total time it takes to travel that distance.

The **instantaneous velocity** of a particle is defined as the limit of the ratio $\Delta x/\Delta t$ as Δt approaches zero. By definition, this limit equals the derivative of x with respect to t, or the time rate of change of the position:

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{2.4}$$

The **instantaneous speed** of a particle is equal to the magnitude of its velocity.

The **average acceleration** of a particle is defined as the ratio of the change in its velocity Δv_x divided by the time interval Δt during which that change occurred:

$$\bar{a}_x \equiv \frac{\Delta v_x}{\Delta t} = \frac{v_{xf} - v_{xi}}{t_f - t_i} \tag{2.5}$$

The **instantaneous acceleration** is equal to the limit of the ratio $\Delta v_x/\Delta t$ as Δt approaches 0. By definition, this limit equals the derivative of v_x with respect to t, or the time rate of change of the velocity:

$$a_x \equiv \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \tag{2.6}$$

The **equations of kinematics** for a particle moving along the x axis with uniform acceleration a_x (constant in magnitude and direction) are

$$v_{xf} = v_{xi} + a_x t \tag{2.8}$$

$$x_f - x_i = \bar{v}_x t = \tfrac{1}{2}(v_{xi} + v_{xf})t \tag{2.10}$$

$$x_f - x_i = v_{xi}t + \tfrac{1}{2}a_x t^2 \tag{2.11}$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i) \tag{2.12}$$

You should be able to use these equations and the definitions in this chapter to analyze the motion of any object moving with constant acceleration.

An object falling freely in the presence of the Earth's gravity experiences a free-fall acceleration directed toward the center of the Earth. If air resistance is neglected, if the motion occurs near the surface of the Earth, and if the range of the motion is small compared with the Earth's radius, then the free-fall acceleration g is constant over the range of motion, where g is equal to 9.80 m/s^2.

Complicated problems are best approached in an organized manner. You should be able to recall and apply the steps of the GOAL strategy when you need them.

QUESTIONS

1. Average velocity and instantaneous velocity are generally different quantities. Can they ever be equal for a specific type of motion? Explain.
2. If the average velocity is nonzero for some time interval, does this mean that the instantaneous velocity is never zero during this interval? Explain.
3. If the average velocity equals zero for some time interval Δt and if $v_x(t)$ is a continuous function, show that the instantaneous velocity must go to zero at some time in this interval. (A sketch of x versus t might be useful in your proof.)
4. Is it possible to have a situation in which the velocity and acceleration have opposite signs? If so, sketch a velocity–time graph to prove your point.
5. If the velocity of a particle is nonzero, can its acceleration be zero? Explain.
6. If the velocity of a particle is zero, can its acceleration be nonzero? Explain.
7. Can an object having constant acceleration ever stop and stay stopped?
8. A stone is thrown vertically upward from the top of a building. Does the stone's displacement depend on the location of the origin of the coordinate system? Does the stone's velocity depend on the origin? (Assume that the coordinate system is stationary with respect to the building.) Explain.
9. A student at the top of a building of height h throws one ball upward with an initial speed v_{yi} and then throws a second ball downward with the same initial speed. How do the final speeds of the balls compare when they reach the ground?
10. Can the magnitude of the instantaneous velocity of an object ever be greater than the magnitude of its average velocity? Can it ever be less?
11. If the average velocity of an object is zero in some time interval, what can you say about the displacement of the object for that interval?
12. A rapidly growing plant doubles in height each week. At the end of the 25th day, the plant reaches the height of a building. At what time was the plant one-fourth the height of the building?
13. Two cars are moving in the same direction in parallel lanes along a highway. At some instant, the velocity of car A exceeds the velocity of car B. Does this mean that the acceleration of car A is greater than that of car B? Explain.
14. An apple is dropped from some height above the Earth's surface. Neglecting air resistance, how much does the apple's speed increase each second during its descent?
15. Consider the following combinations of signs and values for velocity and acceleration of a particle with respect to a one-dimensional x axis:

Velocity	Acceleration
a. Positive	Positive
b. Positive	Negative
c. Positive	Zero
d. Negative	Positive
e. Negative	Negative
f. Negative	Zero
g. Zero	Positive
h. Zero	Negative

Describe what the particle is doing in each case, and give a real-life example for an automobile on an east-west one-dimensional axis, with east considered to be the positive direction.

16. A pebble is dropped into a water well, and the splash is heard 16 s later, as illustrated in Figure Q2.16. Estimate the distance from the rim of the well to the water's surface.
17. Average velocity is an entirely contrived quantity, and other combinations of data may prove useful in other contexts. For example, the ratio $\Delta t/\Delta x$, called the "slowness" of a moving object, is used by geophysicists when discussing the motion of continental plates. Explain what this quantity means.

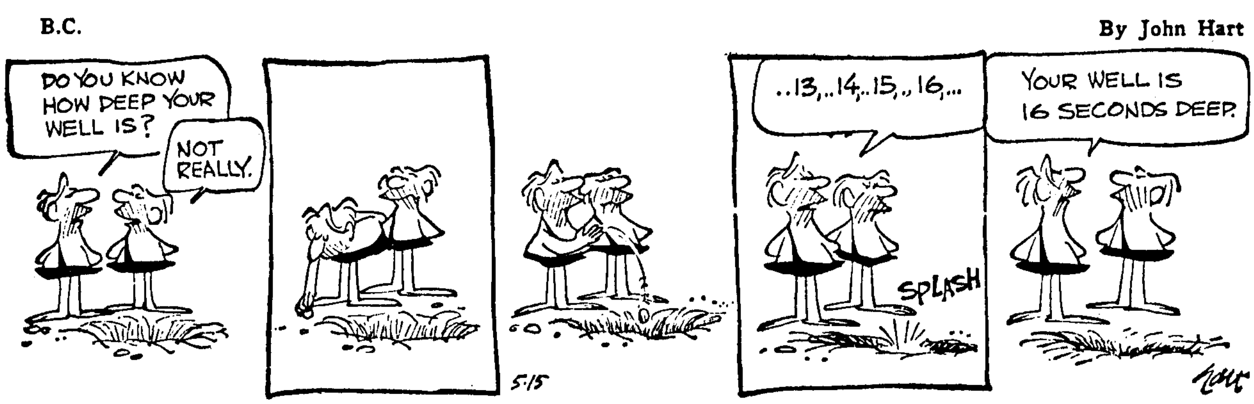


Figure Q2.16

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 2.1 Displacement, Velocity, and Speed

1. The position of a pinewood derby car was observed at various times; the results are summarized in the table below. Find the average velocity of the car for (a) the first second, (b) the last 3 s, and (c) the entire period of observation.

x (m)	0	2.3	9.2	20.7	36.8	57.5
t (s)	0	1.0	2.0	3.0	4.0	5.0

2. A motorist drives north for 35.0 min at 85.0 km/h and then stops for 15.0 min. He then continues north, traveling 130 km in 2.00 h. (a) What is his total displacement? (b) What is his average velocity?

3. The displacement versus time for a certain particle moving along the x axis is shown in Figure P2.3. Find the average velocity in the time intervals (a) 0 to 2 s, (b) 0 to 4 s, (c) 2 s to 4 s, (d) 4 s to 7 s, (e) 0 to 8 s.

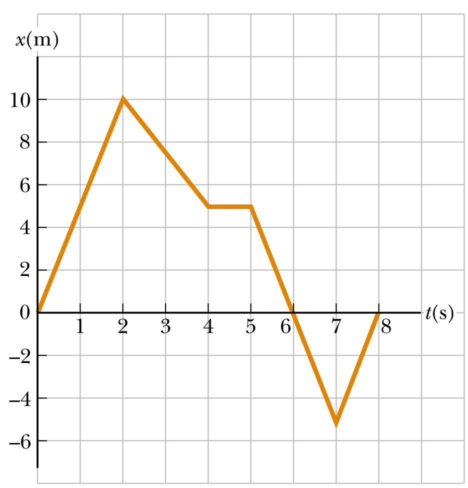


Figure P2.3 Problems 3 and 11.

4. A particle moves according to the equation $x = 10t^2$, where x is in meters and t is in seconds. (a) Find the average velocity for the time interval from 2.0 s to 3.0 s. (b) Find the average velocity for the time interval from 2.0 s to 2.1 s.

5. A person walks first at a constant speed of 5.00 m/s along a straight line from point A to point B and then back along the line from B to A at a constant speed of 3.00 m/s. What are (a) her average speed over the entire trip and (b) her average velocity over the entire trip?

6. A person first walks at a constant speed v_1 along a straight line from A to B and then back along the line from B to A at a constant speed v_2. What are (a) her average speed over the entire trip and (b) her average velocity over the entire trip?

Section 2.2 Instantaneous Velocity and Speed

7. At $t = 1.00$ s, a particle moving with constant velocity is located at $x = -3.00$ m, and at $t = 6.00$ s the particle is located at $x = 5.00$ m. (a) From this information, plot the position as a function of time. (b) Determine the velocity of the particle from the slope of this graph.

8. The position of a particle moving along the x axis varies in time according to the expression $x = 3t^2$, where x is in meters and t is in seconds. Evaluate its position (a) at $t = 3.00$ s and (b) at $3.00\text{ s} + \Delta t$. (c) Evaluate the limit of $\Delta x/\Delta t$ as Δt approaches zero to find the velocity at $t = 3.00$ s.

WEB 9. A position–time graph for a particle moving along the x axis is shown in Figure P2.9. (a) Find the average velocity in the time interval $t = 1.5$ s to $t = 4.0$ s. (b) Determine the instantaneous velocity at $t = 2.0$ s by measuring the slope of the tangent line shown in the graph. (c) At what value of t is the velocity zero?

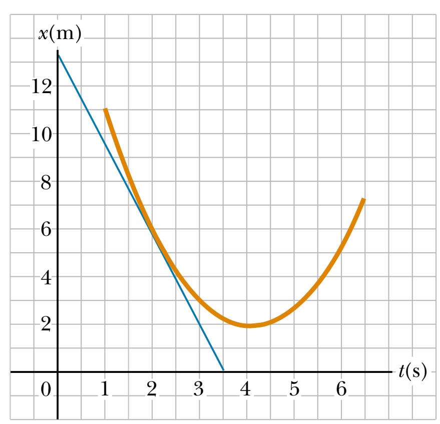


Figure P2.9

10. (a) Use the data in Problem 1 to construct a smooth graph of position versus time. (b) By constructing tangents to the $x(t)$ curve, find the instantaneous velocity of the car at several instants. (c) Plot the instantaneous velocity versus time and, from this, determine the average acceleration of the car. (d) What was the initial velocity of the car?

11. Find the instantaneous velocity of the particle described in Figure P2.3 at the following times: (a) $t = 1.0$ s, (b) $t = 3.0$ s, (c) $t = 4.5$ s, and (d) $t = 7.5$ s.

Section 2.3 Acceleration

12. A particle is moving with a velocity of 60.0 m/s in the positive x direction at $t = 0$. Between $t = 0$ and $t = 15.0$ s, the velocity decreases uniformly to zero. What was the acceleration during this 15.0-s interval? What is the significance of the sign of your answer?

13. A 50.0-g superball traveling at 25.0 m/s bounces off a brick wall and rebounds at 22.0 m/s. A high-speed camera records this event. If the ball is in contact with the wall for 3.50 ms, what is the magnitude of the average acceleration of the ball during this time interval? (*Note:* 1 ms $= 10^{-3}$ s.)

14. A particle starts from rest and accelerates as shown in Figure P2.14. Determine: (a) the particle's speed at $t = 10$ s and at $t = 20$ s, and (b) the distance traveled in the first 20 s.

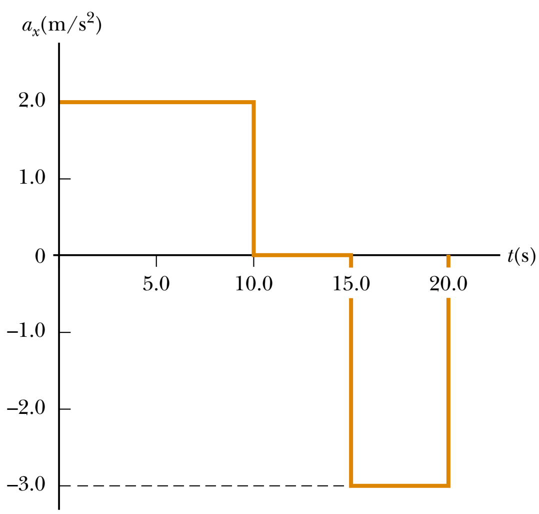


Figure P2.14

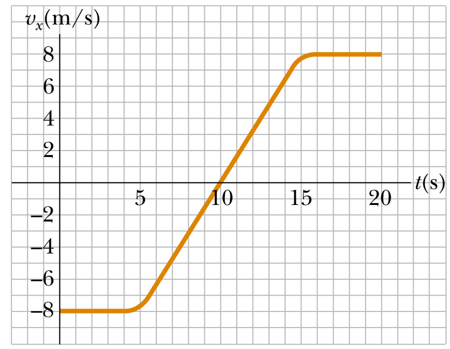


Figure P2.15

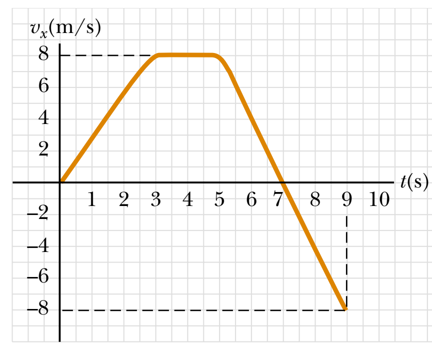


Figure P2.16

15. A velocity–time graph for an object moving along the x axis is shown in Figure P2.15. (a) Plot a graph of the acceleration versus time. (b) Determine the average acceleration of the object in the time intervals $t = 5.00$ s to $t = 15.0$ s and $t = 0$ to $t = 20.0$ s.

16. A student drives a moped along a straight road as described by the velocity–time graph in Figure P2.16. Sketch this graph in the middle of a sheet of graph paper. (a) Directly above your graph, sketch a graph of the position versus time, aligning the time coordinates of the two graphs. (b) Sketch a graph of the acceleration versus time directly below the v_x-t graph, again aligning the time coordinates. On each graph, show the numerical values of x and a_x for all points of inflection. (c) What is the acceleration at $t = 6$ s? (d) Find the position (relative to the starting point) at $t = 6$ s. (e) What is the moped's final position at $t = 9$ s?

WEB 17. A particle moves along the x axis according to the equation $x = 2.00 + 3.00t - t^2$, where x is in meters and t is in seconds. At $t = 3.00$ s, find (a) the position of the particle, (b) its velocity, and (c) its acceleration.

18. An object moves along the x axis according to the equation $x = (3.00t^2 - 2.00t + 3.00)$ m. Determine (a) the average speed between $t = 2.00$ s and $t = 3.00$ s, (b) the instantaneous speed at $t = 2.00$ s and at $t = 3.00$ s, (c) the average acceleration between $t = 2.00$ s and $t = 3.00$ s, and (d) the instantaneous acceleration at $t = 2.00$ s and $t = 3.00$ s.

19. Figure P2.19 shows a graph of v_x versus t for the motion of a motorcyclist as he starts from rest and moves along the road in a straight line. (a) Find the average acceleration for the time interval $t = 0$ to $t = 6.00$ s. (b) Estimate the time at which the acceleration has its greatest positive value and the value of the acceleration at that instant. (c) When is the acceleration zero? (d) Estimate the maximum negative value of the acceleration and the time at which it occurs.

Figure P2.19

Section 2.4 Motion Diagrams

20. Draw motion diagrams for (a) an object moving to the right at constant speed, (b) an object moving to the right and speeding up at a constant rate, (c) an object moving to the right and slowing down at a constant rate, (d) an object moving to the left and speeding up at a constant rate, and (e) an object moving to the left and slowing down at a constant rate. (f) How would your drawings change if the changes in speed were not uniform; that is, if the speed were not changing at a constant rate?

Section 2.5 One-Dimensional Motion with Constant Acceleration

21. Jules Verne in 1865 proposed sending people to the Moon by firing a space capsule from a 220-m-long cannon with a final velocity of 10.97 km/s. What would have been the unrealistically large acceleration experienced by the space travelers during launch? Compare your answer with the free-fall acceleration, 9.80 m/s^2.

22. A certain automobile manufacturer claims that its super-deluxe sports car will accelerate from rest to a speed of 42.0 m/s in 8.00 s. Under the (improbable) assumption that the acceleration is constant, (a) determine the acceleration of the car. (b) Find the distance the car travels in the first 8.00 s. (c) What is the speed of the car 10.0 s after it begins its motion, assuming it continues to move with the same acceleration?

23. A truck covers 40.0 m in 8.50 s while smoothly slowing down to a final speed of 2.80 m/s. (a) Find its original speed. (b) Find its acceleration.

24. The minimum distance required to stop a car moving at 35.0 mi/h is 40.0 ft. What is the minimum stopping distance for the same car moving at 70.0 mi/h, assuming the same rate of acceleration?

WEB **25.** A body moving with uniform acceleration has a velocity of 12.0 cm/s in the positive x direction when its x coordinate is 3.00 cm. If its x coordinate 2.00 s later is -5.00 cm, what is the magnitude of its acceleration?

26. Figure P2.26 represents part of the performance data of a car owned by a proud physics student. (a) Calculate from the graph the total distance traveled. (b) What distance does the car travel between the times $t = 10$ s and $t = 40$ s? (c) Draw a graph of its acceleration versus time between $t = 0$ and $t = 50$ s. (d) Write an equation for x as a function of time for each phase of the motion, represented by (i) $0a$, (ii) ab, (iii) bc. (e) What is the average velocity of the car between $t = 0$ and $t = 50$ s?

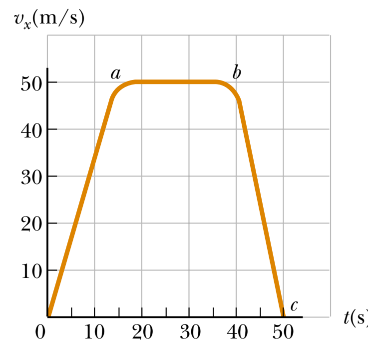


Figure P2.26

27. A particle moves along the x axis. Its position is given by the equation $x = 2.00 + 3.00t - 4.00t^2$ with x in meters and t in seconds. Determine (a) its position at the instant it changes direction and (b) its velocity when it returns to the position it had at $t = 0$.

28. The initial velocity of a body is 5.20 m/s. What is its velocity after 2.50 s (a) if it accelerates uniformly at 3.00 m/s^2 and (b) if it accelerates uniformly at -3.00 m/s^2?

29. A drag racer starts her car from rest and accelerates at 10.0 m/s^2 for the entire distance of 400 m ($\frac{1}{4}$ mi). (a) How long did it take the race car to travel this distance? (b) What is the speed of the race car at the end of the run?

30. A car is approaching a hill at 30.0 m/s when its engine suddenly fails, just at the bottom of the hill. The car moves with a constant acceleration of -2.00 m/s^2 while coasting up the hill. (a) Write equations for the position along the slope and for the velocity as functions of time, taking $x = 0$ at the bottom of the hill, where v_i = 30.0 m/s. (b) Determine the maximum distance the car travels up the hill.

31. A jet plane lands with a speed of 100 m/s and can accelerate at a maximum rate of -5.00 m/s^2 as it comes to rest. (a) From the instant the plane touches the runway, what is the minimum time it needs before it can come to rest? (b) Can this plane land at a small tropical island airport where the runway is 0.800 km long?

32. The driver of a car slams on the brakes when he sees a tree blocking the road. The car slows uniformly with an acceleration of -5.60 m/s^2 for 4.20 s, making straight skid marks 62.4 m long ending at the tree. With what speed does the car then strike the tree?

33. *Help! One of our equations is missing!* We describe constant-acceleration motion with the variables and parameters v_{xi}, v_{xf}, a_x, t, and $x_f - x_i$. Of the equations in Table 2.2, the first does not involve $x_f - x_i$. The second does not contain a_x, the third omits v_{xf}, and the last

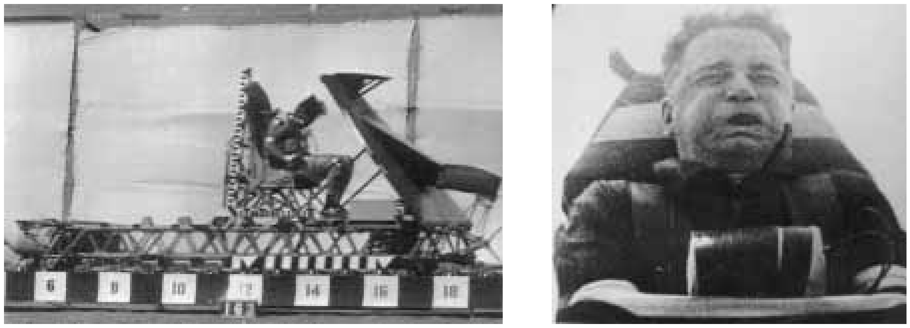


Figure P2.37 *(Left)* Col. John Stapp on rocket sled. *(Courtesy of the U.S. Air Force)* *(Right)* Col. Stapp's face is contorted by the stress of rapid negative acceleration. *(Photri, Inc.)*

leaves out t. So to complete the set there should be an equation *not* involving v_{xi}. Derive it from the others. Use it to solve Problem 32 in one step.

34. An indestructible bullet 2.00 cm long is fired straight through a board that is 10.0 cm thick. The bullet strikes the board with a speed of 420 m/s and emerges with a speed of 280 m/s. (a) What is the average acceleration of the bullet as it passes through the board? (b) What is the total time that the bullet is in contact with the board? (c) What thickness of board (calculated to 0.1 cm) would it take to stop the bullet, assuming the bullet's acceleration through all parts of the board is the same?

35. A truck on a straight road starts from rest, accelerating at 2.00 m/s^2 until it reaches a speed of 20.0 m/s. Then the truck travels for 20.0 s at constant speed until the brakes are applied, stopping the truck in a uniform manner in an additional 5.00 s. (a) How long is the truck in motion? (b) What is the average velocity of the truck for the motion described?

36. A train is traveling down a straight track at 20.0 m/s when the engineer applies the brakes. This results in an acceleration of $-$ 1.00 m/s^2 as long as the train is in motion. How far does the train move during a 40.0-s time interval starting at the instant the brakes are applied?

37. For many years the world's land speed record was held by Colonel John P. Stapp, USAF (Fig. P2.37). On March 19, 1954, he rode a rocket-propelled sled that moved down the track at 632 mi/h. He and the sled were safely brought to rest in 1.40 s. Determine (a) the negative acceleration he experienced and (b) the distance he traveled during this negative acceleration.

38. An electron in a cathode-ray tube (CRT) accelerates uniformly from 2.00×10^4 m/s to 6.00×10^6 m/s over 1.50 cm. (a) How long does the electron take to travel this 1.50 cm? (b) What is its acceleration?

39. A ball starts from rest and accelerates at 0.500 m/s^2 while moving down an inclined plane 9.00 m long. When it reaches the bottom, the ball rolls up another plane, where, after moving 15.0 m, it comes to rest. (a) What is the speed of the ball at the bottom of the first plane? (b) How long does it take to roll down the first plane? (c) What is the acceleration along the second plane? (d) What is the ball's speed 8.00 m along the second plane?

40. Speedy Sue, driving at 30.0 m/s, enters a one-lane tunnel. She then observes a slow-moving van 155 m ahead traveling at 5.00 m/s. Sue applies her brakes but can accelerate only at -2.00 m/s^2 because the road is wet. Will there be a collision? If so, determine how far into the tunnel and at what time the collision occurs. If not, determine the distance of closest approach between Sue's car and the van.

Section 2.6 Freely Falling Objects

Note: In all problems in this section, ignore the effects of air resistance.

41. A golf ball is released from rest from the top of a very tall building. Calculate (a) the position and (b) the velocity of the ball after 1.00 s, 2.00 s, and 3.00 s.

42. *Every morning at seven o'clock*
There's twenty terriers drilling on the rock.
The boss comes around and he says, "Keep still
And bear down heavy on the cast-iron drill
And drill, ye terriers, drill." And drill, ye terriers, drill.
It's work all day for sugar in your tea . . .
And drill, ye terriers, drill.

One day a premature blast went off
And a mile in the air went big Jim Goff. And drill . . .

Then when next payday came around
Jim Goff a dollar short was found.
When he asked what for, came this reply:
"You were docked for the time you were up in the sky." And drill . . .

—American folksong

What was Goff's hourly wage? State the assumptions you make in computing it.

WEB **43.** A student throws a set of keys vertically upward to her sorority sister, who is in a window 4.00 m above. The keys are caught 1.50 s later by the sister's outstretched hand. (a) With what initial velocity were the keys thrown? (b) What was the velocity of the keys just before they were caught?

44. A ball is thrown directly downward with an initial speed of 8.00 m/s from a height of 30.0 m. How many seconds later does the ball strike the ground?

45. Emily challenges her friend David to catch a dollar bill as follows: She holds the bill vertically, as in Figure P2.45, with the center of the bill between David's index finger and thumb. David must catch the bill after Emily releases it without moving his hand downward. If his reaction time is 0.2 s, will he succeed? Explain your reasoning.

Figure P2.45 *(George Semple)*

46. A ball is dropped from rest from a height h above the ground. Another ball is thrown vertically upward from the ground at the instant the first ball is released. Determine the speed of the second ball if the two balls are to meet at a height $h/2$ above the ground.

47. A baseball is hit so that it travels straight upward after being struck by the bat. A fan observes that it takes 3.00 s for the ball to reach its maximum height. Find (a) its initial velocity and (b) the maximum height it reaches.

48. A woman is reported to have fallen 144 ft from the 17th floor of a building, landing on a metal ventilator box, which she crushed to a depth of 18.0 in. She suffered only minor injuries. Calculate (a) the speed of the woman just before she collided with the ventilator box, (b) her average acceleration while in contact with the box, and (c) the time it took to crush the box.

WEB **49.** A daring ranch hand sitting on a tree limb wishes to drop vertically onto a horse galloping under the tree. The speed of the horse is 10.0 m/s, and the distance from the limb to the saddle is 3.00 m. (a) What must be the horizontal distance between the saddle and limb when the ranch hand makes his move? (b) How long is he in the air?

50. A ball thrown vertically upward is caught by the thrower after 20.0 s. Find (a) the initial velocity of the ball and (b) the maximum height it reaches.

51. A ball is thrown vertically upward from the ground with an initial speed of 15.0 m/s. (a) How long does it take the ball to reach its maximum altitude? (b) What is its maximum altitude? (c) Determine the velocity and acceleration of the ball at $t = 2.00$ s.

52. The height of a helicopter above the ground is given by $h = 3.00t^3$, where h is in meters and t is in seconds. After 2.00 s, the helicopter releases a small mailbag. How long after its release does the mailbag reach the ground?

(Optional)

2.7 Kinematic Equations Derived from Calculus

53. Automotive engineers refer to the time rate of change of acceleration as the "jerk." If an object moves in one dimension such that its jerk J is constant, (a) determine expressions for its acceleration a_x, velocity v_x, and position x, given that its initial acceleration, speed, and position are a_{xi}, v_{xi}, and x_i, respectively. (b) Show that $a_x^2 = a_{xi}^2 + 2J(v_x - v_{xi})$.

54. The speed of a bullet as it travels down the barrel of a rifle toward the opening is given by the expression $v = (-5.0 \times 10^7)t^2 + (3.0 \times 10^5)t$, where v is in meters per second and t is in seconds. The acceleration of the bullet just as it leaves the barrel is zero. (a) Determine the acceleration and position of the bullet as a function of time when the bullet is in the barrel. (b) Determine the length of time the bullet is accelerated. (c) Find the speed at which the bullet leaves the barrel. (d) What is the length of the barrel?

55. The acceleration of a marble in a certain fluid is proportional to the speed of the marble squared and is given (in SI units) by $a = -3.00v^2$ for $v > 0$. If the marble enters this fluid with a speed of 1.50 m/s, how long will it take before the marble's speed is reduced to half of its initial value?

ADDITIONAL PROBLEMS

56. A motorist is traveling at 18.0 m/s when he sees a deer in the road 38.0 m ahead. (a) If the maximum negative acceleration of the vehicle is -4.50 m/s^2, what is the maximum reaction time Δt of the motorist that will allow him to avoid hitting the deer? (b) If his reaction time is actually 0.300 s, how fast will he be traveling when he hits the deer?

57. Another scheme to catch the roadrunner has failed. A safe falls from rest from the top of a 25.0-m-high cliff toward Wile E. Coyote, who is standing at the base. Wile first notices the safe after it has fallen 15.0 m. How long does he have to get out of the way?

58. A dog's hair has been cut and is now getting longer by 1.04 mm each day. With winter coming on, this rate of hair growth is steadily increasing by 0.132 mm/day every week. By how much will the dog's hair grow during five weeks?

59. A test rocket is fired vertically upward from a well. A catapult gives it an initial velocity of 80.0 m/s at ground level. Subsequently, its engines fire and it accelerates upward at 4.00 m/s^2 until it reaches an altitude of 1000 m. At that point its engines fail, and the rocket goes into free fall, with an acceleration of -9.80 m/s^2. (a) How long is the rocket in motion above the ground? (b) What is its maximum altitude? (c) What is its velocity just before it collides with the Earth? (*Hint:* Consider the motion while the engine is operating separate from the free-fall motion.)

60. A motorist drives along a straight road at a constant speed of 15.0 m/s. Just as she passes a parked motorcycle police officer, the officer starts to accelerate at 2.00 m/s^2 to overtake her. Assuming the officer maintains this acceleration, (a) determine the time it takes the police officer to reach the motorist. Also find (b) the speed and (c) the total displacement of the officer as he overtakes the motorist.

61. In Figure 2.10a, the area under the velocity–time curve between the vertical axis and time t (vertical dashed line) represents the displacement. As shown, this area consists of a rectangle and a triangle. Compute their areas and compare the sum of the two areas with the expression on the righthand side of Equation 2.11.

62. A commuter train travels between two downtown stations. Because the stations are only 1.00 km apart, the train never reaches its maximum possible cruising speed. The engineer minimizes the time t between the two stations by accelerating at a rate $a_1 = 0.100 \text{ m/s}^2$ for a time t_1 and then by braking with acceleration $a_2 = -0.500 \text{ m/s}^2$ for a time t_2. Find the minimum time of travel t and the time t_1.

63. In a 100-m race, Maggie and Judy cross the finish line in a dead heat, both taking 10.2 s. Accelerating uniformly, Maggie took 2.00 s and Judy 3.00 s to attain maximum speed, which they maintained for the rest of the race. (a) What was the acceleration of each sprinter? (b) What were their respective maximum speeds? (c) Which sprinter was ahead at the 6.00-s mark, and by how much?

64. A hard rubber ball, released at chest height, falls to the pavement and bounces back to nearly the same height. When it is in contact with the pavement, the lower side of the ball is temporarily flattened. Suppose the maximum depth of the dent is on the order of 1 cm. Compute an order-of-magnitude estimate for the maximum acceleration of the ball while it is in contact with the pavement. State your assumptions, the quantities you estimate, and the values you estimate for them.

65. A teenager has a car that speeds up at 3.00 m/s^2 and slows down at -4.50 m/s^2. On a trip to the store, he accelerates from rest to 12.0 m/s, drives at a constant speed for 5.00 s, and then comes to a momentary stop at an intersection. He then accelerates to 18.0 m/s, drives at a constant speed for 20.0 s, slows down for 2.67 s, continues for 4.00 s at this speed, and then comes to a stop. (a) How long does the trip take? (b) How far has he traveled? (c) What is his average speed for the trip? (d) How long would it take to walk to the store and back if he walks at 1.50 m/s?

66. A rock is dropped from rest into a well. (a) If the sound of the splash is heard 2.40 s later, how far below the top of the well is the surface of the water? The speed of sound in air (at the ambient temperature) is 336 m/s. (b) If the travel time for the sound is neglected, what percentage error is introduced when the depth of the well is calculated?

67. An inquisitive physics student and mountain climber climbs a 50.0-m cliff that overhangs a calm pool of water. He throws two stones vertically downward, 1.00 s apart, and observes that they cause a single splash. The first stone has an initial speed of 2.00 m/s. (a) How long after release of the first stone do the two stones hit the water? (b) What was the initial velocity of the second stone? (c) What is the velocity of each stone at the instant the two hit the water?

68. A car and train move together along parallel paths at 25.0 m/s, with the car adjacent to the rear of the train. Then, because of a red light, the car undergoes a uniform acceleration of -2.50 m/s^2 and comes to rest. It remains at rest for 45.0 s and then accelerates back to a speed of 25.0 m/s at a rate of 2.50 m/s^2. How far behind the rear of the train is the car when it reaches the speed of 25.0 m/s, assuming that the speed of the train has remained 25.0 m/s?

69. Kathy Kool buys a sports car that can accelerate at the rate of 4.90 m/s^2. She decides to test the car by racing with another speedster, Stan Speedy. Both start from rest, but experienced Stan leaves the starting line 1.00 s before Kathy. If Stan moves with a constant acceleration of 3.50 m/s^2 and Kathy maintains an acceleration of 4.90 m/s^2, find (a) the time it takes Kathy to overtake Stan, (b) the distance she travels before she catches up with him, and (c) the speeds of both cars at the instant she overtakes him.

70. To protect his food from hungry bears, a boy scout raises his food pack with a rope that is thrown over a tree limb at height h above his hands. He walks away from the vertical rope with constant velocity v_{boy}, holding the free end of the rope in his hands (Fig. P2.70).

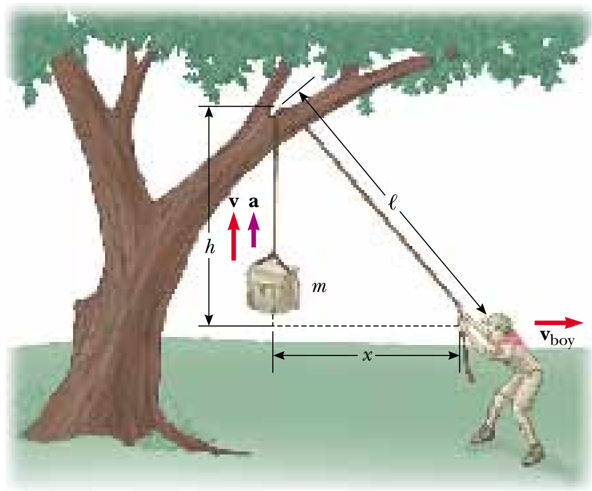


Figure P2.70

(a) Show that the speed v of the food pack is $x(x^2 + h^2)^{-1/2}\, v_{boy}$, where x is the distance he has walked away from the vertical rope. (b) Show that the acceleration a of the food pack is $h^2(x^2 + h^2)^{-3/2}\, v_{boy}{}^2$. (c) What values do the acceleration and velocity have shortly after he leaves the point under the pack ($x = 0$)? (d) What values do the pack's velocity and acceleration approach as the distance x continues to increase?

71. In Problem 70, let the height h equal 6.00 m and the speed v_{boy} equal 2.00 m/s. Assume that the food pack starts from rest. (a) Tabulate and graph the speed–time graph. (b) Tabulate and graph the acceleration–time graph. (Let the range of time be from 0 to 5.00 s and the time intervals be 0.500 s.)

72. Astronauts on a distant planet toss a rock into the air. With the aid of a camera that takes pictures at a steady rate, they record the height of the rock as a function of time as given in Table P2.72. (a) Find the average velocity of the rock in the time interval between each measurement and the next. (b) Using these average velocities to approximate instantaneous velocities at the midpoints of the time intervals, make a graph of velocity as a function of time. Does the rock move with constant acceleration? If so, plot a straight line of best fit on the graph and calculate its slope to find the acceleration.

TABLE P2.72 Height of a Rock versus Time

Time (s)	Height (m)	Time (s)	Height (m)
0.00	5.00	2.75	7.62
0.25	5.75	3.00	7.25
0.50	6.40	3.25	6.77
0.75	6.94	3.50	6.20
1.00	7.38	3.75	5.52
1.25	7.72	4.00	4.73
1.50	7.96	4.25	3.85
1.75	8.10	4.50	2.86
2.00	8.13	4.75	1.77
2.25	8.07	5.00	0.58
2.50	7.90		

73. Two objects, A and B, are connected by a rigid rod that has a length L. The objects slide along perpendicular guide rails, as shown in Figure P2.73. If A slides to the left with a constant speed v, find the speed of B when $\alpha = 60.0°$.

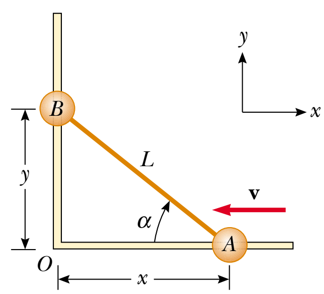


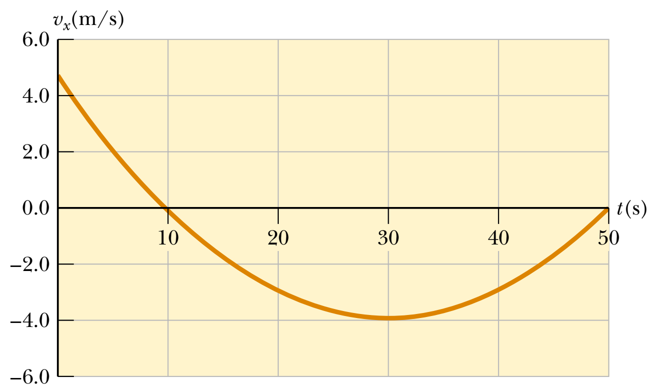
Figure P2.73

ANSWERS TO QUICK QUIZZES

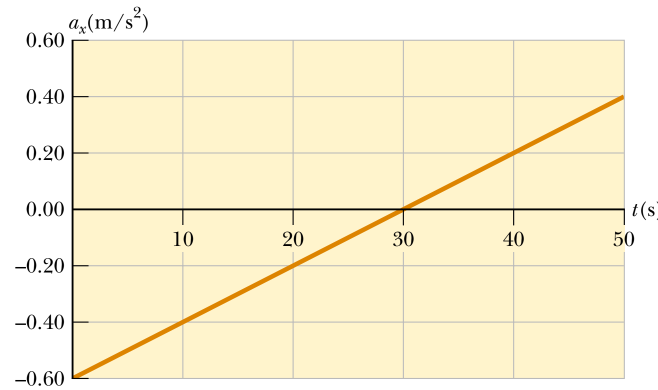
2.1 Your graph should look something like the one in (a). This v_x-t graph shows that the maximum speed is about 5.0 m/s, which is 18 km/h (= 11 mi/h), and so the driver was not speeding. Can you derive the acceleration–time graph from the velocity–time graph? It should look something like the one in (b).

2.2 (a) Yes. This occurs when the car is slowing down, so that the direction of its acceleration is opposite the direction of its motion. (b) Yes. If the motion is in the direction chosen as negative, a positive acceleration causes a decrease in speed.

2.3 The left side represents the final velocity of an object. The first term on the right side is the velocity that the object had initially when we started watching it. The second term is the change in that initial velocity that is caused by the acceleration. If this second term is positive, then the initial velocity has increased ($v_{xf} > v_{xi}$). If this term is negative, then the initial velocity has decreased ($v_{xf} < v_{xi}$).



(a)



(b)

2.4 Graph (a) has a constant slope, indicating a constant acceleration; this is represented by graph (e).

Graph (b) represents a speed that is increasing constantly but not at a uniform rate. Thus, the acceleration must be increasing, and the graph that best indicates this is (d).

Graph (c) depicts a velocity that first increases at a constant rate, indicating constant acceleration. Then the velocity stops increasing and becomes constant, indicating zero acceleration. The best match to this situation is graph (f).

2.5 (c). As can be seen from Figure 2.13b, the ball is at rest for an infinitesimally short time at these three points. Nonetheless, gravity continues to act even though the ball is instantaneously not moving.

PUZZLER

When this honeybee gets back to its hive, it will tell the other bees how to return to the food it has found. By moving in a special, very precisely defined pattern, the bee conveys to other workers the information they need to find a flower bed. Bees communicate by “speaking in vectors.” What does the bee have to tell the other bees in order to specify where the flower bed is located relative to the hive? *(E. Webber/Visuals Unlimited)*

chapter

3 Vectors

Chapter Outline

We often need to work with physical quantities that have both numerical and directional properties. As noted in Section 2.1, quantities of this nature are represented by vectors. This chapter is primarily concerned with vector algebra and with some general properties of vector quantities. We discuss the addition and subtraction of vector quantities, together with some common applications to physical situations.

Vector quantities are used throughout this text, and it is therefore imperative that you master both their graphical and their algebraic properties.

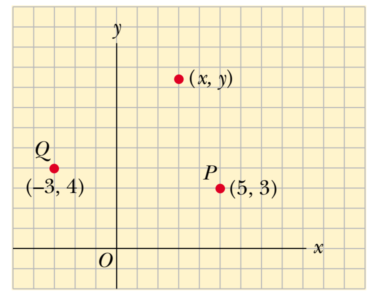


Figure 3.1 Designation of points in a cartesian coordinate system. Every point is labeled with coordinates (x, y).

3.1 COORDINATE SYSTEMS

Many aspects of physics deal in some form or other with locations in space. In Chapter 2, for example, we saw that the mathematical description of an object's motion requires a method for describing the object's position at various times. This description is accomplished with the use of coordinates, and in Chapter 2 we used the cartesian coordinate system, in which horizontal and vertical axes intersect at a point taken to be the origin (Fig. 3.1). Cartesian coordinates are also called *rectangular coordinates.*

2.2 Sometimes it is more convenient to represent a point in a plane by its *plane polar coordinates* (r, θ), as shown in Figure 3.2a. In this *polar coordinate system,* r is the distance from the origin to the point having cartesian coordinates (x, y), and θ is the angle between r and a fixed axis. This fixed axis is usually the positive x axis, and θ is usually measured counterclockwise from it. From the right triangle in Figure 3.2b, we find that $\sin\theta = y/r$ and that $\cos\theta = x/r$. (A review of trigonometric functions is given in Appendix B.4.) Therefore, starting with the plane polar coordinates of any point, we can obtain the cartesian coordinates, using the equations

$$x = r\cos\theta \quad (3.1)$$

$$y = r\sin\theta \quad (3.2)$$

Furthermore, the definitions of trigonometry tell us that

$$\tan\theta = \frac{y}{x} \quad (3.3)$$

$$r = \sqrt{x^2 + y^2} \quad (3.4)$$

These four expressions relating the coordinates (x, y) to the coordinates (r, θ) apply only when θ is defined, as shown in Figure 3.2a—in other words, when positive θ is an angle measured *counterclockwise* from the positive x axis. (Some scientific calculators perform conversions between cartesian and polar coordinates based on these standard conventions.) If the reference axis for the polar angle θ is chosen to be one other than the positive x axis or if the sense of increasing θ is chosen differently, then the expressions relating the two sets of coordinates will change.

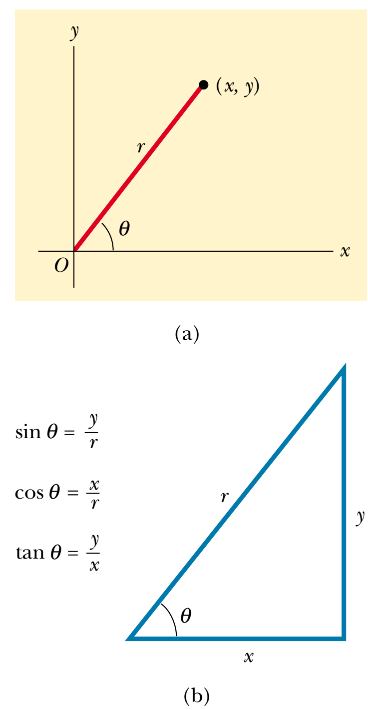


Figure 3.2 (a) The plane polar coordinates of a point are represented by the distance r and the angle θ, where θ is measured counterclockwise from the positive x axis. (b) The right triangle used to relate (x, y) to (r, θ).

Quick Quiz 3.1

Would the honeybee at the beginning of the chapter use cartesian or polar coordinates when specifying the location of the flower? Why? What is the honeybee using as an origin of coordinates?

You may want to read *Talking Apes and Dancing Bees* (1997) by Betsy Wyckoff.

EXAMPLE 3.1 Polar Coordinates

The cartesian coordinates of a point in the xy plane are $(x, y) = (-3.50, -2.50)$ m, as shown in Figure 3.3. Find the polar coordinates of this point.

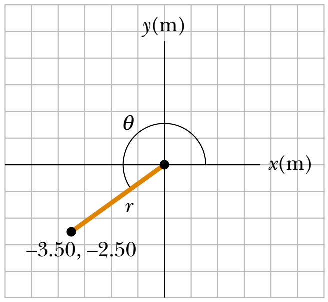


Figure 3.3 Finding polar coordinates when cartesian coordinates are given.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} = \boxed{4.30\text{ m}}$$

$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = \boxed{216°}$$

Note that you must use the signs of x and y to find that the point lies in the third quadrant of the coordinate system. That is, $\theta = 216°$ and not $35.5°$.

3.2 VECTOR AND SCALAR QUANTITIES

2.3

As noted in Chapter 2, some physical quantities are scalar quantities whereas others are vector quantities. When you want to know the temperature outside so that you will know how to dress, the only information you need is a number and the unit "degrees C" or "degrees F." Temperature is therefore an example of a **scalar quantity,** which is defined as a quantity that is completely specified by a number and appropriate units. That is,

> A scalar quantity is specified by a single value with an appropriate unit and has no direction.

Other examples of scalar quantities are volume, mass, and time intervals. The rules of ordinary arithmetic are used to manipulate scalar quantities.

If you are getting ready to pilot a small plane and need to know the wind velocity, you must know both the speed of the wind and its direction. Because direction is part of the information it gives, velocity is a **vector quantity,** which is defined as a physical quantity that is completely specified by a number and appropriate units plus a direction. That is,

> A vector quantity has both magnitude and direction.

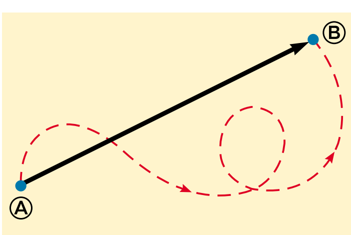


Figure 3.4 As a particle moves from Ⓐ to Ⓑ along an arbitrary path represented by the broken line, its displacement is a vector quantity shown by the arrow drawn from Ⓐ to Ⓑ.

Another example of a vector quantity is displacement, as you know from Chapter 2. Suppose a particle moves from some point Ⓐ to some point Ⓑ along a straight path, as shown in Figure 3.4. We represent this displacement by drawing an arrow from Ⓐ to Ⓑ, with the tip of the arrow pointing away from the starting point. The direction of the arrowhead represents the direction of the displacement, and the length of the arrow represents the magnitude of the displacement. If the particle travels along some other path from Ⓐ to Ⓑ, such as the broken line in Figure 3.4, its displacement is still the arrow drawn from Ⓐ to Ⓑ.

(a)

(b)

(c)

(a) The number of apples in the basket is one example of a scalar quantity. Can you think of other examples? *(Superstock)* (b) Jennifer pointing to the right. A vector quantity is one that must be specified by both magnitude and direction. *(Photo by Ray Serway)* (c) An anemometer is a device meteorologists use in weather forecasting. The cups spin around and reveal the magnitude of the wind velocity. The pointer indicates the direction. *(Courtesy of Peet Bros.Company, 1308 Doris Avenue, Ocean, NJ 07712)*

In this text, we use a boldface letter, such as **A**, to represent a vector quantity. Another common method for vector notation that you should be aware of is the use of an arrow over a letter, such as $\vec{A}$. The magnitude of the vector **A** is written either A or $|\mathbf{A}|$. The magnitude of a vector has physical units, such as meters for displacement or meters per second for velocity.

3.3 SOME PROPERTIES OF VECTORS

Equality of Two Vectors

For many purposes, two vectors **A** and **B** may be defined to be equal if they have the same magnitude and point in the same direction. That is, $\mathbf{A} = \mathbf{B}$ only if $A = B$ *and* if **A** and **B** point in the same direction along parallel lines. For example, all the vectors in Figure 3.5 are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

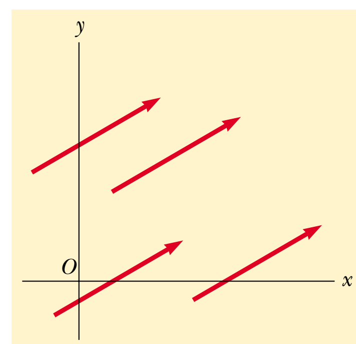


Figure 3.5 These four vectors are equal because they have equal lengths and point in the same direction.

Adding Vectors

2.4 The rules for adding vectors are conveniently described by geometric methods. To add vector **B** to vector **A**, first draw vector **A**, with its magnitude represented by a convenient scale, on graph paper and then draw vector **B** to the same scale with its tail starting from the tip of **A**, as shown in Figure 3.6. The **resultant vector R** = **A** + **B** is the vector drawn from the tail of **A** to the tip of **B**. This procedure is known as the **triangle method of addition.**

For example, if you walked 3.0 m toward the east and then 4.0 m toward the north, as shown in Figure 3.7, you would find yourself 5.0 m from where you

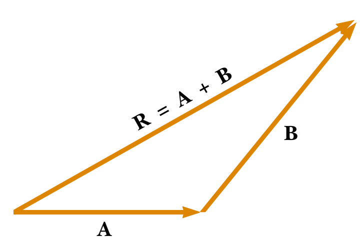


Figure 3.6 When vector **B** is added to vector **A**, the resultant **R** is the vector that runs from the tail of **A** to the tip of **B**.

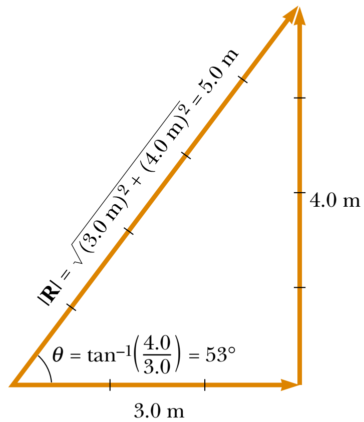


Figure 3.7 Vector addition. Walking first 3.0 m due east and then 4.0 m due north leaves you |**R**| = 5.0 m from your starting point.

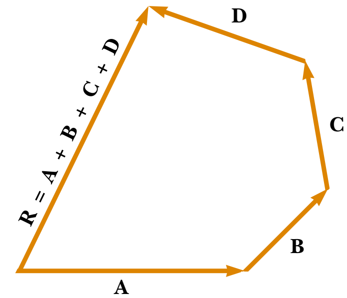


Figure 3.8 Geometric construction for summing four vectors. The resultant vector **R** is by definition the one that completes the polygon.

started, measured at an angle of 53° north of east. Your total displacement is the vector sum of the individual displacements.

A geometric construction can also be used to add more than two vectors. This is shown in Figure 3.8 for the case of four vectors. The resultant vector **R** = **A** + **B** + **C** + **D** is the vector that completes the polygon. In other words, **R is the vector drawn from the tail of the first vector to the tip of the last vector.**

An alternative graphical procedure for adding two vectors, known as the **parallelogram rule of addition,** is shown in Figure 3.9a. In this construction, the tails of the two vectors **A** and **B** are joined together and the resultant vector **R** is the diagonal of a parallelogram formed with **A** and **B** as two of its four sides.

When two vectors are added, the sum is independent of the order of the addition. (This fact may seem trivial, but as you will see in Chapter 11, the order is important when vectors are multiplied). This can be seen from the geometric construction in Figure 3.9b and is known as the **commutative law of addition:**

Commutative law

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad (3.5)$$

When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together. A geometric proof of this rule

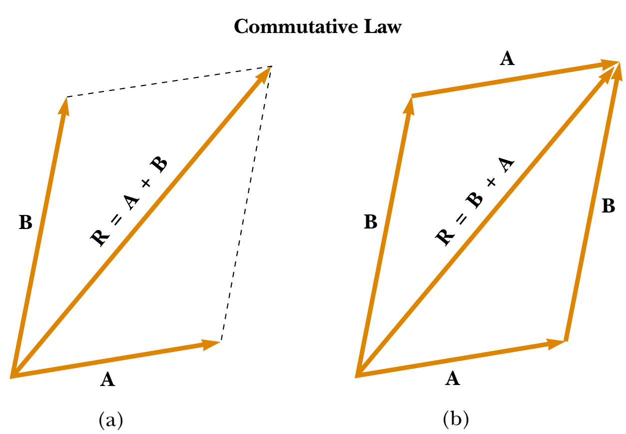


Figure 3.9 (a) In this construction, the resultant **R** is the diagonal of a parallelogram having sides **A** and **B**. (b) This construction shows that **A** + **B** = **B** + **A**—in other words, that vector addition is commutative.

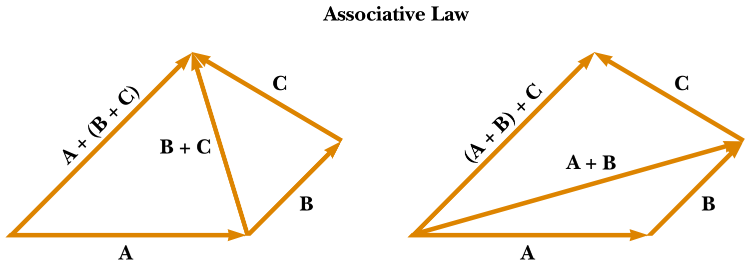


Figure 3.10 Geometric constructions for verifying the associative law of addition.

for three vectors is given in Figure 3.10. This is called the **associative law of addition:**

Associative law

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{3.6}$$

In summary, **a vector quantity has both magnitude and direction and also obeys the laws of vector addition** as described in Figures 3.6 to 3.10. When two or more vectors are added together, *all* of them *must* have the same units. It would be meaningless to add a velocity vector (for example, 60 km/h to the east) to a displacement vector (for example, 200 km to the north) because they represent different physical quantities. The same rule also applies to scalars. For example, it would be meaningless to add time intervals to temperatures.

Negative of a Vector

The negative of the vector **A** is defined as the vector that when added to **A** gives zero for the vector sum. That is, $\mathbf{A} + (-\mathbf{A}) = 0$. The vectors **A** and $-\mathbf{A}$ have the same magnitude but point in opposite directions.

Subtracting Vectors

The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation $\mathbf{A} - \mathbf{B}$ as vector $-\mathbf{B}$ added to vector **A**:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{3.7}$$

The geometric construction for subtracting two vectors in this way is illustrated in Figure 3.11a.

Another way of looking at vector subtraction is to note that the difference $\mathbf{A} - \mathbf{B}$ between two vectors **A** and **B** is what you have to add to the second vector to obtain the first. In this case, the vector $\mathbf{A} - \mathbf{B}$ points from the tip of the second vector to the tip of the first, as Figure 3.11b shows.

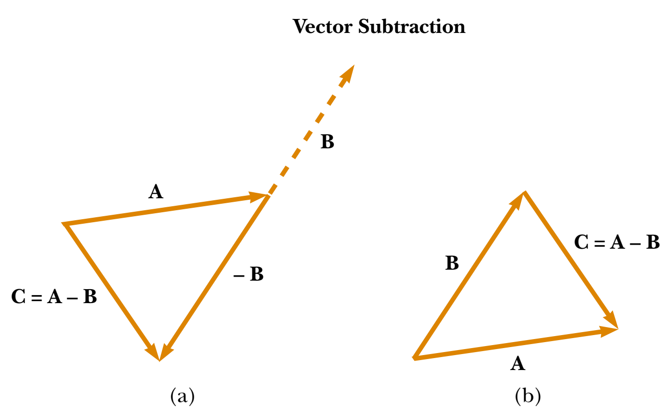


Figure 3.11 (a) This construction shows how to subtract vector **B** from vector **A**. The vector $-\mathbf{B}$ is equal in magnitude to vector **B** and points in the opposite direction. To subtract **B** from **A**, apply the rule of vector addition to the combination of **A** and $-\mathbf{B}$: Draw **A** along some convenient axis, place the tail of $-\mathbf{B}$ at the tip of **A**, and **C** is the difference $\mathbf{A} - \mathbf{B}$. (b) A second way of looking at vector subtraction. The difference vector $\mathbf{C} = \mathbf{A} - \mathbf{B}$ is the vector that we must add to **B** to obtain **A**.

EXAMPLE 3.2 A Vacation Trip

A car travels 20.0 km due north and then 35.0 km in a direction 60.0° west of north, as shown in Figure 3.12. Find the magnitude and direction of the car's resultant displacement.

Solution In this example, we show two ways to find the resultant of two vectors. We can solve the problem geometrically, using graph paper and a protractor, as shown in Figure 3.12. (In fact, even when you know you are going to be carrying out a calculation, you should sketch the vectors to check your results.) The displacement **R** is the resultant when the two individual displacements **A** and **B** are added.

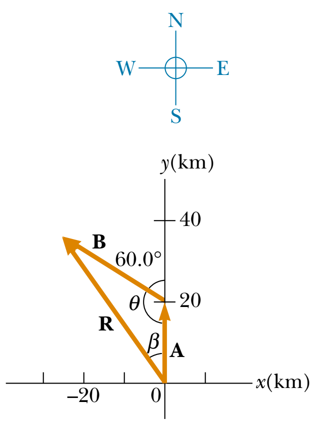


Figure 3.12 Graphical method for finding the resultant displacement vector **R** = **A** + **B**.

To solve the problem algebraically, we note that the magnitude of **R** can be obtained from the law of cosines as applied to the triangle (see Appendix B.4). With $\theta = 180° - 60° = 120°$ and $R^2 = A^2 + B^2 - 2AB\cos\theta$, we find that

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$$= \sqrt{(20.0\text{ km})^2 + (35.0\text{ km})^2 - 2(20.0\text{ km})(35.0\text{ km})\cos 120°}$$

$$= \boxed{48.2\text{ km}}$$

The direction of **R** measured from the northerly direction can be obtained from the law of sines (Appendix B.4):

$$\frac{\sin\beta}{B} = \frac{\sin\theta}{R}$$

$$\sin\beta = \frac{B}{R}\sin\theta = \frac{35.0\text{ km}}{48.2\text{ km}}\sin 120° = 0.629$$

$$\beta = \boxed{38.9°}$$

The resultant displacement of the car is 48.2 km in a direction 38.9° west of north. This result matches what we found graphically.

Multiplying a Vector by a Scalar

If vector **A** is multiplied by a positive scalar quantity m, then the product m**A** is a vector that has the same direction as **A** and magnitude mA. If vector **A** is multiplied by a negative scalar quantity $-m$, then the product $-m$**A** is directed opposite **A**. For example, the vector 5**A** is five times as long as **A** and points in the same direction as **A**; the vector $-\frac{1}{3}$**A** is one-third the length of **A** and points in the direction opposite **A**.

Quick Quiz 3.2

If vector **B** is added to vector **A**, under what condition does the resultant vector **A** + **B** have magnitude $A + B$? Under what conditions is the resultant vector equal to zero?

3.4 COMPONENTS OF A VECTOR AND UNIT VECTORS

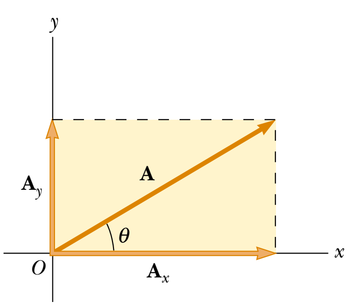


Figure 3.13 Any vector **A** lying in the xy plane can be represented by a vector $\mathbf{A}_x$ lying along the x axis and by a vector $\mathbf{A}_y$ lying along the y axis, where $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$.

2.5

The geometric method of adding vectors is not recommended whenever great accuracy is required or in three-dimensional problems. In this section, we describe a method of adding vectors that makes use of the *projections* of vectors along coordinate axes. These projections are called the **components** of the vector. Any vector can be completely described by its components.

Consider a vector **A** lying in the xy plane and making an arbitrary angle θ with the positive x axis, as shown in Figure 3.13. This vector can be expressed as the sum of two other vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. From Figure 3.13, we see that the three vectors form a right triangle and that $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$. (If you cannot see why this equality holds, go back to Figure 3.9 and review the parallelogram rule.) We shall often refer to the "components of a vector **A**," written A_x and A_y (without the boldface notation). The component A_x represents the projection of **A** along the x axis, and the component A_y represents the projection of **A** along the y axis. These components can be positive or negative. The component A_x is positive if $\mathbf{A}_x$ points in the positive x direction and is negative if $\mathbf{A}_x$ points in the negative x direction. The same is true for the component A_y.

From Figure 3.13 and the definition of sine and cosine, we see that $\cos\theta = A_x/A$ and that $\sin\theta = A_y/A$. Hence, the components of **A** are

Components of the vector **A**

$$A_x = A\cos\theta \tag{3.8}$$

$$A_y = A\sin\theta \tag{3.9}$$

These components form two sides of a right triangle with a hypotenuse of length A. Thus, it follows that the magnitude and direction of **A** are related to its components through the expressions

Magnitude of **A**

$$A = \sqrt{A_x^2 + A_y^2} \tag{3.10}$$

Direction of **A**

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{3.11}$$

Note that **the signs of the components A_x and A_y depend on the angle θ.** For example, if $\theta = 120°$, then A_x is negative and A_y is positive. If $\theta = 225°$, then both A_x and A_y are negative. Figure 3.14 summarizes the signs of the components when **A** lies in the various quadrants.

When solving problems, you can specify a vector **A** either with its components A_x and A_y or with its magnitude and direction A and θ.

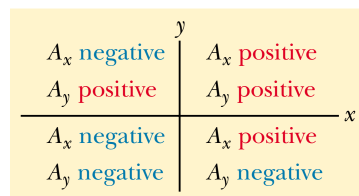


Figure 3.14 The signs of the components of a vector **A** depend on the quadrant in which the vector is located.

Quick Quiz 3.3

Can the component of a vector ever be greater than the magnitude of the vector?

Suppose you are working a physics problem that requires resolving a vector into its components. In many applications it is convenient to express the components in a coordinate system having axes that are not horizontal and vertical but are still perpendicular to each other. If you choose reference axes or an angle other than the axes and angle shown in Figure 3.13, the components must be modified accordingly. Suppose a vector **B** makes an angle θ' with the x' axis defined in Figure 3.15. The components of **B** along the x' and y' axes are $B_{x'} = B\cos\theta'$ and $B_{y'} = B\sin\theta'$, as specified by Equations 3.8 and 3.9. The magnitude and direction of **B** are obtained from expressions equivalent to Equations 3.10 and 3.11. Thus, we can express the components of a vector in *any* coordinate system that is convenient for a particular situation.

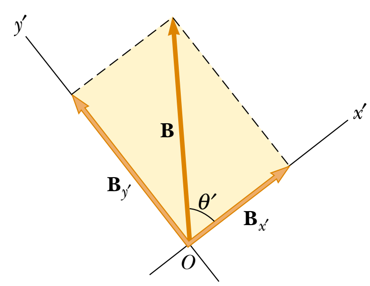


Figure 3.15 The component vectors of **B** in a coordinate system that is tilted.

Unit Vectors

Vector quantities often are expressed in terms of unit vectors. **A unit vector is a dimensionless vector having a magnitude of exactly 1.** Unit vectors are used to specify a given direction and have no other physical significance. They are used solely as a convenience in describing a direction in space. We shall use the symbols

i, **j**, and **k** to represent unit vectors pointing in the positive x, y, and z directions, respectively. The unit vectors **i**, **j**, and **k** form a set of mutually perpendicular vectors in a right-handed coordinate system, as shown in Figure 3.16a. The magnitude of each unit vector equals 1; that is, $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

Consider a vector **A** lying in the xy plane, as shown in Figure 3.16b. The product of the component A_x and the unit vector **i** is the vector $A_x\mathbf{i}$, which lies on the x axis and has magnitude $|A_x|$. (The vector $A_x\mathbf{i}$ is an alternative representation of vector $\mathbf{A}_x$.) Likewise, $A_y\mathbf{j}$ is a vector of magnitude $|A_y|$ lying on the y axis. (Again, vector $A_y\mathbf{j}$ is an alternative representation of vector $\mathbf{A}_y$.) Thus, the unit–vector notation for the vector **A** is

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} \tag{3.12}$$

For example, consider a point lying in the xy plane and having cartesian coordinates (x, y), as in Figure 3.17. The point can be specified by the **position vector r,** which in unit–vector form is given by

Position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{3.13}$$

This notation tells us that the components of **r** are the lengths x and y.

Now let us see how to use components to add vectors when the geometric method is not sufficiently accurate. Suppose we wish to add vector **B** to vector **A**, where vector **B** has components B_x and B_y. All we do is add the x and y components separately. The resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$ is therefore

$$\mathbf{R} = (A_x\mathbf{i} + A_y\mathbf{j}) + (B_x\mathbf{i} + B_y\mathbf{j})$$

or

$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} \tag{3.14}$$

Because $\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$, we see that the components of the resultant vector are

$$\begin{aligned} R_x &= A_x + B_x \\ R_y &= A_y + B_y \end{aligned} \tag{3.15}$$

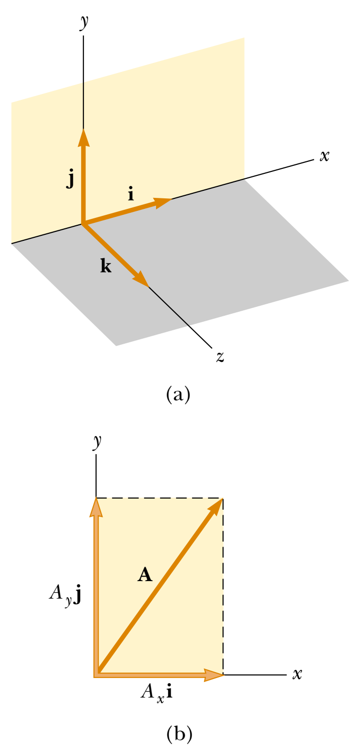


Figure 3.16 (a) The unit vectors **i**, **j**, and **k** are directed along the x, y, and z axes, respectively. (b) Vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$ lying in the xy plane has components A_x and A_y.

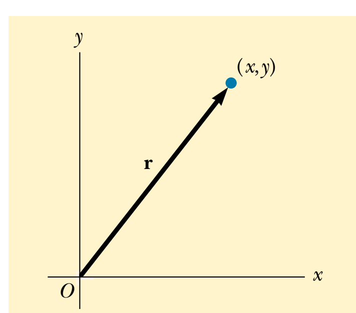


Figure 3.17 The point whose cartesian coordinates are (x, y) can be represented by the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

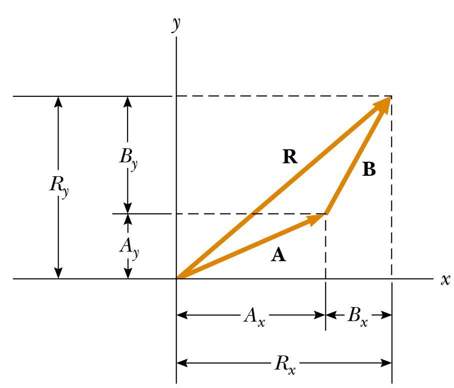


Figure 3.18 This geometric construction for the sum of two vectors shows the relationship between the components of the resultant **R** and the components of the individual vectors.

We obtain the magnitude of **R** and the angle it makes with the *x* axis from its components, using the relationships

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \quad \textbf{(3.16)}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x} \quad \textbf{(3.17)}$$

We can check this addition by components with a geometric construction, as shown in Figure 3.18. Remember that you must note the *signs* of the components when using either the algebraic or the geometric method.

At times, we need to consider situations involving motion in three component directions. The extension of our methods to three-dimensional vectors is straightforward. If **A** and **B** both have *x*, *y*, and *z* components, we express them in the form

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \quad \textbf{(3.18)}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k} \quad \textbf{(3.19)}$$

The sum of **A** and **B** is

$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \quad \textbf{(3.20)}$$

Note that Equation 3.20 differs from Equation 3.14: in Equation 3.20, the resultant vector also has a *z* component $R_z = A_z + B_z$.

QuickLab

Write an expression for the vector describing the displacement of a fly that moves from one corner of the floor of the room that you are in to the opposite corner of the room, near the ceiling.

Quick Quiz 3.4

If one component of a vector is not zero, can the magnitude of the vector be zero? Explain.

Quick Quiz 3.5

If $\mathbf{A} + \mathbf{B} = 0$, what can you say about the components of the two vectors?

Problem-Solving Hints

Adding Vectors

When you need to add two or more vectors, use this step-by-step procedure:

- Select a coordinate system that is convenient. (Try to reduce the number of components you need to find by choosing axes that line up with as many vectors as possible.)
- Draw a labeled sketch of the vectors described in the problem.
- Find the *x* and *y* components of all vectors and the resultant components (the algebraic sum of the components) in the *x* and *y* directions.
- If necessary, use the Pythagorean theorem to find the magnitude of the resultant vector and select a suitable trigonometric function to find the angle that the resultant vector makes with the *x* axis.

EXAMPLE 3.3 The Sum of Two Vectors

Find the sum of two vectors **A** and **B** lying in the *xy* plane and given by

$$\mathbf{A} = (2.0\mathbf{i} + 2.0\mathbf{j})\text{ m} \quad \text{and} \quad \mathbf{B} = (2.0\mathbf{i} - 4.0\mathbf{j})\text{ m}$$

Solution Comparing this expression for **A** with the general expression $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$, we see that $A_x = 2.0$ m and that $A_y = 2.0$ m. Likewise, $B_x = 2.0$ m and $B_y = -4.0$ m. We obtain the resultant vector **R**, using Equation 3.14:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (2.0 + 2.0)\mathbf{i}\text{ m} + (2.0 - 4.0)\mathbf{j}\text{ m}$$
$$= (4.0\mathbf{i} - 2.0\mathbf{j})\text{ m}$$

or

$$R_x = 4.0\text{ m} \qquad R_y = -2.0\text{ m}$$

The magnitude of **R** is given by Equation 3.16:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0\text{ m})^2 + (-2.0\text{ m})^2} = \sqrt{20}\text{ m}$$
$$= 4.5\text{ m}$$

We can find the direction of **R** from Equation 3.17:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0\text{ m}}{4.0\text{ m}} = -0.50$$

Your calculator likely gives the answer $-27°$ for $\theta = \tan^{-1}(-0.50)$. This answer is correct if we interpret it to mean 27° clockwise from the *x* axis. Our standard form has been to quote the angles measured counterclockwise from the $+x$ axis, and that angle for this vector is $\theta = 333°$.

EXAMPLE 3.4 The Resultant Displacement

A particle undergoes three consecutive displacements: $\mathbf{d}_1 = (15\mathbf{i} + 30\mathbf{j} + 12\mathbf{k})$ cm, $\mathbf{d}_2 = (23\mathbf{i} - 14\mathbf{j} - 5.0\mathbf{k})$ cm, and $\mathbf{d}_3 = (-13\mathbf{i} + 15\mathbf{j})$ cm. Find the components of the resultant displacement and its magnitude.

Solution Rather than looking at a sketch on flat paper, visualize the problem as follows: Start with your fingertip at the front left corner of your horizontal desktop. Move your fingertip 15 cm to the right, then 30 cm toward the far side of the desk, then 12 cm vertically upward, then 23 cm to the right, then 14 cm horizontally toward the front edge of the desk, then 5.0 cm vertically toward the desk, then 13 cm to the left, and (finally!) 15 cm toward the back of the desk. The mathematical calculation keeps track of this motion along the three perpendicular axes:

$$\mathbf{R} = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$$
$$= (15 + 23 - 13)\mathbf{i}\text{ cm} + (30 - 14 + 15)\mathbf{j}\text{ cm}$$
$$+ (12 - 5.0 + 0)\mathbf{k}\text{ cm}$$
$$= (25\mathbf{i} + 31\mathbf{j} + 7.0\mathbf{k})\text{ cm}$$

The resultant displacement has components $R_x = 25$ cm, $R_y = 31$ cm, and $R_z = 7.0$ cm. Its magnitude is

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$
$$= \sqrt{(25\text{ cm})^2 + (31\text{ cm})^2 + (7.0\text{ cm})^2} = 40\text{ cm}$$

EXAMPLE 3.5 Taking a Hike

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower. (a) Determine the components of the hiker's displacement for each day.

Solution If we denote the displacement vectors on the first and second days by **A** and **B**, respectively, and use the car as the origin of coordinates, we obtain the vectors shown in Figure 3.19. Displacement **A** has a magnitude of 25.0 km and is directed 45.0° below the positive *x* axis. From Equations 3.8 and 3.9, its components are

$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = 17.7\text{ km}$$

$$A_y = A\sin(-45.0°) = -(25.0\text{ km})(0.707) = -17.7\text{ km}$$

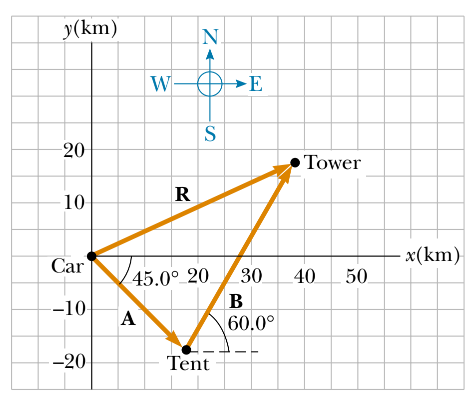


Figure 3.19 The total displacement of the hiker is the vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$.

The negative value of A_y indicates that the hiker walks in the negative y direction on the first day. The signs of A_x and A_y also are evident from Figure 3.19.

The second displacement **B** has a magnitude of 40.0 km and is 60.0° north of east. Its components are

$$B_x = B\cos 60.0° = (40.0 \text{ km})(0.500) = \boxed{20.0 \text{ km}}$$

$$B_y = B\sin 60.0° = (40.0 \text{ km})(0.866) = \boxed{34.6 \text{ km}}$$

(b) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.

Solution The resultant displacement for the trip **R** = **A** + **B** has components given by Equation 3.15:

$$R_x = A_x + B_x = 17.7 \text{ km} + 20.0 \text{ km} = \boxed{37.7 \text{ km}}$$

$$R_y = A_y + B_y = -17.7 \text{ km} + 34.6 \text{ km} = \boxed{16.9 \text{ km}}$$

In unit–vector form, we can write the total displacement as

$$\boxed{\mathbf{R} = (37.7\mathbf{i} + 16.9\mathbf{j}) \text{ km}}$$

Exercise Determine the magnitude and direction of the total displacement.

Answer 41.3 km, 24.1° north of east from the car.

EXAMPLE 3.6 Let's Fly Away!

A commuter airplane takes the route shown in Figure 3.20. First, it flies from the origin of the coordinate system shown to city A, located 175 km in a direction 30.0° north of east. Next, it flies 153 km 20.0° west of north to city B. Finally, it flies 195 km due west to city C. Find the location of city C relative to the origin.

Solution It is convenient to choose the coordinate system shown in Figure 3.20, where the x axis points to the east and the y axis points to the north. Let us denote the three consecutive displacements by the vectors **a**, **b**, and **c**. Displacement **a** has a magnitude of 175 km and the components

$$a_x = a\cos(30.0°) = (175 \text{ km})(0.866) = 152 \text{ km}$$

$$a_y = a\sin(30.0°) = (175 \text{ km})(0.500) = 87.5 \text{ km}$$

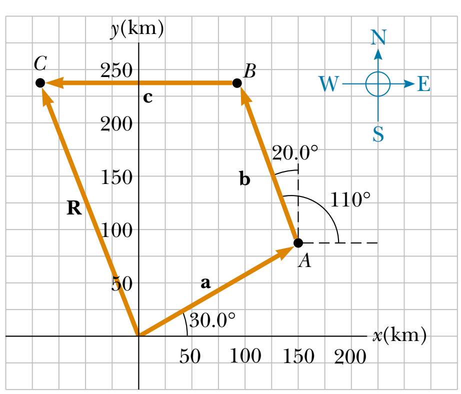


Figure 3.20 The airplane starts at the origin, flies first to city A, then to city B, and finally to city C.

Displacement **b**, whose magnitude is 153 km, has the components

$$b_x = b\cos(110°) = (153 \text{ km})(-0.342) = -52.3 \text{ km}$$

$$b_y = b\sin(110°) = (153 \text{ km})(0.940) = 144 \text{ km}$$

Finally, displacement **c**, whose magnitude is 195 km, has the components

$$c_x = c\cos(180°) = (195 \text{ km})(-1) = -195 \text{ km}$$

$$c_y = c\sin(180°) = 0$$

Therefore, the components of the position vector **R** from the starting point to city C are

$$R_x = a_x + b_x + c_x = 152 \text{ km} - 52.3 \text{ km} - 195 \text{ km} = \boxed{-95.3 \text{ km}}$$

$$R_y = a_y + b_y + c_y = 87.5 \text{ km} + 144 \text{ km} + 0 = \boxed{232 \text{ km}}$$

In unit–vector notation, $\boxed{\mathbf{R} = (-95.3\mathbf{i} + 232\mathbf{j}) \text{ km.}}$ That is, the airplane can reach city C from the starting point by first traveling 95.3 km due west and then by traveling 232 km due north.

Exercise Find the magnitude and direction of **R**.

Answer 251 km, 22.3° west of north.

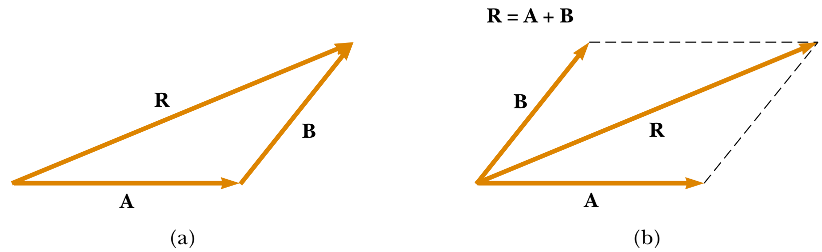


Figure 3.21 (a) Vector addition by the triangle method. (b) Vector addition by the parallelogram rule.

SUMMARY

Scalar quantities are those that have only magnitude and no associated direction. **Vector quantities** have both magnitude and direction and obey the laws of vector addition.

We can add two vectors **A** and **B** graphically, using either the triangle method or the parallelogram rule. In the triangle method (Fig. 3.21a), the resultant vector **R** = **A** + **B** runs from the tail of **A** to the tip of **B**. In the parallelogram method (Fig. 3.21b), **R** is the diagonal of a parallelogram having **A** and **B** as two of its sides. You should be able to add or subtract vectors, using these graphical methods.

The x component A_x of the vector **A** is equal to the projection of **A** along the x axis of a coordinate system, as shown in Figure 3.22, where $A_x = A \cos \theta$. The y component A_y of **A** is the projection of **A** along the y axis, where $A_y = A \sin \theta$. Be sure you can determine which trigonometric functions you should use in all situations, especially when θ is defined as something other than the counterclockwise angle from the positive x axis.

If a vector **A** has an x component A_x and a y component A_y, the vector can be expressed in unit–vector form as $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$. In this notation, **i** is a unit vector pointing in the positive x direction, and **j** is a unit vector pointing in the positive y direction. Because **i** and **j** are unit vectors, $|\mathbf{i}| = |\mathbf{j}| = 1$.

We can find the resultant of two or more vectors by resolving all vectors into their x and y components, adding their resultant x and y components, and then using the Pythagorean theorem to find the magnitude of the resultant vector. We can find the angle that the resultant vector makes with respect to the x axis by using a suitable trigonometric function.

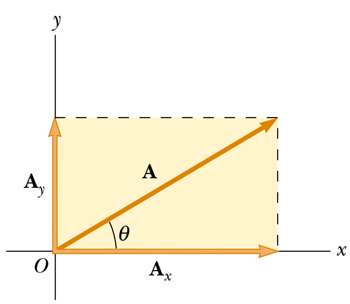


Figure 3.22 The addition of the two vectors $\mathbf{A}_x$ and $\mathbf{A}_y$ gives vector **A**. Note that $\mathbf{A}_x = A_x\mathbf{i}$ and $\mathbf{A}_y = A_y\mathbf{j}$, where A_x and A_y are the *components* of vector **A**.

QUESTIONS

1. Two vectors have unequal magnitudes. Can their sum be zero? Explain.

2. Can the magnitude of a particle's displacement be greater than the distance traveled? Explain.

3. The magnitudes of two vectors **A** and **B** are $A = 5$ units and $B = 2$ units. Find the largest and smallest values possible for the resultant vector **R** = **A** + **B**.

4. Vector **A** lies in the xy plane. For what orientations of vector **A** will both of its components be negative? For what orientations will its components have opposite signs?

5. If the component of vector **A** along the direction of vector **B** is zero, what can you conclude about these two vectors?

6. Can the magnitude of a vector have a negative value? Explain.

7. Which of the following are vectors and which are not: force, temperature, volume, ratings of a television show, height, velocity, age?

8. Under what circumstances would a nonzero vector lying in the xy plane ever have components that are equal in magnitude?

9. Is it possible to add a vector quantity to a scalar quantity? Explain.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 3.1 Coordinate Systems

WEB **1.** The polar coordinates of a point are $r = 5.50$ m and $\theta = 240°$. What are the cartesian coordinates of this point?

2. Two points in the *xy* plane have cartesian coordinates (2.00, − 4.00) m and (− 3.00, 3.00) m. Determine (a) the distance between these points and (b) their polar coordinates.

3. If the cartesian coordinates of a point are given by (2, *y*) and its polar coordinates are (*r*, 30°), determine *y* and *r*.

4. Two points in a plane have polar coordinates (2.50 m, 30.0°) and (3.80 m, 120.0°). Determine (a) the cartesian coordinates of these points and (b) the distance between them.

5. A fly lands on one wall of a room. The lower left-hand corner of the wall is selected as the origin of a two-dimensional cartesian coordinate system. If the fly is located at the point having coordinates (2.00, 1.00) m, (a) how far is it from the corner of the room? (b) what is its location in polar coordinates?

6. If the polar coordinates of the point (*x*, *y*) are (*r*, θ), determine the polar coordinates for the points (a) (− *x*, *y*), (b) (− 2*x*, − 2*y*), and (c) (3*x*, − 3*y*).

Section 3.2 Vector and Scalar Quantities

Section 3.3 Some Properties of Vectors

7. An airplane flies 200 km due west from city A to city B and then 300 km in the direction 30.0° north of west from city B to city C. (a) In straight-line distance, how far is city C from city A? (b) Relative to city A, in what direction is city C?

8. A pedestrian moves 6.00 km east and then 13.0 km north. Using the graphical method, find the magnitude and direction of the resultant displacement vector.

9. A surveyor measures the distance across a straight river by the following method: Starting directly across from a tree on the opposite bank, she walks 100 m along the riverbank to establish a baseline. Then she sights across to the tree. The angle from her baseline to the tree is 35.0°. How wide is the river?

10. A plane flies from base camp to lake A, a distance of 280 km at a direction 20.0° north of east. After dropping off supplies, it flies to lake B, which is 190 km and 30.0° west of north from lake A. Graphically determine the distance and direction from lake B to the base camp.

11. Vector **A** has a magnitude of 8.00 units and makes an angle of 45.0° with the positive *x* axis. Vector **B** also has a magnitude of 8.00 units and is directed along the negative *x* axis. Using graphical methods, find (a) the vector sum **A** + **B** and (b) the vector difference **A** − **B**.

12. A force $\mathbf{F}_1$ of magnitude 6.00 units acts at the origin in a direction 30.0° above the positive *x* axis. A second force $\mathbf{F}_2$ of magnitude 5.00 units acts at the origin in the direction of the positive *y* axis. Find graphically the magnitude and direction of the resultant force $\mathbf{F}_1 + \mathbf{F}_2$.

WEB **13.** A person walks along a circular path of radius 5.00 m. If the person walks around one half of the circle, find (a) the magnitude of the displacement vector and (b) how far the person walked. (c) What is the magnitude of the displacement if the person walks all the way around the circle?

14. A dog searching for a bone walks 3.50 m south, then 8.20 m at an angle 30.0° north of east, and finally 15.0 m west. Using graphical techniques, find the dog's resultant displacement vector.

WEB **15.** Each of the displacement vectors **A** and **B** shown in Figure P3.15 has a magnitude of 3.00 m. Find graphically (a) **A** + **B**, (b) **A** − **B**, (c) **B** − **A**, (d) **A** − 2**B**. Report all angles counterclockwise from the positive *x* axis.

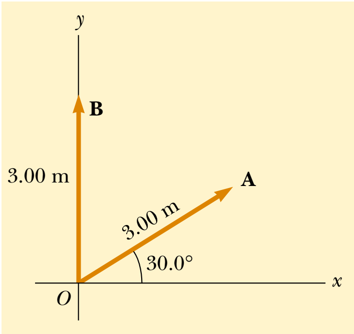


Figure P3.15 Problems 15 and 39.

16. Arbitrarily define the "instantaneous vector height" of a person as the displacement vector from the point halfway between the feet to the top of the head. Make an order-of-magnitude estimate of the total vector height of all the people in a city of population 100 000 (a) at 10 a.m. on a Tuesday and (b) at 5 a.m. on a Saturday. Explain your reasoning.

17. A roller coaster moves 200 ft horizontally and then rises 135 ft at an angle of 30.0° above the horizontal. It then travels 135 ft at an angle of 40.0° downward. What is its displacement from its starting point? Use graphical techniques.

18. The driver of a car drives 3.00 km north, 2.00 km northeast (45.0° east of north), 4.00 km west, and then

3.00 km southeast (45.0° east of south). Where does he end up relative to his starting point? Work out your answer graphically. Check by using components. (The car is not near the North Pole or the South Pole.)

19. Fox Mulder is trapped in a maze. To find his way out, he walks 10.0 m, makes a 90.0° right turn, walks 5.00 m, makes another 90.0° right turn, and walks 7.00 m. What is his displacement from his initial position?

Section 3.4 Components of a Vector and Unit Vectors

20. Find the horizontal and vertical components of the 100-m displacement of a superhero who flies from the top of a tall building following the path shown in Figure P3.20.

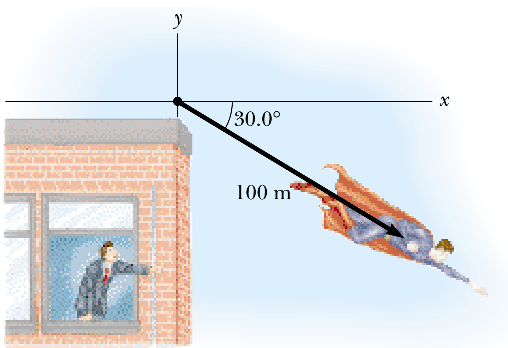


Figure P3.20

21. A person walks 25.0° north of east for 3.10 km. How far would she have to walk due north and due east to arrive at the same location?

22. While exploring a cave, a spelunker starts at the entrance and moves the following distances: She goes 75.0 m north, 250 m east, 125 m at an angle 30.0° north of east, and 150 m south. Find the resultant displacement from the cave entrance.

23. In the assembly operation illustrated in Figure P3.23, a robot first lifts an object upward along an arc that forms one quarter of a circle having a radius of 4.80 cm and lying in an east–west vertical plane. The robot then moves the object upward along a second arc that forms one quarter of a circle having a radius of 3.70 cm and lying in a north–south vertical plane. Find (a) the magnitude of the total displacement of the object and (b) the angle the total displacement makes with the vertical.

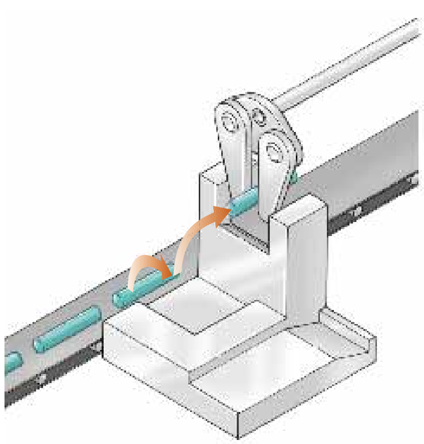


Figure P3.23

24. Vector **B** has *x*, *y*, and *z* components of 4.00, 6.00, and 3.00 units, respectively. Calculate the magnitude of **B** and the angles that **B** makes with the coordinate axes.

WEB **25.** A vector has an *x* component of -25.0 units and a *y* component of 40.0 units. Find the magnitude and direction of this vector.

26. A map suggests that Atlanta is 730 mi in a direction 5.00° north of east from Dallas. The same map shows that Chicago is 560 mi in a direction 21.0° west of north from Atlanta. Assuming that the Earth is flat, use this information to find the displacement from Dallas to Chicago.

27. A displacement vector lying in the *xy* plane has a magnitude of 50.0 m and is directed at an angle of 120° to the positive *x* axis. Find the *x* and *y* components of this vector and express the vector in unit–vector notation.

28. If $\mathbf{A} = 2.00\mathbf{i} + 6.00\mathbf{j}$ and $\mathbf{B} = 3.00\mathbf{i} - 2.00\mathbf{j}$, (a) sketch the vector sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and the vector difference $\mathbf{D} = \mathbf{A} - \mathbf{B}$. (b) Find solutions for **C** and **D**, first in terms of unit vectors and then in terms of polar coordinates, with angles measured with respect to the $+x$ axis.

29. Find the magnitude and direction of the resultant of three displacements having *x* and *y* components (3.00, 2.00) m, (-5.00, 3.00) m, and (6.00, 1.00) m.

30. Vector **A** has *x* and *y* components of -8.70 cm and 15.0 cm, respectively; vector **B** has *x* and *y* components of 13.2 cm and -6.60 cm, respectively. If $\mathbf{A} - \mathbf{B} + 3\mathbf{C} = 0$, what are the components of **C**?

31. Consider two vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} - 4\mathbf{j}$. Calculate (a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$, (c) $|\mathbf{A} + \mathbf{B}|$, (d) $|\mathbf{A} - \mathbf{B}|$, (e) the directions of $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$.

32. A boy runs 3.00 blocks north, 4.00 blocks northeast, and 5.00 blocks west. Determine the length and direction of the displacement vector that goes from the starting point to his final position.

33. Obtain expressions in component form for the position vectors having polar coordinates (a) 12.8 m, 150°; (b) 3.30 cm, 60.0°; (c) 22.0 in., 215°.

34. Consider the displacement vectors $\mathbf{A} = (3\mathbf{i} + 3\mathbf{j})$ m, $\mathbf{B} = (\mathbf{i} - 4\mathbf{j})$ m, and $\mathbf{C} = (-2\mathbf{i} + 5\mathbf{j})$ m. Use the component method to determine (a) the magnitude and direction of the vector $\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C}$ and (b) the magnitude and direction of $\mathbf{E} = -\mathbf{A} - \mathbf{B} + \mathbf{C}$.

35. A particle undergoes the following consecutive displacements: 3.50 m south, 8.20 m northeast, and 15.0 m west. What is the resultant displacement?

36. In a game of American football, a quarterback takes the ball from the line of scrimmage, runs backward for 10.0 yards, and then sideways parallel to the line of scrimmage for 15.0 yards. At this point, he throws a forward pass 50.0 yards straight downfield perpendicular to the line of scrimmage. What is the magnitude of the football's resultant displacement?

37. The helicopter view in Figure P3.37 shows two people pulling on a stubborn mule. Find (a) the single force that is equivalent to the two forces shown and (b) the force that a third person would have to exert on the mule to make the resultant force equal to zero. The forces are measured in units of newtons.

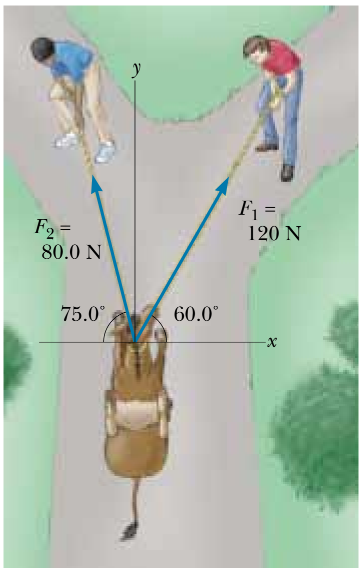


Figure P3.37

38. A novice golfer on the green takes three strokes to sink the ball. The successive displacements are 4.00 m to the north, 2.00 m northeast, and 1.00 m 30.0° west of south. Starting at the same initial point, an expert golfer could make the hole in what single displacement?

39. Find the x and y components of the vectors **A** and **B** shown in Figure P3.15; then derive an expression for the resultant vector **A** + **B** in unit–vector notation.

40. You are standing on the ground at the origin of a coordinate system. An airplane flies over you with constant velocity parallel to the x axis and at a constant height of 7.60×10^3 m. At $t = 0$, the airplane is directly above you, so that the vector from you to it is given by $\mathbf{P}_0 = (7.60 \times 10^3 \text{ m})\mathbf{j}$. At $t = 30.0$ s, the position vector leading from you to the airplane is $\mathbf{P}_{30} = (8.04 \times 10^3 \text{ m})\mathbf{i} + (7.60 \times 10^3 \text{ m})\mathbf{j}$. Determine the magnitude and orientation of the airplane's position vector at $t = 45.0$ s.

41. A particle undergoes two displacements. The first has a magnitude of 150 cm and makes an angle of 120° with the positive x axis. The *resultant* displacement has a magnitude of 140 cm and is directed at an angle of 35.0° to the positive x axis. Find the magnitude and direction of the second displacement.

42. Vectors **A** and **B** have equal magnitudes of 5.00. If the sum of **A** and **B** is the vector 6.00 **j**, determine the angle between **A** and **B**.

43. The vector **A** has x, y, and z components of 8.00, 12.0, and − 4.00 units, respectively. (a) Write a vector expression for **A** in unit–vector notation. (b) Obtain a unit–vector expression for a vector **B** one-fourth the length of **A** pointing in the same direction as **A**. (c) Obtain a unit–vector expression for a vector **C** three times the length of **A** pointing in the direction opposite the direction of **A**.

44. Instructions for finding a buried treasure include the following: Go 75.0 paces at 240°, turn to 135° and walk 125 paces, then travel 100 paces at 160°. The angles are measured counterclockwise from an axis pointing to the east, the + x direction. Determine the resultant displacement from the starting point.

45. Given the displacement vectors $\mathbf{A} = (3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})$ m and $\mathbf{B} = (2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})$ m, find the magnitudes of the vectors (a) $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and (b) $\mathbf{D} = 2\mathbf{A} - \mathbf{B}$, also expressing each in terms of its x, y, and z components.

46. A radar station locates a sinking ship at range 17.3 km and bearing 136° clockwise from north. From the same station a rescue plane is at horizontal range 19.6 km, 153° clockwise from north, with elevation 2.20 km. (a) Write the vector displacement from plane to ship, letting **i** represent east, **j** north, and **k** up. (b) How far apart are the plane and ship?

47. As it passes over Grand Bahama Island, the eye of a hurricane is moving in a direction 60.0° north of west with a speed of 41.0 km/h. Three hours later, the course of the hurricane suddenly shifts due north and its speed slows to 25.0 km/h. How far from Grand Bahama is the eye 4.50 h after it passes over the island?

48. (a) Vector **E** has magnitude 17.0 cm and is directed 27.0° counterclockwise from the + x axis. Express it in unit–vector notation. (b) Vector **F** has magnitude 17.0 cm and is directed 27.0° counterclockwise from the + y axis. Express it in unit–vector notation. (c) Vector **G** has magnitude 17.0 cm and is directed 27.0° clockwise from the + y axis. Express it in unit–vector notation.

49. Vector **A** has a negative x component 3.00 units in length and a positive y component 2.00 units in length. (a) Determine an expression for **A** in unit–vector notation. (b) Determine the magnitude and direction of **A**. (c) What vector **B**, when added to vector **A**, gives a resultant vector with no x component and a negative y component 4.00 units in length?

50. An airplane starting from airport A flies 300 km east, then 350 km at 30.0° west of north, and then 150 km north to arrive finally at airport B. (a) The next day, another plane flies directly from airport A to airport B in a straight line. In what direction should the pilot travel in this direct flight? (b) How far will the pilot travel in this direct flight? Assume there is no wind during these flights.

WEB **51.** Three vectors are oriented as shown in Figure P3.51, where $|\mathbf{A}| = 20.0$ units, $|\mathbf{B}| = 40.0$ units, and $|\mathbf{C}| = 30.0$ units. Find (a) the x and y components of the resultant vector (expressed in unit–vector notation) and (b) the magnitude and direction of the resultant vector.

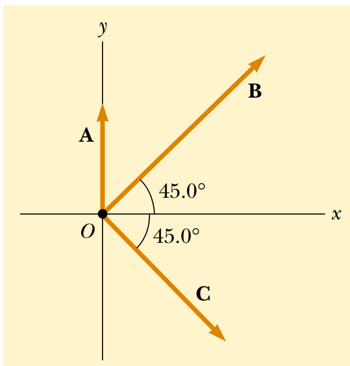


Figure P3.51

52. If $\mathbf{A} = (6.00\mathbf{i} - 8.00\mathbf{j})$ units, $\mathbf{B} = (-8.00\mathbf{i} + 3.00\mathbf{j})$ units, and $\mathbf{C} = (26.0\mathbf{i} + 19.0\mathbf{j})$ units, determine a and b such that $a\mathbf{A} + b\mathbf{B} + \mathbf{C} = 0$.

ADDITIONAL PROBLEMS

53. Two vectors **A** and **B** have precisely equal magnitudes. For the magnitude of $\mathbf{A} + \mathbf{B}$ to be 100 times greater than the magnitude of $\mathbf{A} - \mathbf{B}$, what must be the angle between them?

54. Two vectors **A** and **B** have precisely equal magnitudes. For the magnitude of $\mathbf{A} + \mathbf{B}$ to be greater than the magnitude of $\mathbf{A} - \mathbf{B}$ by the factor n, what must be the angle between them?

55. A vector is given by $\mathbf{R} = 2.00\mathbf{i} + 1.00\mathbf{j} + 3.00\mathbf{k}$. Find (a) the magnitudes of the x, y, and z components, (b) the magnitude of **R**, and (c) the angles between **R** and the x, y, and z axes.

56. Find the sum of these four vector forces: 12.0 N to the right at 35.0° above the horizontal, 31.0 N to the left at 55.0° above the horizontal, 8.40 N to the left at 35.0° below the horizontal, and 24.0 N to the right at 55.0° below the horizontal. (*Hint:* Make a drawing of this situation and select the best axes for x and y so that you have the least number of components. Then add the vectors, using the component method.)

57. A person going for a walk follows the path shown in Figure P3.57. The total trip consists of four straight-line paths. At the end of the walk, what is the person's resultant displacement measured from the starting point?

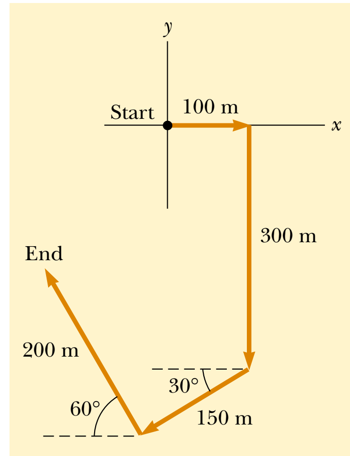


Figure P3.57

58. In general, the instantaneous position of an object is specified by its position vector **P** leading from a fixed origin to the location of the object. Suppose that for a certain object the position vector is a function of time, given by $\mathbf{P} = 4\mathbf{i} + 3\mathbf{j} - 2t\mathbf{j}$, where P is in meters and t is in seconds. Evaluate $d\mathbf{P}/dt$. What does this derivative represent about the object?

59. A jet airliner, moving initially at 300 mi/h to the east, suddenly enters a region where the wind is blowing at 100 mi/h in a direction 30.0° north of east. What are the new speed and direction of the aircraft relative to the ground?

60. A pirate has buried his treasure on an island with five trees located at the following points: A(30.0 m, −20.0 m), B(60.0 m, 80.0 m), C(−10.0 m, −10.0 m), D(40.0 m, −30.0 m), and E(−70.0 m, 60.0 m). All points are measured relative to some origin, as in Figure P3.60. Instructions on the map tell you to start at A and move toward B, but to cover only one-half the distance between A and B. Then, move toward C, covering one-third the distance between your current location and C. Next, move toward D, covering one-fourth the distance between where you are and D. Finally, move toward E, covering one-fifth the distance between you and E, stop, and dig. (a) What are the coordinates of the point where the pirate's treasure is buried? (b) Re-

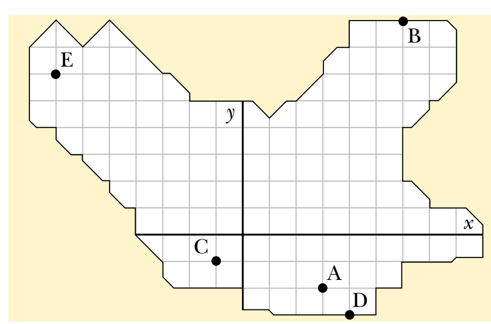


Figure P3.60

arrange the order of the trees, (for instance, B(30.0 m, − 20.0 m), A(60.0 m, 80.0 m), E(− 10.0 m, − 10.0 m), C(40.0 m, − 30.0 m), and D(− 70.0 m, 60.0 m), and repeat the calculation to show that the answer does not depend on the order of the trees.

61. A rectangular parallelepiped has dimensions a, b, and c, as in Figure P3.61. (a) Obtain a vector expression for the face diagonal vector $\mathbf{R}_1$. What is the magnitude of this vector? (b) Obtain a vector expression for the body diagonal vector $\mathbf{R}_2$. Note that $\mathbf{R}_1$, $c\mathbf{k}$, and $\mathbf{R}_2$ make a right triangle, and prove that the magnitude of $\mathbf{R}_2$ is $\sqrt{a^2 + b^2 + c^2}$.

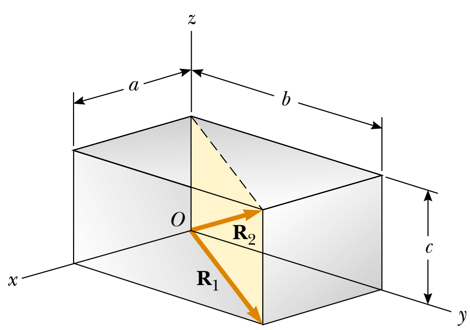


Figure P3.61

62. A point lying in the xy plane and having coordinates (x, y) can be described by the position vector given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. (a) Show that the displacement vector for a particle moving from (x_1, y_1) to (x_2, y_2) is given by $\mathbf{d} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$. (b) Plot the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and the displacement vector $\mathbf{d}$, and verify by the graphical method that $\mathbf{d} = \mathbf{r}_2 - \mathbf{r}_1$.

63. A point P is described by the coordinates (x, y) with respect to the normal cartesian coordinate system shown in Figure P3.63. Show that (x', y'), the coordinates of this point in the rotated coordinate system, are related to (x, y) and the rotation angle α by the expressions

$$x' = x \cos \alpha + y \sin \alpha$$

$$y' = -x \sin \alpha + y \cos \alpha$$

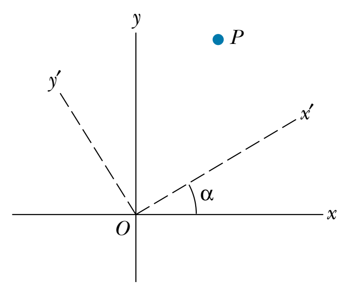


Figure P3.63

ANSWERS TO QUICK QUIZZES

3.1 The honeybee needs to communicate to the other honeybees how far it is to the flower and in what direction they must fly. This is exactly the kind of information that polar coordinates convey, as long as the origin of the coordinates is the beehive.

3.2 The resultant has magnitude $A + B$ when vector $\mathbf{A}$ is oriented in the same direction as vector $\mathbf{B}$. The resultant vector is $\mathbf{A} + \mathbf{B} = 0$ when vector $\mathbf{A}$ is oriented in the direction opposite vector $\mathbf{B}$ and $A = B$.

3.3 No. In two dimensions, a vector and its components form a right triangle. The vector is the hypotenuse and must be longer than either side. Problem 61 extends this concept to three dimensions.

3.4 No. The magnitude of a vector $\mathbf{A}$ is equal to $\sqrt{A_x^2 + A_y^2 + A_z^2}$. Therefore, if any component is nonzero, A cannot be zero. This generalization of the Pythagorean theorem is left for you to prove in Problem 61.

3.5 The fact that $\mathbf{A} + \mathbf{B} = 0$ tells you that $\mathbf{A} = -\mathbf{B}$. Therefore, the components of the two vectors must have opposite signs and equal magnitudes: $A_x = -B_x$, $A_y = -B_y$, and $A_z = -B_z$.

PUZZLER

This airplane is used by NASA for astronaut training. When it flies along a certain curved path, anything inside the plane that is not strapped down begins to float. What causes this strange effect? *(NASA)*

web

For more information on microgravity in general and on this airplane, visit **http://microgravity.msfc.nasa.gov/** and **http://www.jsc.nasa.gov/coop/kc135/kc135.html**

chapter

4 Motion in Two Dimensions

Chapter Outline

In this chapter we deal with the kinematics of a particle moving in two dimensions. Knowing the basics of two-dimensional motion will allow us to examine—in future chapters—a wide variety of motions, ranging from the motion of satellites in orbit to the motion of electrons in a uniform electric field. We begin by studying in greater detail the vector nature of displacement, velocity, and acceleration. As in the case of one-dimensional motion, we derive the kinematic equations for two-dimensional motion from the fundamental definitions of these three quantities. We then treat projectile motion and uniform circular motion as special cases of motion in two dimensions. We also discuss the concept of relative motion, which shows why observers in different frames of reference may measure different displacements, velocities, and accelerations for a given particle.

4.1 THE DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

In Chapter 2 we found that the motion of a particle moving along a straight line is completely known if its position is known as a function of time. Now let us extend this idea to motion in the *xy* plane. We begin by describing the position of a particle by its position vector **r**, drawn from the origin of some coordinate system to the particle located in the *xy* plane, as in Figure 4.1. At time t_i the particle is at point Ⓐ, and at some later time t_f it is at point Ⓑ. The path from Ⓐ to Ⓑ is not necessarily a straight line. As the particle moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$, its position vector changes from $\mathbf{r}_i$ to $\mathbf{r}_f$. As we learned in Chapter 2, displacement is a vector, and the displacement of the particle is the difference between its final position and its initial position. We now formally define the **displacement vector Δr** for the particle of Figure 4.1 as being the difference between its final position vector and its initial position vector:

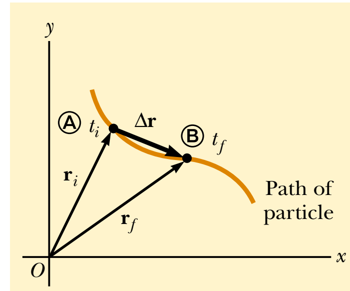


Figure 4.1 A particle moving in the *xy* plane is located with the position vector **r** drawn from the origin to the particle. The displacement of the particle as it moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$ is equal to the vector $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$.

Displacement vector

$$\Delta\mathbf{r} \equiv \mathbf{r}_f - \mathbf{r}_i \tag{4.1}$$

The direction of **Δr** is indicated in Figure 4.1. As we see from the figure, the magnitude of **Δr** is *less* than the distance traveled along the curved path followed by the particle.

As we saw in Chapter 2, it is often useful to quantify motion by looking at the ratio of a displacement divided by the time interval during which that displacement occurred. In two-dimensional (or three-dimensional) kinematics, everything is the same as in one-dimensional kinematics except that we must now use vectors rather than plus and minus signs to indicate the direction of motion.

We define the **average velocity** of a particle during the time interval Δt as the displacement of the particle divided by that time interval:

Average velocity

$$\bar{\mathbf{v}} \equiv \frac{\Delta\mathbf{r}}{\Delta t} \tag{4.2}$$

Multiplying or dividing a vector quantity by a scalar quantity changes only the magnitude of the vector, not its direction. Because displacement is a vector quantity and the time interval is a scalar quantity, we conclude that the average velocity is a vector quantity directed along **Δr**.

Note that the average velocity between points is *independent of the path* taken. This is because average velocity is proportional to displacement, which depends

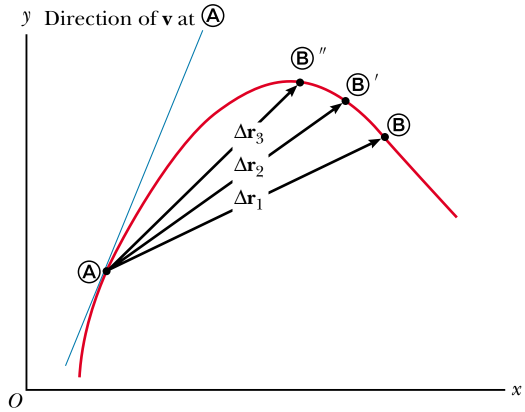


Figure 4.2 As a particle moves between two points, its average velocity is in the direction of the displacement vector $\Delta \mathbf{r}$. As the end point of the path is moved from Ⓑ to Ⓑ′ to Ⓑ″, the respective displacements and corresponding time intervals become smaller and smaller. In the limit that the end point approaches Ⓐ, Δt approaches zero, and the direction of $\Delta \mathbf{r}$ approaches that of the line tangent to the curve at Ⓐ. By definition, the instantaneous velocity at Ⓐ is in the direction of this tangent line.

only on the initial and final position vectors and not on the path taken. As we did with one-dimensional motion, we conclude that if a particle starts its motion at some point and returns to this point via any path, its average velocity is zero for this trip because its displacement is zero.

Consider again the motion of a particle between two points in the *xy* plane, as shown in Figure 4.2. As the time interval over which we observe the motion becomes smaller and smaller, the direction of the displacement approaches that of the line tangent to the path at Ⓐ.

Instantaneous velocity

The **instantaneous velocity v** is defined as the limit of the average velocity $\Delta \mathbf{r}/\Delta t$ as Δt approaches zero:

$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \quad \textbf{(4.3)}$$

That is, the instantaneous velocity equals the derivative of the position vector with respect to time. The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion (Fig. 4.3).

The magnitude of the instantaneous velocity vector $v = |\mathbf{v}|$ is called the *speed,* which, as you should remember, is a scalar quantity.

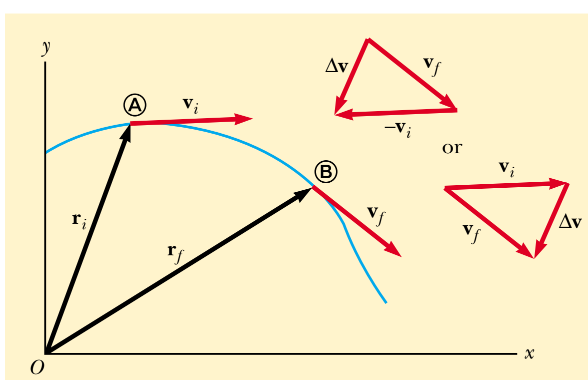


Figure 4.3 A particle moves from position Ⓐ to position Ⓑ. Its velocity vector changes from $\mathbf{v}_i$ to $\mathbf{v}_f$. The vector diagrams at the upper right show two ways of determining the vector $\Delta \mathbf{v}$ from the initial and final velocities.

As a particle moves from one point to another along some path, its instantaneous velocity vector changes from $\mathbf{v}_i$ at time t_i to $\mathbf{v}_f$ at time t_f. Knowing the velocity at these points allows us to determine the average acceleration of the particle:

The **average acceleration** of a particle as it moves from one position to another is defined as the change in the instantaneous velocity vector $\Delta\mathbf{v}$ divided by the time Δt during which that change occurred:

$$\overline{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta\mathbf{v}}{\Delta t} \quad \textbf{(4.4)}$$

Average acceleration

Because it is the ratio of a vector quantity $\Delta\mathbf{v}$ and a scalar quantity Δt, we conclude that average acceleration $\overline{\mathbf{a}}$ is a vector quantity directed along $\Delta\mathbf{v}$. As indicated in Figure 4.3, the direction of $\Delta\mathbf{v}$ is found by adding the vector $-\mathbf{v}_i$ (the negative of $\mathbf{v}_i$) to the vector $\mathbf{v}_f$, because by definition $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$.

When the average acceleration of a particle changes during different time intervals, it is useful to define its instantaneous acceleration $\mathbf{a}$:

The **instantaneous acceleration a** is defined as the limiting value of the ratio $\Delta\mathbf{v}/\Delta t$ as Δt approaches zero:

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \quad \textbf{(4.5)}$$

Instantaneous acceleration

In other words, the instantaneous acceleration equals the derivative of the velocity
3.5 vector with respect to time.

It is important to recognize that various changes can occur when a particle accelerates. First, the magnitude of the velocity vector (the speed) may change with time as in straight-line (one-dimensional) motion. Second, the direction of the velocity vector may change with time even if its magnitude (speed) remains constant, as in curved-path (two-dimensional) motion. Finally, both the magnitude and the direction of the velocity vector may change simultaneously.

Quick Quiz 4.1

The gas pedal in an automobile is called the *accelerator.* (a) Are there any other controls in an automobile that can be considered accelerators? (b) When is the gas pedal not an accelerator?

4.2 TWO-DIMENSIONAL MOTION WITH CONSTANT ACCELERATION

Let us consider two-dimensional motion during which the acceleration remains constant in both magnitude and direction.

The position vector for a particle moving in the xy plane can be written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \quad \textbf{(4.6)}$$

where x, y, and $\mathbf{r}$ change with time as the particle moves while $\mathbf{i}$ and $\mathbf{j}$ remain constant. If the position vector is known, the velocity of the particle can be obtained from Equations 4.3 and 4.6, which give

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} \quad \textbf{(4.7)}$$

Because $\mathbf{a}$ is assumed constant, its components a_x and a_y also are constants. Therefore, we can apply the equations of kinematics to the x and y components of the velocity vector. Substituting $v_{xf} = v_{xi} + a_x t$ and $v_{yf} = v_{yi} + a_y t$ into Equation 4.7 to determine the final velocity at any time t, we obtain

Velocity vector as a function of time

$$\mathbf{v}_f = (v_{xi} + a_x t)\mathbf{i} + (v_{yi} + a_y t)\mathbf{j}$$
$$= (v_{xi}\mathbf{i} + v_{yi}\mathbf{j}) + (a_x\mathbf{i} + a_y\mathbf{j})t$$
$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \quad \textbf{(4.8)}$$

This result states that the velocity of a particle at some time t equals the vector sum of its initial velocity $\mathbf{v}_i$ and the additional velocity $\mathbf{a}t$ acquired in the time t as a result of constant acceleration.

Similarly, from Equation 2.11 we know that the x and y coordinates of a particle moving with constant acceleration are

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

Substituting these expressions into Equation 4.6 (and labeling the final position vector $\mathbf{r}_f$) gives

Position vector as a function of time

$$\mathbf{r}_f = (x_i + v_{xi}t + \tfrac{1}{2}a_x t^2)\mathbf{i} + (y_i + v_{yi}t + \tfrac{1}{2}a_y t^2)\mathbf{j}$$
$$= (x_i\mathbf{i} + y_i\mathbf{j}) + (v_{xi}\mathbf{i} + v_{yi}\mathbf{j})t + \tfrac{1}{2}(a_x\mathbf{i} + a_y\mathbf{j})t^2$$
$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \quad \textbf{(4.9)}$$

This equation tells us that the displacement vector $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$ is the vector sum of a displacement $\mathbf{v}_i t$ arising from the initial velocity of the particle and a displacement $\frac{1}{2}\mathbf{a}t^2$ resulting from the uniform acceleration of the particle.

Graphical representations of Equations 4.8 and 4.9 are shown in Figure 4.4. For simplicity in drawing the figure, we have taken $\mathbf{r}_i = 0$ in Figure 4.4a. That is, we assume the particle is at the origin at $t = t_i = 0$. Note from Figure 4.4a that $\mathbf{r}_f$ is generally not along the direction of either $\mathbf{v}_i$ or $\mathbf{a}$ because the relationship between these quantities is a vector expression. For the same reason, from Figure 4.4b we see that $\mathbf{v}_f$ is generally not along the direction of $\mathbf{v}_i$ or $\mathbf{a}$. Finally, note that $\mathbf{v}_f$ and $\mathbf{r}_f$ are generally not in the same direction.

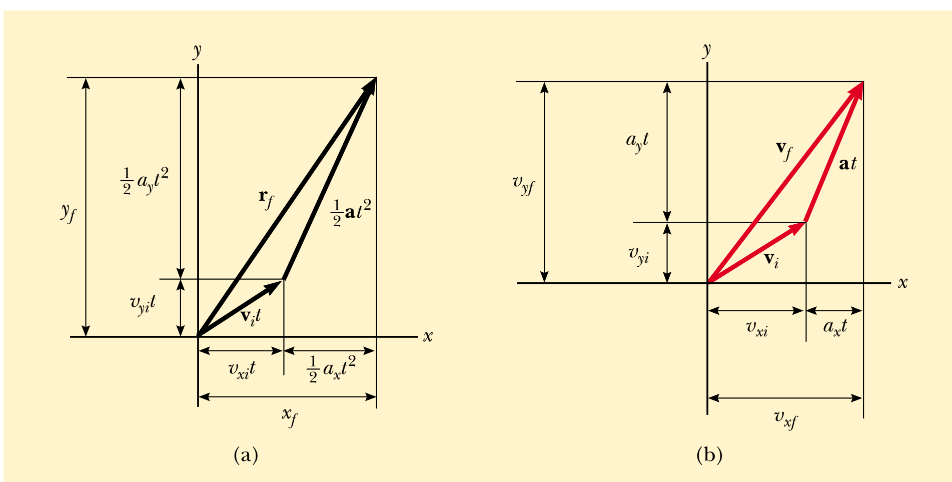


Figure 4.4 Vector representations and components of (a) the displacement and (b) the velocity of a particle moving with a uniform acceleration $\mathbf{a}$. To simplify the drawing, we have set $\mathbf{r}_i = 0$.

Because Equations 4.8 and 4.9 are vector expressions, we may write them in component form:

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \qquad \begin{cases} v_{xf} = v_{xi} + a_x t \\ v_{yf} = v_{yi} + a_y t \end{cases} \tag{4.8a}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \qquad \begin{cases} x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \\ y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 \end{cases} \tag{4.9a}$$

These components are illustrated in Figure 4.4. The component form of the equations for $\mathbf{v}_f$ and $\mathbf{r}_f$ show us that two-dimensional motion at constant acceleration is equivalent to two *independent* motions—one in the x direction and one in the y direction—having constant accelerations a_x and a_y.

EXAMPLE 4.1 Motion in a Plane

A particle starts from the origin at $t = 0$ with an initial velocity having an x component of 20 m/s and a y component of -15 m/s. The particle moves in the xy plane with an x component of acceleration only, given by $a_x = 4.0\ \text{m/s}^2$. (a) Determine the components of the velocity vector at any time and the total velocity vector at any time.

Solution After carefully reading the problem, we realize we can set $v_{xi} = 20$ m/s, $v_{yi} = -15$ m/s, $a_x = 4.0\ \text{m/s}^2$, and $a_y = 0$. This allows us to sketch a rough motion diagram of the situation. The x component of velocity starts at 20 m/s and increases by 4.0 m/s every second. The y component of velocity never changes from its initial value of -15 m/s. From this information we sketch some velocity vectors as shown in Figure 4.5. Note that the spacing between successive images increases as time goes on because the velocity is increasing.

The equations of kinematics give

$$v_{xf} = v_{xi} + a_x t = (20 + 4.0t)\ \text{m/s}$$

$$v_{yf} = v_{yi} + a_y t = -15\ \text{m/s} + 0 = -15\ \text{m/s}$$

Therefore,

$$\mathbf{v}_f = v_{xf}\mathbf{i} + v_{yf}\mathbf{j} = \boxed{[(20 + 4.0t)\mathbf{i} - 15\mathbf{j}]\ \text{m/s}}$$

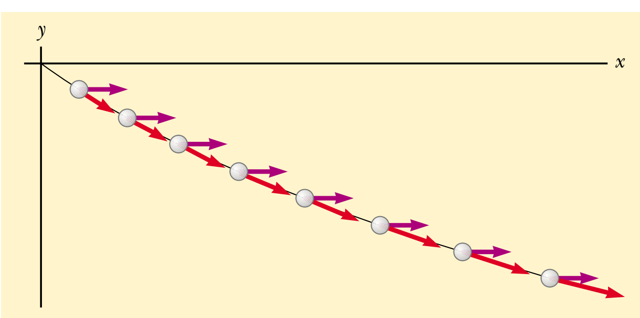


Figure 4.5 Motion diagram for the particle.

We could also obtain this result using Equation 4.8 directly, noting that $\mathbf{a} = 4.0\mathbf{i}\ \text{m/s}^2$ and $\mathbf{v}_i = (20\mathbf{i} - 15\mathbf{j})$ m/s. According to this result, the x component of velocity increases while the y component remains constant; this is consistent with what we predicted. After a long time, the x component will be so great that the y component will be negligible. If we were to extend the object's path in Figure 4.5, eventually it would become nearly parallel to the x axis. It is always helpful to make comparisons between final answers and initial stated conditions.

(b) Calculate the velocity and speed of the particle at $t = 5.0$ s.

Solution With $t = 5.0$ s, the result from part (a) gives

$$\mathbf{v}_f = \{[20 + 4.0(5.0)]\mathbf{i} - 15\mathbf{j}\}\ \text{m/s} = \boxed{(40\mathbf{i} - 15\mathbf{j})\ \text{m/s}}$$

This result tells us that at $t = 5.0$ s, $v_{xf} = 40$ m/s and $v_{yf} = -15$ m/s. Knowing these two components for this two-dimensional motion, we can find both the direction and the magnitude of the velocity vector. To determine the angle θ that $\mathbf{v}$ makes with the x axis at $t = 5.0$ s, we use the fact that $\tan\theta = v_{yf}/v_{xf}$:

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \text{m/s}}{40\ \text{m/s}}\right) = \boxed{-21^\circ}$$

where the minus sign indicates an angle of 21° below the positive x axis. The speed is the magnitude of $\mathbf{v}_f$:

$$v_f = |\mathbf{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ \text{m/s} = \boxed{43\ \text{m/s}}$$

In looking over our result, we notice that if we calculate v_i from the x and y components of $\mathbf{v}_i$, we find that $v_f > v_i$. Does this make sense?

(c) Determine the x and y coordinates of the particle at any time t and the position vector at this time.

Solution Because $x_i = y_i = 0$ at $t = 0$, Equation 2.11 gives

$$x_f = v_{xi}t + \tfrac{1}{2}a_x t^2 = (20t + 2.0t^2)\ \text{m}$$

$$y_f = v_{yi}t = (-15t)\ \text{m}$$

Therefore, the position vector at any time t is

$$\mathbf{r}_f = x_f\mathbf{i} + y_f\mathbf{j} = [(20t + 2.0t^2)\mathbf{i} - 15t\mathbf{j}]\ \text{m}$$

(Alternatively, we could obtain $\mathbf{r}_f$ by applying Equation 4.9 directly, with $\mathbf{v}_i = (20\mathbf{i} - 15\mathbf{j})$ m/s and $\mathbf{a} = 4.0\mathbf{i}$ m/s^2. Try it!) Thus, for example, at $t = 5.0$ s, $x = 150$ m, $y = -75$ m, and $\mathbf{r}_f = (150\mathbf{i} - 75\mathbf{j})$ m. The magnitude of the displacement of the particle from the origin at $t = 5.0$ s is the magnitude of $\mathbf{r}_f$ at this time:

$$r_f = |\mathbf{r}_f| = \sqrt{(150)^2 + (-75)^2}\ \text{m} = 170\ \text{m}$$

Note that this is *not* the distance that the particle travels in this time! Can you determine this distance from the available data?

4.3 PROJECTILE MOTION

Assumptions of projectile motion

Anyone who has observed a baseball in motion (or, for that matter, any other object thrown into the air) has observed projectile motion. The ball moves in a curved path, and its motion is simple to analyze if we make two assumptions: (1) the free-fall acceleration $\mathbf{g}$ is constant over the range of motion and is directed downward,[1] and (2) the effect of air resistance is negligible.[2] With these assumptions, we find that the path of a projectile, which we call its *trajectory,* is *always* a parabola. **We use these assumptions throughout this chapter.**

To show that the trajectory of a projectile is a parabola, let us choose our reference frame such that the y direction is vertical and positive is upward. Because air resistance is neglected, we know that $a_y = -g$ (as in one-dimensional free fall) and that $a_x = 0$. Furthermore, let us assume that at $t = 0$, the projectile leaves the origin ($x_i = y_i = 0$) with speed v_i, as shown in Figure 4.6. The vector $\mathbf{v}_i$ makes an angle θ_i with the horizontal, where θ_i is the angle at which the projectile leaves the origin. From the definitions of the cosine and sine functions we have

3.5

$$\cos\theta_i = v_{xi}/v_i \qquad \sin\theta_i = v_{yi}/v_i$$

Therefore, the initial x and y components of velocity are

$$v_{xi} = v_i\cos\theta_i \qquad v_{yi} = v_i\sin\theta_i$$

Substituting the x component into Equation 4.9a with $x_i = 0$ and $a_x = 0$, we find that

Horizontal position component

$$x_f = v_{xi}t = (v_i\cos\theta_i)t \tag{4.10}$$

Repeating with the y component and using $y_i = 0$ and $a_y = -g$, we obtain

Vertical position component

$$y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = (v_i\sin\theta_i)t - \tfrac{1}{2}gt^2 \tag{4.11}$$

Next, we solve Equation 4.10 for $t = x_f/(v_i\cos\theta_i)$ and substitute this expression for t into Equation 4.11; this gives

$$y = (\tan\theta_i)x - \left(\frac{g}{2v_i^2\cos^2\theta_i}\right)x^2 \tag{4.12}$$

[1] This assumption is reasonable as long as the range of motion is small compared with the radius of the Earth (6.4×10^6 m). In effect, this assumption is equivalent to assuming that the Earth is flat over the range of motion considered.

[2] This assumption is generally *not* justified, especially at high velocities. In addition, any spin imparted to a projectile, such as that applied when a pitcher throws a curve ball, can give rise to some very interesting effects associated with aerodynamic forces, which will be discussed in Chapter 15.

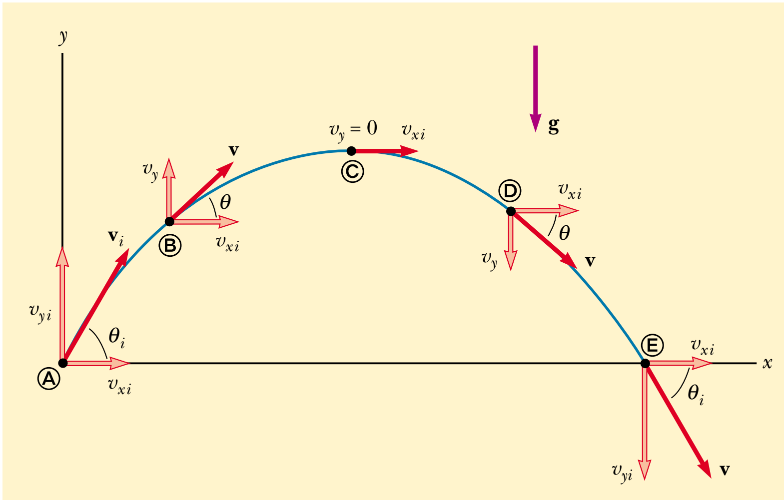


Figure 4.6 The parabolic path of a projectile that leaves the origin with a velocity $\mathbf{v}_i$. The velocity vector $\mathbf{v}$ changes with time in both magnitude and direction. This change is the result of acceleration in the negative y direction. The x component of velocity remains constant in time because there is no acceleration along the horizontal direction. The y component of velocity is zero at the peak of the path.

A welder cuts holes through a heavy metal construction beam with a hot torch. The sparks generated in the process follow parabolic paths.

This equation is valid for launch angles in the range $0 < \theta_i < \pi/2$. We have left the subscripts off the x and y because the equation is valid for any point (x, y) along the path of the projectile. The equation is of the form $y = ax - bx^2$, which is the equation of a parabola that passes through the origin. Thus, we have shown that the trajectory of a projectile is a parabola. Note that the trajectory is completely specified if both the initial speed v_i and the launch angle θ_i are known.

The vector expression for the position vector of the projectile as a function of time follows directly from Equation 4.9, with $\mathbf{r}_i = 0$ and $\mathbf{a} = \mathbf{g}$:

$$\mathbf{r} = \mathbf{v}_i t + \tfrac{1}{2}\mathbf{g}t^2$$

This expression is plotted in Figure 4.7.

QuickLab

Place two tennis balls at the edge of a tabletop. Sharply snap one ball horizontally off the table with one hand while gently tapping the second ball off with your other hand. Compare how long it takes the two to reach the floor. Explain your results.

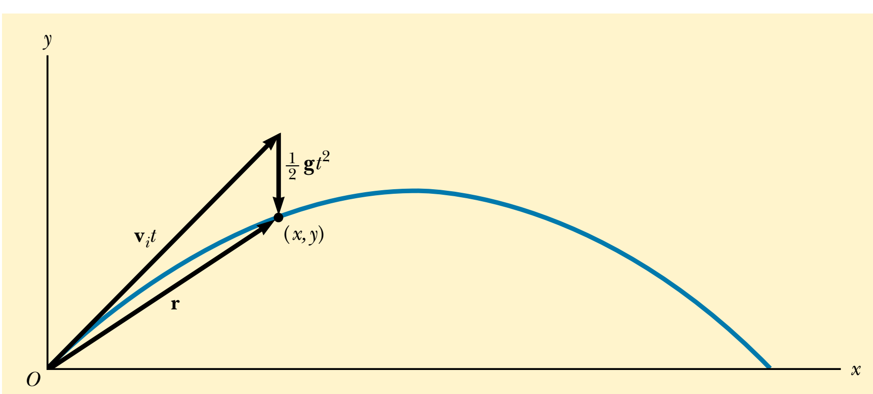


Figure 4.7 The position vector $\mathbf{r}$ of a projectile whose initial velocity at the origin is $\mathbf{v}_i$. The vector $\mathbf{v}_i t$ would be the displacement of the projectile if gravity were absent, and the vector $\frac{1}{2}\mathbf{g}t^2$ is its vertical displacement due to its downward gravitational acceleration.

Multiflash exposure of a tennis player executing a forehand swing. Note that the ball follows a parabolic path characteristic of a projectile. Such photographs can be used to study the quality of sports equipment and the performance of an athlete.

It is interesting to realize that the motion of a particle can be considered the superposition of the term $\mathbf{v}_i t$, the displacement if no acceleration were present, and the term $\frac{1}{2}\mathbf{g}t^2$, which arises from the acceleration due to gravity. In other words, if there were no gravitational acceleration, the particle would continue to move along a straight path in the direction of $\mathbf{v}_i$. Therefore, the vertical distance $\frac{1}{2}\mathbf{g}t^2$ through which the particle "falls" off the straight-line path is the same distance that a freely falling body would fall during the same time interval. We conclude that **projectile motion is the superposition of two motions:** (1) **constant-velocity motion in the horizontal direction** and (2) **free-fall motion in the vertical direction.** Except for t, the time of flight, the horizontal and vertical components of a projectile's motion are completely independent of each other.

EXAMPLE 4.2 Approximating Projectile Motion

A ball is thrown in such a way that its initial vertical and horizontal components of velocity are 40 m/s and 20 m/s, respectively. Estimate the total time of flight and the distance the ball is from its starting point when it lands.

Solution We start by remembering that the two velocity components are independent of each other. By considering the vertical motion first, we can determine how long the ball remains in the air. Then, we can use the time of flight to estimate the horizontal distance covered.

A motion diagram like Figure 4.8 helps us organize what we know about the problem. The acceleration vectors are all the same, pointing downward with a magnitude of nearly 10 m/s^2. The velocity vectors change direction. Their hori-

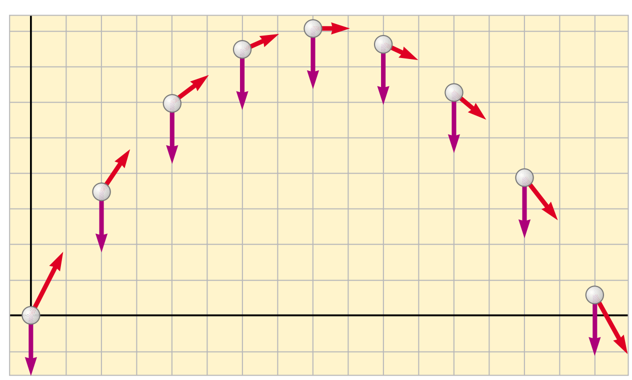


Figure 4.8 Motion diagram for a projectile.

zontal components are all the same: 20 m/s. Because the vertical motion is free fall, the vertical components of the velocity vectors change, second by second, from 40 m/s to roughly 30, 20, and 10 m/s in the upward direction, and then to 0 m/s. Subsequently, its velocity becomes 10, 20, 30, and 40 m/s in the downward direction. Thus it takes the ball about 4 s to go up and another 4 s to come back down, for a total time of flight of approximately 8 s. Because the horizontal component of velocity is 20 m/s, and because the ball travels at this speed for 8 s, it ends up approximately 160 m from its starting point.

Horizontal Range and Maximum Height of a Projectile

Let us assume that a projectile is fired from the origin at $t_i = 0$ with a positive v_{yi} component, as shown in Figure 4.9. Two points are especially interesting to analyze: the peak point Ⓐ, which has cartesian coordinates $(R/2, h)$, and the point Ⓑ, which has coordinates $(R, 0)$. The distance R is called the *horizontal range* of the projectile, and the distance h is its *maximum height.* Let us find h and R in terms of v_i, θ_i, and g.

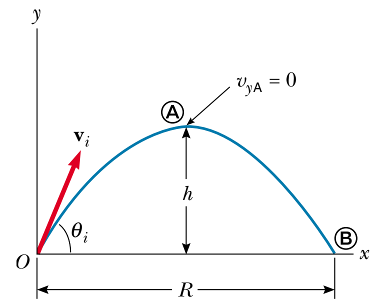


Figure 4.9 A projectile fired from the origin at $t_i = 0$ with an initial velocity $\mathbf{v}_i$. The maximum height of the projectile is h, and the horizontal range is R. At Ⓐ, the peak of the trajectory, the particle has coordinates $(R/2, h)$.

We can determine h by noting that at the peak, $v_{y\text{A}} = 0$. Therefore, we can use Equation 4.8a to determine the time t_{A} it takes the projectile to reach the peak:

$$v_{yf} = v_{yi} + a_y t$$

$$0 = v_i \sin\theta_i - g t_{\text{A}}$$

$$t_{\text{A}} = \frac{v_i \sin\theta_i}{g}$$

Substituting this expression for t_{A} into the y part of Equation 4.9a and replacing $y_f = y_{\text{A}}$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \tfrac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{v_i^2 \sin^2\theta_i}{2g} \qquad \textbf{(4.13)}$$

Maximum height of projectile

The range R is the horizontal distance that the projectile travels in twice the time it takes to reach its peak, that is, in a time $t_{\text{B}} = 2t_{\text{A}}$. Using the x part of Equation 4.9a, noting that $v_{xi} = v_{x\text{B}} = v_i \cos\theta_i$, and setting $R \equiv x_{\text{B}}$ at $t = 2t_{\text{A}}$, we find that

$$R = v_{xi}t_{\text{B}} = (v_i \cos\theta_i)2t_{\text{A}}$$

$$= (v_i \cos\theta_i)\frac{2v_i \sin\theta_i}{g} = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g}$$

Using the identity $\sin 2\theta = 2 \sin\theta \cos\theta$ (see Appendix B.4), we write R in the more compact form

$$R = \frac{v_i^2 \sin 2\theta_i}{g} \qquad \textbf{(4.14)}$$

Range of projectile

Keep in mind that Equations 4.13 and 4.14 are useful for calculating h and R only if v_i and θ_i are known (which means that only $\mathbf{v}_i$ has to be specified) and if the projectile lands at the same height from which it started, as it does in Figure 4.9.

The maximum value of R from Equation 4.14 is $R_{\text{max}} = v_i^2/g$. This result follows from the fact that the maximum value of $\sin 2\theta_i$ is 1, which occurs when $2\theta_i = 90°$. Therefore, R is a maximum when $\theta_i = 45°$.

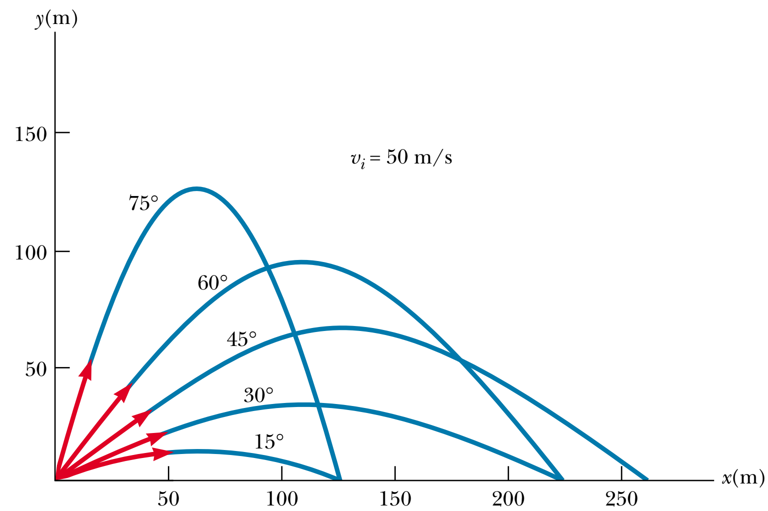


Figure 4.10 A projectile fired from the origin with an initial speed of 50 m/s at various angles of projection. Note that complementary values of θ_i result in the same value of x (range of the projectile).

QuickLab

To carry out this investigation, you need to be outdoors with a small ball, such as a tennis ball, as well as a wristwatch. Throw the ball straight up as hard as you can and determine the initial speed of your throw and the approximate maximum height of the ball, using only your watch. What happens when you throw the ball at some angle $\theta \neq 90°$? Does this change the time of flight (perhaps because it is easier to throw)? Can you still determine the maximum height and initial speed?

Figure 4.10 illustrates various trajectories for a projectile having a given initial speed but launched at different angles. As you can see, the range is a maximum for $\theta_i = 45°$. In addition, for any θ_i other than 45°, a point having cartesian coordinates $(R, 0)$ can be reached by using either one of two complementary values of θ_i, such as 75° and 15°. Of course, the maximum height and time of flight for one of these values of θ_i are different from the maximum height and time of flight for the complementary value.

Quick Quiz 4.2

As a projectile moves in its parabolic path, is there any point along the path where the velocity and acceleration vectors are (a) perpendicular to each other? (b) parallel to each other? (c) Rank the five paths in Figure 4.10 with respect to time of flight, from the shortest to the longest.

Problem-Solving Hints

Projectile Motion

We suggest that you use the following approach to solving projectile motion problems:

- Select a coordinate system and resolve the initial velocity vector into x and y components.
- Follow the techniques for solving constant-velocity problems to analyze the horizontal motion. Follow the techniques for solving constant-acceleration problems to analyze the vertical motion. The x and y motions share the same time of flight t.

EXAMPLE 4.3 The Long-Jump

A long-jumper leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s. (a) How far does he jump in the horizontal direction? (Assume his motion is equivalent to that of a particle.)

Solution Because the initial speed and launch angle are given, the most direct way of solving this problem is to use the range formula given by Equation 4.14. However, it is more instructive to take a more general approach and use Figure 4.9. As before, we set our origin of coordinates at the takeoff point and label the peak as Ⓐ and the landing point as Ⓑ. The horizontal motion is described by Equation 4.10:

$$x_f = x_B = (v_i \cos \theta_i) t_B = (11.0 \text{ m/s})(\cos 20.0°) t_B$$

The value of x_B can be found if the total time of the jump is known. We are able to find t_B by remembering that $a_y = -g$ and by using the y part of Equation 4.8a. We also note that at the top of the jump the vertical component of velocity v_{yA} is zero:

$$v_{yf} = v_{yA} = v_i \sin \theta_i - g t_A$$
$$0 = (11.0 \text{ m/s}) \sin 20.0° - (9.80 \text{ m/s}^2) t_A$$
$$t_A = 0.384 \text{ s}$$

This is the time needed to reach the *top* of the jump. Because of the symmetry of the vertical motion, an identical time interval passes before the jumper returns to the ground. Therefore, the *total time* in the air is $t_B = 2t_A = 0.768$ s. Substituting this value into the above expression for x_f gives

$$x_f = x_B = (11.0 \text{ m/s})(\cos 20.0°)(0.768 \text{ s}) = \boxed{7.94 \text{ m}}$$

This is a reasonable distance for a world-class athlete.

(b) What is the maximum height reached?

Solution We find the maximum height reached by using Equation 4.11:

$$y_{max} = y_A = (v_i \sin \theta_i) t_A - \tfrac{1}{2} g t_A^2$$
$$= (11.0 \text{ m/s})(\sin 20.0°)(0.384 \text{ s})$$
$$- \tfrac{1}{2}(9.80 \text{ m/s}^2)(0.384 \text{ s})^2$$
$$= \boxed{0.722 \text{ m}}$$

Treating the long-jumper as a particle is an oversimplification. Nevertheless, the values obtained are reasonable.

Exercise To check these calculations, use Equations 4.13 and 4.14 to find the maximum height and horizontal range.

In a long-jump event, 1993 United States champion Mike Powell can leap horizontal distances of at least 8 m.

EXAMPLE 4.4 A Bull's-Eye Every Time

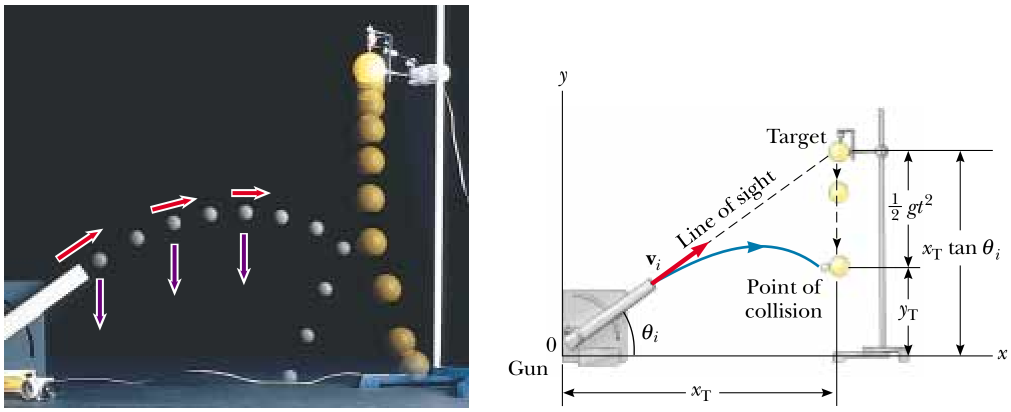
In a popular lecture demonstration, a projectile is fired at a target in such a way that the projectile leaves the gun at the same time the target is dropped from rest, as shown in Figure 4.11. Show that if the gun is initially aimed at the stationary target, the projectile hits the target.

Solution We can argue that a collision results under the conditions stated by noting that, as soon as they are released, the projectile and the target experience the same acceleration $a_y = -g$. First, note from Figure 4.11b that the initial y coordinate of the target is $x_T \tan \theta_i$ and that it falls through a distance $\frac{1}{2}gt^2$ in a time t. Therefore, the y coordinate of the target at any moment after release is

$$y_T = x_T \tan \theta_i - \tfrac{1}{2} g t^2$$

Now if we use Equation 4.9a to write an expression for the y coordinate of the projectile at any moment, we obtain

$$y_P = x_P \tan \theta_i - \tfrac{1}{2} g t^2$$



(a) (b)

Figure 4.11 (a) Multiflash photograph of projectile–target demonstration. If the gun is aimed directly at the target and is fired at the same instant the target begins to fall, the projectile will hit the target. Note that the velocity of the projectile (red arrows) changes in direction and magnitude, while the downward acceleration (violet arrows) remains constant. *(Central Scientific Company.)* (b) Schematic diagram of the projectile–target demonstration. Both projectile and target fall through the same vertical distance in a time t because both experience the same acceleration $a_y = -g$.

Thus, by comparing the two previous equations, we see that when the y coordinates of the projectile and target are the same, their x coordinates are the same and a collision results. That is, when $y_P = y_T$, $x_P = x_T$. You can obtain the same result, using expressions for the position vectors for the projectile and target.

Note that a collision will *not* always take place owing to a further restriction: A collision can result only when $v_i \sin\theta_i \geq \sqrt{gd/2}$, where d is the initial elevation of the target above the *floor*. If $v_i \sin\theta_i$ is less than this value, the projectile will strike the floor before reaching the target.

EXAMPLE 4.5 That's Quite an Arm!

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal and with an initial speed of 20.0 m/s, as shown in Figure 4.12. If the height of the building is 45.0 m, (a) how long is it before the stone hits the ground?

Solution We have indicated the various parameters in Figure 4.12. When working problems on your own, you should always make a sketch such as this and label it.

The initial x and y components of the stone's velocity are

$$v_{xi} = v_i \cos\theta_i = (20.0 \text{ m/s})(\cos 30.0°) = 17.3 \text{ m/s}$$

$$v_{yi} = v_i \sin\theta_i = (20.0 \text{ m/s})(\sin 30.0°) = 10.0 \text{ m/s}$$

To find t, we can use $y_f = v_{yi}t + \frac{1}{2}a_y t^2$ (Eq. 4.9a) with $y_f = -45.0$ m, $a_y = -g$, and $v_{yi} = 10.0$ m/s (there is a minus sign on the numerical value of y_f because we have chosen the top of the building as the origin):

$$-45.0 \text{ m} = (10.0 \text{ m/s})t - \tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$

Solving the quadratic equation for t gives, for the positive root, $t =$ 4.22 s. Does the negative root have any physical

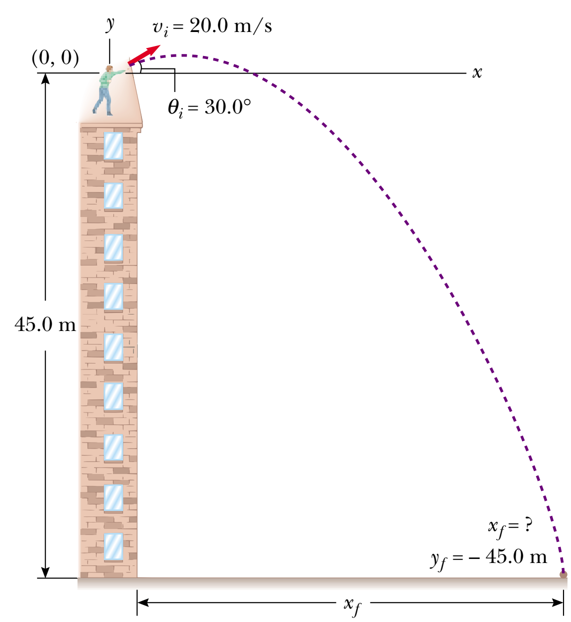


Figure 4.12

meaning? (Can you think of another way of finding t from the information given?)

(b) What is the speed of the stone just before it strikes the ground?

Solution We can use Equation 4.8a, $v_{yf} = v_{yi} + a_y t$, with $t = 4.22$ s to obtain the y component of the velocity just before the stone strikes the ground:

$$v_{yf} = 10.0 \text{ m/s} - (9.80 \text{ m/s}^2)(4.22 \text{ s}) = -31.4 \text{ m/s}$$

The negative sign indicates that the stone is moving downward. Because $v_{xf} = v_{xi} = 17.3$ m/s, the required speed is

$$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(17.3)^2 + (-31.4)^2} \text{ m/s} = \boxed{35.9 \text{ m/s}}$$

Exercise Where does the stone strike the ground?

Answer 73.0 m from the base of the building.

EXAMPLE 4.6 The Stranded Explorers

An Alaskan rescue plane drops a package of emergency rations to a stranded party of explorers, as shown in Figure 4.13. If the plane is traveling horizontally at 40.0 m/s and is 100 m above the ground, where does the package strike the ground relative to the point at which it was released?

Solution For this problem we choose the coordinate system shown in Figure 4.13, in which the origin is at the point of release of the package. Consider first the horizontal motion of the package. The only equation available to us for finding the distance traveled along the horizontal direction is $x_f = v_{xi}t$ (Eq. 4.9a). The initial x component of the package velocity is the same as that of the plane when the package is released: 40.0 m/s. Thus, we have

$$x_f = (40.0 \text{ m/s})t$$

If we know t, the length of time the package is in the air, then we can determine x_f, the distance the package travels in the horizontal direction. To find t, we use the equations that describe the vertical motion of the package. We know that at the instant the package hits the ground, its y coordinate is $y_f = -100$ m. We also know that the initial vertical component of the package velocity v_{yi} is zero because at the moment of release, the package had only a horizontal component of velocity.

From Equation 4.9a, we have

$$y_f = -\tfrac{1}{2}gt^2$$

$$-100 \text{ m} = -\tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$

$$t = 4.52 \text{ s}$$

Substitution of this value for the time of flight into the equation for the x coordinate gives

$$x_f = (40.0 \text{ m/s})(4.52 \text{ s}) = \boxed{181 \text{ m}}$$

The package hits the ground 181 m to the right of the drop point.

Exercise What are the horizontal and vertical components of the velocity of the package just before it hits the ground?

Answer $v_{xf} = 40.0$ m/s; $v_{yf} = -44.3$ m/s.

Exercise Where is the plane when the package hits the ground? (Assume that the plane does not change its speed or course.)

Answer Directly over the package.

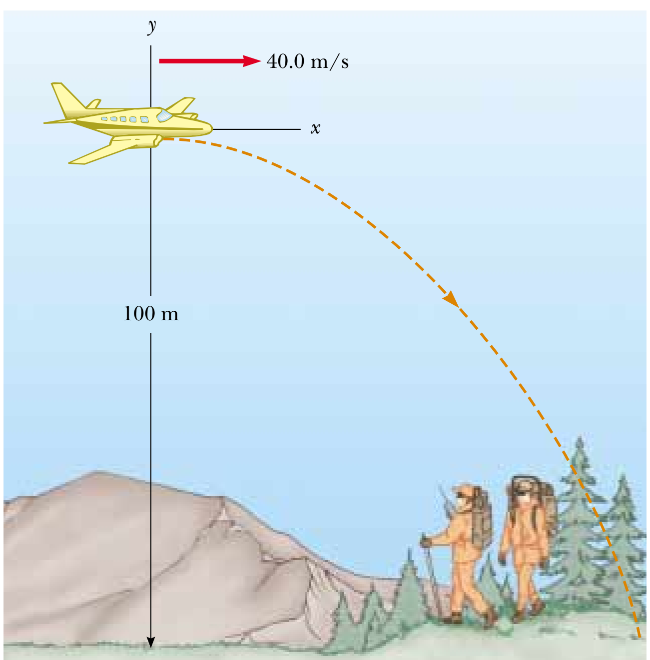


Figure 4.13

EXAMPLE 4.7 The End of the Ski Jump

A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s, as shown in Figure 4.14. The landing incline below him falls off with a slope of 35.0°. Where does he land on the incline?

Solution It is reasonable to expect the skier to be airborne for less than 10 s, and so he will not go farther than 250 m horizontally. We should expect the value of d, the distance traveled along the incline, to be of the same order of magnitude. It is convenient to select the beginning of the jump as the origin ($x_i = 0, y_i = 0$). Because $v_{xi} = 25.0$ m/s and $v_{yi} = 0$, the x and y component forms of Equation 4.9a are

$$(1) \qquad x_f = v_{xi}t = (25.0 \text{ m/s})t$$

$$(2) \qquad y_f = \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$

From the right triangle in Figure 4.14, we see that the jumper's x and y coordinates at the landing point are $x_f = d\cos 35.0°$ and $y_f = -d\sin 35.0°$. Substituting these relationships into (1) and (2), we obtain

$$(3) \qquad d\cos 35.0° = (25.0 \text{ m/s})t$$

$$(4) \qquad -d\sin 35.0° = -\tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$

Solving (3) for t and substituting the result into (4), we find that $d = 109$ m. Hence, the x and y coordinates of the point at which he lands are

$$x_f = d\cos 35.0° = (109 \text{ m})\cos 35.0° = \boxed{89.3 \text{ m}}$$

$$y_f = -d\sin 35.0° = -(109 \text{ m})\sin 35.0° = \boxed{-62.5 \text{ m}}$$

Exercise Determine how long the jumper is airborne and his vertical component of velocity just before he lands.

Answer 3.57 s; −35.0 m/s.

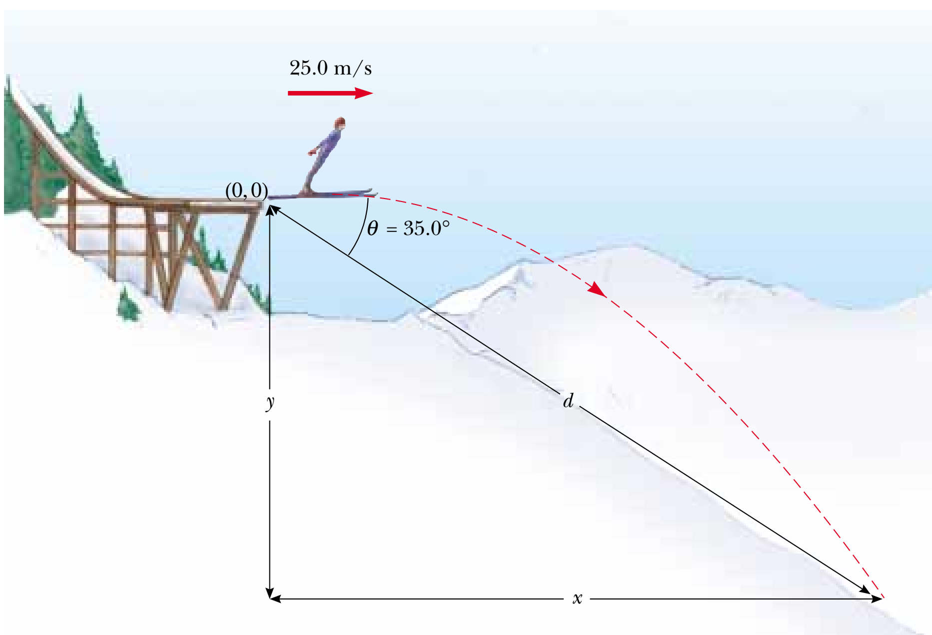


Figure 4.14

What would have occurred if the skier in the last example happened to be carrying a stone and let go of it while in midair? Because the stone has the same initial velocity as the skier, it will stay near him as he moves—that is, it floats alongside him. This is a technique that NASA uses to train astronauts. The plane pictured at the beginning of the chapter flies in the same type of projectile path that the skier and stone follow. The passengers and cargo in the plane fall along-

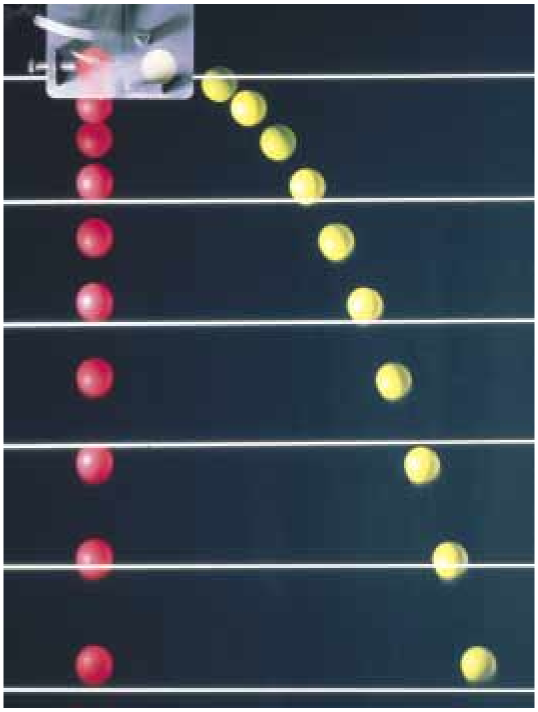


Figure 4.15 This multiflash photograph of two balls released simultaneously illustrates both free fall (red ball) and projectile motion (yellow ball). The yellow ball was projected horizontally, while the red ball was released from rest. *(Richard Megna/Fundamental Photographs)*

QuickLab

Armed with nothing but a ruler and the knowledge that the time between images was 1/30 s, find the horizontal speed of the yellow ball in Figure 4.15. (*Hint:* Start by analyzing the motion of the red ball. Because you know its vertical acceleration, you can calibrate the distances depicted in the photograph. Then you can find the horizontal speed of the yellow ball.)

side each other; that is, they have the same trajectory. An astronaut can release a piece of equipment and it will float freely alongside her hand. The same thing happens in the space shuttle. The craft and everything in it are falling as they orbit the Earth.

4.4 UNIFORM CIRCULAR MOTION

Figure 4.16a shows a car moving in a circular path with constant linear speed v.
3.6 Such motion is called **uniform circular motion.** Because the car's direction of motion changes, the car has an acceleration, as we learned in Section 4.1. For any motion, the velocity vector is tangent to the path. Consequently, when an object moves in a circular path, its velocity vector is perpendicular to the radius of the circle.

We now show that the acceleration vector in uniform circular motion is always perpendicular to the path and always points toward the center of the circle. An ac-

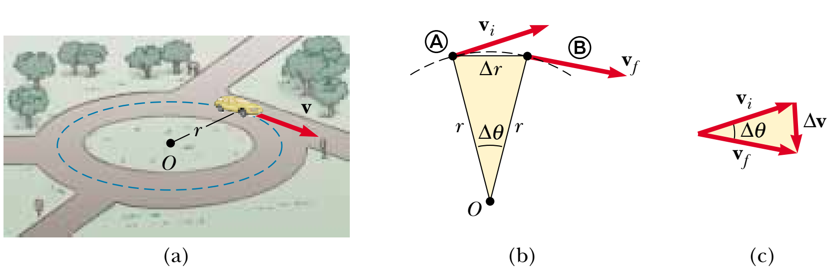


Figure 4.16 (a) A car moving along a circular path at constant speed experiences uniform circular motion. (b) As a particle moves from Ⓐ to Ⓑ, its velocity vector changes from $\mathbf{v}_i$ to $\mathbf{v}_f$. (c) The construction for determining the direction of the change in velocity $\Delta\mathbf{v}$, which is toward the center of the circle for small $\Delta\mathbf{r}$.

celeration of this nature is called a **centripetal** (center-seeking) acceleration, and its magnitude is

$$a_r = \frac{v^2}{r} \tag{4.15}$$

where r is the radius of the circle and the notation a_r is used to indicate that the centripetal acceleration is along the radial direction.

To derive Equation 4.15, consider Figure 4.16b, which shows a particle first at point Ⓐ and then at point Ⓑ. The particle is at Ⓐ at time t_i, and its velocity at that time is $\mathbf{v}_i$. It is at Ⓑ at some later time t_f, and its velocity at that time is $\mathbf{v}_f$. Let us assume here that $\mathbf{v}_i$ and $\mathbf{v}_f$ differ only in direction; their magnitudes (speeds) are the same (that is, $v_i = v_f = v$). To calculate the acceleration of the particle, let us begin with the defining equation for average acceleration (Eq. 4.4):

$$\overline{\mathbf{a}} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

This equation indicates that we must subtract $\mathbf{v}_i$ from $\mathbf{v}_f$, being sure to treat them as vectors, where $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$ is the change in the velocity. Because $\mathbf{v}_i + \Delta\mathbf{v} = \mathbf{v}_f$, we can find the vector $\Delta\mathbf{v}$, using the vector triangle in Figure 4.16c.

Now consider the triangle in Figure 4.16b, which has sides Δr and r. This triangle and the one in Figure 4.16c, which has sides Δv and v, are similar. This fact enables us to write a relationship between the lengths of the sides:

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

This equation can be solved for Δv and the expression so obtained substituted into $\overline{a} = \Delta v/\Delta t$ (Eq. 4.4) to give

$$\overline{a} = \frac{v\,\Delta r}{r\,\Delta t}$$

Now imagine that points Ⓐ and Ⓑ in Figure 4.16b are extremely close together. In this case $\Delta\mathbf{v}$ points toward the center of the circular path, and because the acceleration is in the direction of $\Delta\mathbf{v}$, it too points toward the center. Furthermore, as Ⓐ and Ⓑ approach each other, Δt approaches zero, and the ratio $\Delta r/\Delta t$ approaches the speed v. Hence, in the limit $\Delta t \rightarrow 0$, the magnitude of the acceleration is

$$a_r = \frac{v^2}{r}$$

Thus, we conclude that in uniform circular motion, the acceleration is directed toward the center of the circle and has a magnitude given by v^2/r, where v is the speed of the particle and r is the radius of the circle. You should be able to show that the dimensions of a_r are L/T^2. We shall return to the discussion of circular motion in Section 6.1.

4.5 TANGENTIAL AND RADIAL ACCELERATION

3.6 Now let us consider a particle moving along a curved path where the velocity changes both in direction and in magnitude, as shown in Figure 4.17. As is always the case, the velocity vector is tangent to the path, but now the direction of the ac-

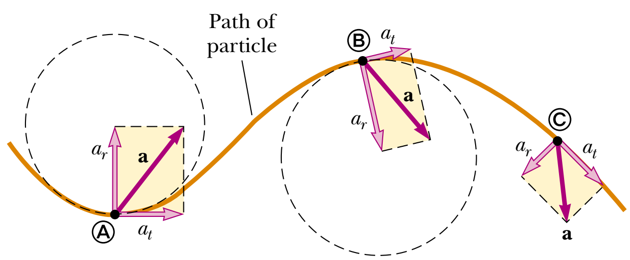


Figure 4.17 The motion of a particle along an arbitrary curved path lying in the *xy* plane. If the velocity vector **v** (always tangent to the path) changes in direction and magnitude, the component vectors of the acceleration **a** are a tangential component a_t and a radial component a_r.

celeration vector **a** changes from point to point. This vector can be resolved into two component vectors: a radial component vector $\mathbf{a}_r$ and a tangential component vector $\mathbf{a}_t$. Thus, **a** can be written as the vector sum of these component vectors:

$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t \tag{4.16}$$

Total acceleration

The tangential acceleration causes the change in the speed of the particle. It is parallel to the instantaneous velocity, and its magnitude is

$$a_t = \frac{d|\mathbf{v}|}{dt} \tag{4.17}$$

Tangential acceleration

The radial acceleration arises from the change in direction of the velocity vector as described earlier and has an absolute magnitude given by

$$a_r = \frac{v^2}{r} \tag{4.18}$$

Radial acceleration

where r is the radius of curvature of the path at the point in question. Because $\mathbf{a}_r$ and $\mathbf{a}_t$ are mutually perpendicular component vectors of **a**, it follows that $a = \sqrt{a_r^2 + a_t^2}$. As in the case of uniform circular motion, $\mathbf{a}_r$ in nonuniform circular motion always points toward the center of curvature, as shown in Figure 4.17. Also, at a given speed, a_r is large when the radius of curvature is small (as at points Ⓐ and Ⓑ in Figure 4.17) and small when r is large (such as at point Ⓒ). The direction of $\mathbf{a}_t$ is either in the same direction as **v** (if v is increasing) or opposite **v** (if v is decreasing).

In uniform circular motion, where v is constant, $a_t = 0$ and the acceleration is always completely radial, as we described in Section 4.4. (*Note:* Eq. 4.18 is identical to Eq. 4.15.) In other words, uniform circular motion is a special case of motion along a curved path. Furthermore, if the direction of **v** does not change, then there is no radial acceleration and the motion is one-dimensional (in this case, $a_r = 0$, but a_t may not be zero).

Quick Quiz 4.3

(a) Draw a motion diagram showing velocity and acceleration vectors for an object moving with constant speed counterclockwise around a circle. Draw similar diagrams for an object moving counterclockwise around a circle but (b) slowing down at constant tangential acceleration and (c) speeding up at constant tangential acceleration.

It is convenient to write the acceleration of a particle moving in a circular path in terms of unit vectors. We do this by defining the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ shown in

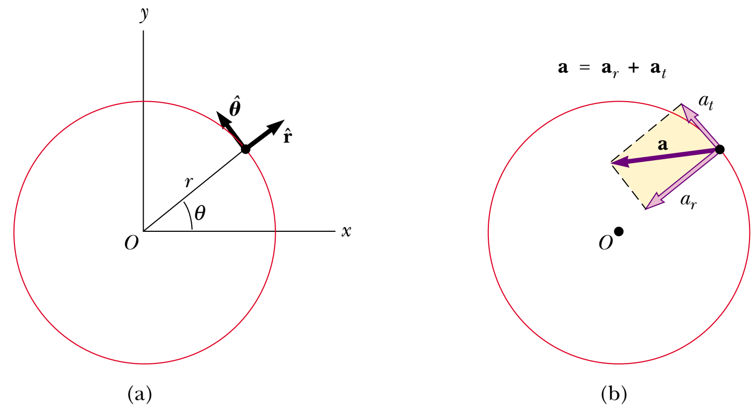


Figure 4.18 (a) Descriptions of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. (b) The total acceleration **a** of a particle moving along a curved path (which at any instant is part of a circle of radius r) is the sum of radial and tangential components. The radial component is directed toward the center of curvature. If the tangential component of acceleration becomes zero, the particle follows uniform circular motion.

Figure 4.18a, where $\hat{\mathbf{r}}$ is a unit vector lying along the radius vector and directed radially outward from the center of the circle and $\hat{\boldsymbol{\theta}}$ is a unit vector tangent to the circle. The direction of $\hat{\boldsymbol{\theta}}$ is in the direction of increasing θ, where θ is measured counterclockwise from the positive x axis. Note that both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ "move along with the particle" and so vary in time. Using this notation, we can express the total acceleration as

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d|\mathbf{v}|}{dt}\,\hat{\boldsymbol{\theta}} - \frac{v^2}{r}\,\hat{\mathbf{r}} \qquad \textbf{(4.19)}$$

These vectors are described in Figure 4.18b. The negative sign on the v^2/r term in Equation 4.19 indicates that the radial acceleration is always directed radially inward, *opposite* $\hat{\mathbf{r}}$.

Quick Quiz 4.4

Based on your experience, draw a motion diagram showing the position, velocity, and acceleration vectors for a pendulum that, from an initial position 45° to the right of a central vertical line, swings in an arc that carries it to a final position 45° to the left of the central vertical line. The arc is part of a circle, and you should use the center of this circle as the origin for the position vectors.

EXAMPLE 4.8 The Swinging Ball

A ball tied to the end of a string 0.50 m in length swings in a vertical circle under the influence of gravity, as shown in Figure 4.19. When the string makes an angle $\theta = 20°$ with the vertical, the ball has a speed of 1.5 m/s. (a) Find the magnitude of the radial component of acceleration at this instant.

Solution The diagram from the answer to Quick Quiz 4.4 (p. 109) applies to this situation, and so we have a good idea of how the acceleration vector varies during the motion. Figure 4.19 lets us take a closer look at the situation. The radial acceleration is given by Equation 4.18. With $v = 1.5$ m/s and $r = 0.50$ m, we find that

$$a_r = \frac{v^2}{r} = \frac{(1.5\ \text{m/s})^2}{0.50\ \text{m}} = \boxed{4.5\ \text{m/s}^2}$$

(b) What is the magnitude of the tangential acceleration when $\theta = 20°$?

Solution When the ball is at an angle θ to the vertical, it has a tangential acceleration of magnitude $g \sin \theta$ (the component of **g** tangent to the circle). Therefore, at $\theta = 20°$, $a_t = g \sin 20° = \boxed{3.4 \text{ m/s}^2}$.

(c) Find the magnitude and direction of the total acceleration **a** at $\theta = 20°$.

Solution Because $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t$, the magnitude of **a** at $\theta = 20°$ is

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(4.5)^2 + (3.4)^2} \text{ m/s}^2 = \boxed{5.6 \text{ m/s}^2}$$

If ϕ is the angle between **a** and the string, then

$$\phi = \tan^{-1} \frac{a_t}{a_r} = \tan^{-1} \left(\frac{3.4 \text{ m/s}^2}{4.5 \text{ m/s}^2} \right) = \boxed{37°}$$

Note that **a**, $\mathbf{a}_t$, and $\mathbf{a}_r$ all change in direction *and* magnitude as the ball swings through the circle. When the ball is at its lowest elevation ($\theta = 0$), $a_t = 0$ because there is no tangential component of **g** at this angle; also, a_r is a *maximum* because v is a maximum. If the ball has enough speed to reach its highest position ($\theta = 180°$), then a_t is again zero but a_r is a minimum because v is now a minimum. Finally, in the two horizontal positions ($\theta = 90°$ and $270°$), $|\mathbf{a}_t| = g$ and a_r has a value between its minimum and maximum values.

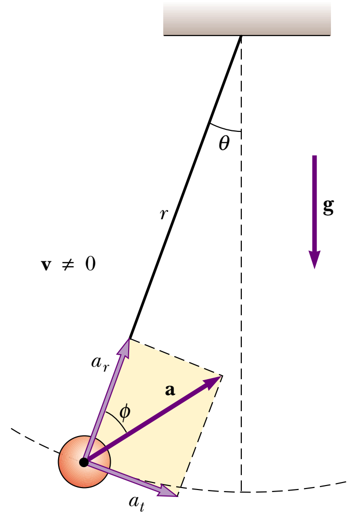


Figure 4.19 Motion of a ball suspended by a string of length r. The ball swings with nonuniform circular motion in a vertical plane, and its acceleration **a** has a radial component a_r and a tangential component a_t.

4.6 RELATIVE VELOCITY AND RELATIVE ACCELERATION

In this section, we describe how observations made by different observers in different frames of reference are related to each other. We find that observers in different frames of reference may measure different displacements, velocities, and accelerations for a given particle. That is, two observers moving relative to each other generally do not agree on the outcome of a measurement.

3.7 For example, suppose two cars are moving in the same direction with speeds of 50 mi/h and 60 mi/h. To a passenger in the slower car, the speed of the faster car is 10 mi/h. Of course, a stationary observer will measure the speed of the faster car to be 60 mi/h, not 10 mi/h. Which observer is correct? They both are! This simple example demonstrates that the velocity of an object depends on the frame of reference in which it is measured.

Suppose a person riding on a skateboard (observer A) throws a ball in such a way that it appears in this person's frame of reference to move first straight upward and then straight downward along the same vertical line, as shown in Figure 4.20a. A stationary observer B sees the path of the ball as a parabola, as illustrated in Figure 4.20b. Relative to observer B, the ball has a vertical component of velocity (resulting from the initial upward velocity and the downward acceleration of gravity) *and* a horizontal component.

Another example of this concept that of is a package dropped from an airplane flying with a constant velocity; this is the situation we studied in Example 4.6. An observer on the airplane sees the motion of the package as a straight line toward the Earth. The stranded explorer on the ground, however, sees the trajectory of the package as a parabola. If, once it drops the package, the airplane con-

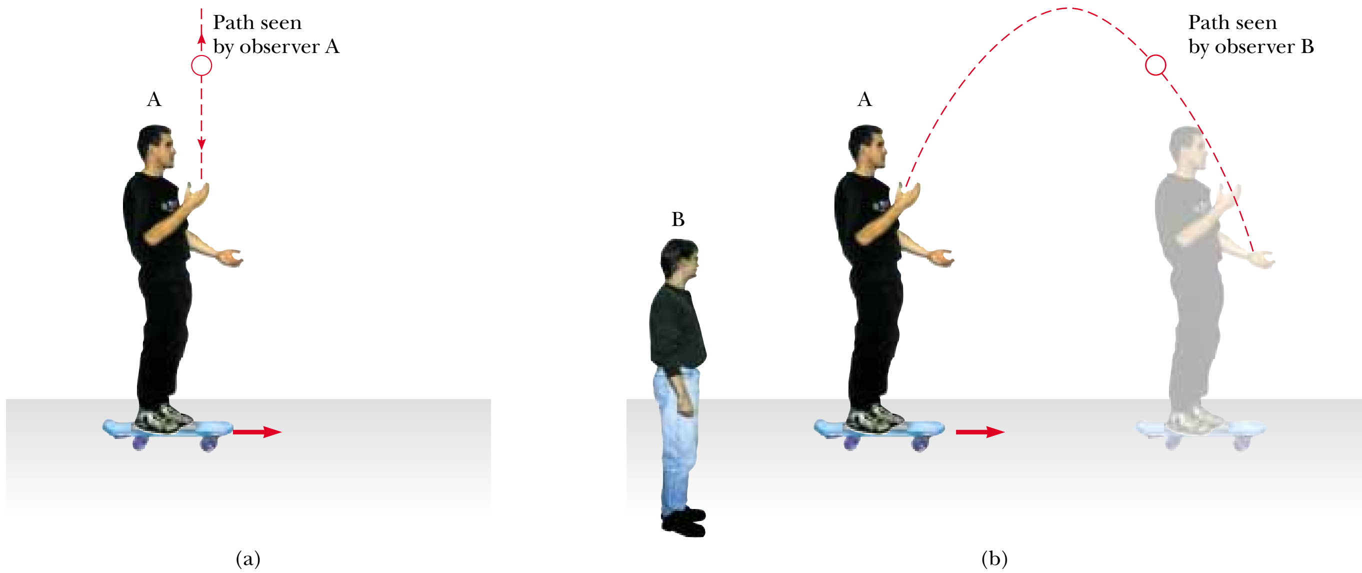


Figure 4.20 (a) Observer A on a moving vehicle throws a ball upward and sees it rise and fall in a straight-line path. (b) Stationary observer B sees a parabolic path for the same ball.

tinues to move horizontally with the same velocity, then the package hits the ground directly beneath the airplane (if we assume that air resistance is neglected)!

In a more general situation, consider a particle located at point Ⓐ in Figure 4.21. Imagine that the motion of this particle is being described by two observers, one in reference frame S, fixed relative to the Earth, and another in reference frame S', moving to the right relative to S (and therefore relative to the Earth) with a constant velocity $\mathbf{v}_0$. (Relative to an observer in S', S moves to the left with a velocity $-\mathbf{v}_0$.) Where an observer stands in a reference frame is irrelevant in this discussion, but for purposes of this discussion let us place each observer at her or his respective origin.

We label the position of the particle relative to the S frame with the position vector $\mathbf{r}$ and that relative to the S' frame with the position vector $\mathbf{r}'$, both after some time t. The vectors $\mathbf{r}$ and $\mathbf{r}'$ are related to each other through the expression $\mathbf{r} = \mathbf{r}' + \mathbf{v}_0 t$, or

Galilean coordinate transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t \tag{4.20}$$

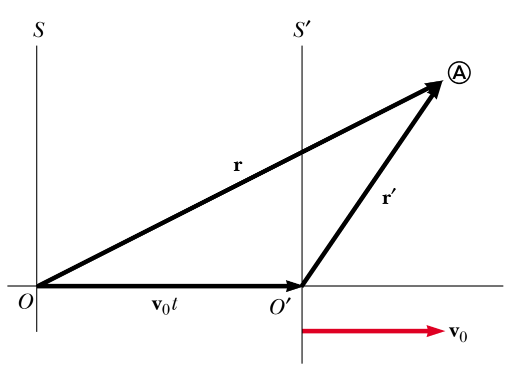
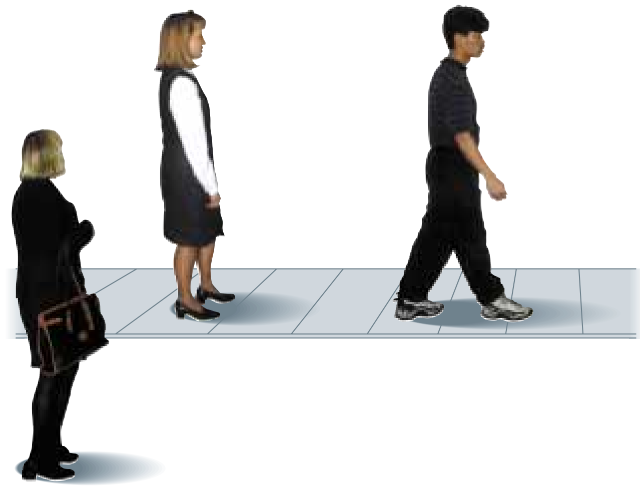


Figure 4.21 A particle located at Ⓐ is described by two observers, one in the fixed frame of reference S, and the other in the frame S', which moves to the right with a constant velocity $\mathbf{v}_0$. The vector $\mathbf{r}$ is the particle's position vector relative to S, and $\mathbf{r}'$ is its position vector relative to S'.



The woman standing on the beltway sees the walking man pass by at a slower speed than the woman standing on the stationary floor does.

That is, after a time t, the S' frame is displaced to the right of the S frame by an amount $\mathbf{v}_0 t$.

If we differentiate Equation 4.20 with respect to time and note that $\mathbf{v}_0$ is constant, we obtain

$$\frac{d\mathbf{r}'}{dt} = \frac{d\mathbf{r}}{dt} - \mathbf{v}_0$$

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \tag{4.21}$$

Galilean velocity transformation

where $\mathbf{v}'$ is the velocity of the particle observed in the S' frame and $\mathbf{v}$ is its velocity observed in the S frame. Equations 4.20 and 4.21 are known as **Galilean transformation equations.** They relate the coordinates and velocity of a particle as measured in a frame fixed relative to the Earth to those measured in a frame moving with uniform motion relative to the Earth.

Although observers in two frames measure different velocities for the particle, they measure the *same acceleration* when $\mathbf{v}_0$ is constant. We can verify this by taking the time derivative of Equation 4.21:

$$\frac{d\mathbf{v}'}{dt} = \frac{d\mathbf{v}}{dt} - \frac{d\mathbf{v}_0}{dt}$$

Because $\mathbf{v}_0$ is constant, $d\mathbf{v}_0/dt = 0$. Therefore, we conclude that $\mathbf{a}' = \mathbf{a}$ because $\mathbf{a}' = d\mathbf{v}'/dt$ and $\mathbf{a} = d\mathbf{v}/dt$. That is, **the acceleration of the particle measured by an observer in the Earth's frame of reference is the same as that measured by any other observer moving with constant velocity relative to the Earth's frame.**

Quick Quiz 4.5

A passenger in a car traveling at 60 mi/h pours a cup of coffee for the tired driver. Describe the path of the coffee as it moves from a Thermos bottle into a cup as seen by (a) the passenger and (b) someone standing beside the road and looking in the window of the car as it drives past. (c) What happens if the car accelerates while the coffee is being poured?

EXAMPLE 4.9 A Boat Crossing a River

A boat heading due north crosses a wide river with a speed of 10.0 km/h relative to the water. The water in the river has a uniform speed of 5.00 km/h due east relative to the Earth. Determine the velocity of the boat relative to an observer standing on either bank.

Solution We know $\mathbf{v}_{br}$, the velocity of the *boat* relative to the *river*, and $\mathbf{v}_{rE}$, the velocity of the *river* relative to the *Earth*. What we need to find is $\mathbf{v}_{bE}$, the velocity of the *boat* relative to the *Earth*. The relationship between these three quantities is

$$\mathbf{v}_{bE} = \mathbf{v}_{br} + \mathbf{v}_{rE}$$

The terms in the equation must be manipulated as vector quantities; the vectors are shown in Figure 4.22. The quantity $\mathbf{v}_{br}$ is due north, $\mathbf{v}_{rE}$ is due east, and the vector sum of the two, $\mathbf{v}_{bE}$, is at an angle θ, as defined in Figure 4.22. Thus, we can find the speed v_{bE} of the boat relative to the Earth by using the Pythagorean theorem:

$$v_{bE} = \sqrt{v_{br}^2 + v_{rE}^2} = \sqrt{(10.0)^2 + (5.00)^2}\ \text{km/h}$$

$$= \boxed{11.2\ \text{km/h}}$$

The direction of $\mathbf{v}_{bE}$ is

$$\theta = \tan^{-1}\left(\frac{v_{rE}}{v_{br}}\right) = \tan^{-1}\left(\frac{5.00}{10.0}\right) = 26.6^\circ$$

The boat is moving at a speed of 11.2 km/h in the direction 26.6° east of north relative to the Earth.

Exercise If the width of the river is 3.0 km, find the time it takes the boat to cross it.

Answer 18 min.

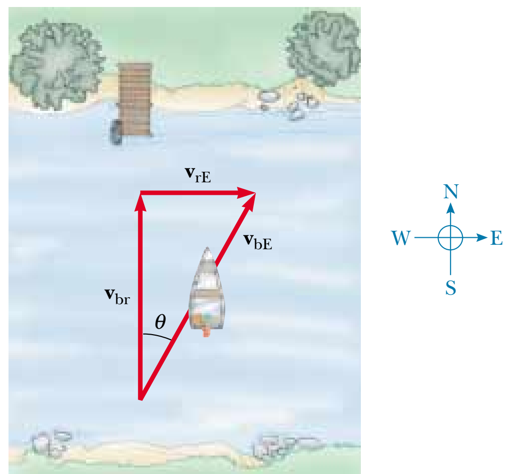


Figure 4.22

EXAMPLE 4.10 Which Way Should We Head?

If the boat of the preceding example travels with the same speed of 10.0 km/h relative to the river and is to travel due north, as shown in Figure 4.23, what should its heading be?

Solution As in the previous example, we know $\mathbf{v}_{rE}$ and the magnitude of the vector $\mathbf{v}_{br}$, and we want $\mathbf{v}_{bE}$ to be directed across the river. Figure 4.23 shows that the boat must head upstream in order to travel directly northward across the river. Note the difference between the triangle in Figure 4.22 and the one in Figure 4.23—specifically, that the hypotenuse in Figure 4.23 is no longer $\mathbf{v}_{bE}$. Therefore, when we use the Pythagorean theorem to find $\mathbf{v}_{bE}$ this time, we obtain

$$v_{bE} = \sqrt{v_{br}^2 - v_{rE}^2} = \sqrt{(10.0)^2 - (5.00)^2}\ \text{km/h} = 8.66\ \text{km/h}$$

Now that we know the magnitude of $\mathbf{v}_{bE}$, we can find the direction in which the boat is heading:

$$\theta = \tan^{-1}\left(\frac{v_{rE}}{v_{bE}}\right) = \tan^{-1}\left(\frac{5.00}{8.66}\right) = \boxed{30.0^\circ}$$

The boat must steer a course 30.0° west of north.

Exercise If the width of the river is 3.0 km, find the time it takes the boat to cross it.

Answer 21 min.

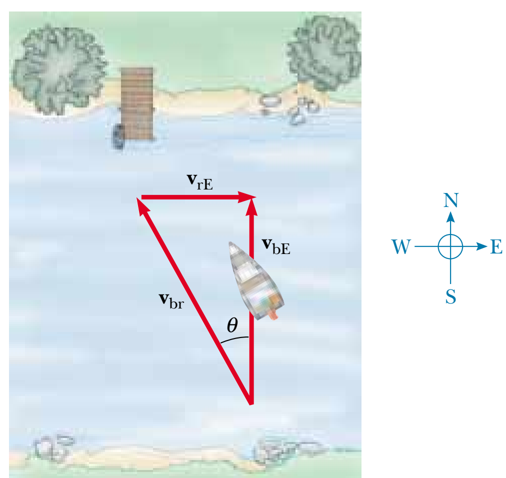


Figure 4.23

SUMMARY

If a particle moves with *constant* acceleration $\mathbf{a}$ and has velocity $\mathbf{v}_i$ and position $\mathbf{r}_i$ at $t = 0$, its velocity and position vectors at some later time t are

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \quad \textbf{(4.8)}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \quad \textbf{(4.9)}$$

For two-dimensional motion in the xy plane under constant acceleration, each of these vector expressions is equivalent to two component expressions—one for the motion in the x direction and one for the motion in the y direction. You should be able to break the two-dimensional motion of any object into these two components.

Projectile motion is one type of two-dimensional motion under constant acceleration, where $a_x = 0$ and $a_y = -g$. It is useful to think of projectile motion as the superposition of two motions: (1) constant-velocity motion in the x direction and (2) free-fall motion in the vertical direction subject to a constant downward acceleration of magnitude $g = 9.80 \text{ m/s}^2$. You should be able to analyze the motion in terms of separate horizontal and vertical components of velocity, as shown in Figure 4.24.

A particle moving in a circle of radius r with constant speed v is in **uniform circular motion**. It undergoes a centripetal (or radial) acceleration $\mathbf{a}_r$ because the direction of $\mathbf{v}$ changes in time. The magnitude of $\mathbf{a}_r$ is

$$a_r = \frac{v^2}{r} \quad \textbf{(4.18)}$$

and its direction is always toward the center of the circle.

If a particle moves along a curved path in such a way that both the magnitude and the direction of $\mathbf{v}$ change in time, then the particle has an acceleration vector that can be described by two component vectors: (1) a radial component vector $\mathbf{a}_r$ that causes the change in direction of $\mathbf{v}$ and (2) a tangential component vector $\mathbf{a}_t$ that causes the change in magnitude of $\mathbf{v}$. The magnitude of $\mathbf{a}_r$ is v^2/r, and the magnitude of $\mathbf{a}_t$ is $d|\mathbf{v}|/dt$. You should be able to sketch motion diagrams for an object following a curved path and show how the velocity and acceleration vectors change as the object's motion varies.

The velocity $\mathbf{v}$ of a particle measured in a fixed frame of reference S can be related to the velocity $\mathbf{v}'$ of the same particle measured in a moving frame of reference S' by

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \quad \textbf{(4.21)}$$

where $\mathbf{v}_0$ is the velocity of S' relative to S. You should be able to translate back and forth between different frames of reference.

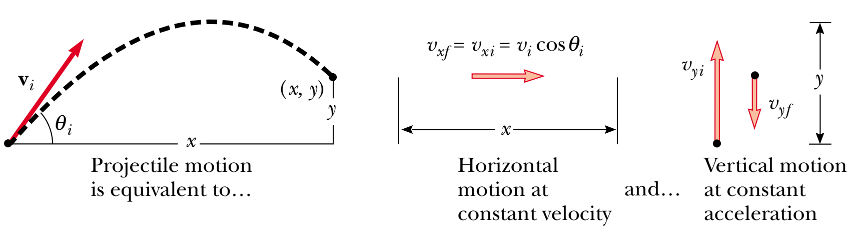


Figure 4.24 Analyzing projectile motion in terms of horizontal and vertical components.

QUESTIONS

1. Can an object accelerate if its speed is constant? Can an object accelerate if its velocity is constant?
2. If the average velocity of a particle is zero in some time interval, what can you say about the displacement of the particle for that interval?
3. If you know the position vectors of a particle at two points along its path and also know the time it took to get from one point to the other, can you determine the particle's instantaneous velocity? Its average velocity? Explain.
4. Describe a situation in which the velocity of a particle is always perpendicular to the position vector.
5. Explain whether or not the following particles have an acceleration: (a) a particle moving in a straight line with constant speed and (b) a particle moving around a curve with constant speed.
6. Correct the following statement: "The racing car rounds the turn at a constant velocity of 90 mi/h."
7. Determine which of the following moving objects have an approximately parabolic trajectory: (a) a ball thrown in an arbitrary direction, (b) a jet airplane, (c) a rocket leaving the launching pad, (d) a rocket whose engines fail a few minutes after launch, (e) a tossed stone moving to the bottom of a pond.
8. A rock is dropped at the same instant that a ball at the same initial elevation is thrown horizontally. Which will have the greater speed when it reaches ground level?
9. A spacecraft drifts through space at a constant velocity. Suddenly, a gas leak in the side of the spacecraft causes a constant acceleration of the spacecraft in a direction perpendicular to the initial velocity. The orientation of the spacecraft does not change, and so the acceleration remains perpendicular to the original direction of the velocity. What is the shape of the path followed by the spacecraft in this situation?
10. A ball is projected horizontally from the top of a building. One second later another ball is projected horizontally from the same point with the same velocity. At what point in the motion will the balls be closest to each other? Will the first ball always be traveling faster than the second ball? How much time passes between the moment the first ball hits the ground and the moment the second one hits the ground? Can the horizontal projection velocity of the second ball be changed so that the balls arrive at the ground at the same time?
11. A student argues that as a satellite orbits the Earth in a circular path, the satellite moves with a constant velocity and therefore has no acceleration. The professor claims that the student is wrong because the satellite must have a centripetal acceleration as it moves in its circular orbit. What is wrong with the student's argument?
12. What is the fundamental difference between the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ and the unit vectors $\mathbf{i}$ and $\mathbf{j}$?
13. At the end of its arc, the velocity of a pendulum is zero. Is its acceleration also zero at this point?
14. If a rock is dropped from the top of a sailboat's mast, will it hit the deck at the same point regardless of whether the boat is at rest or in motion at constant velocity?
15. A stone is thrown upward from the top of a building. Does the stone's displacement depend on the location of the origin of the coordinate system? Does the stone's velocity depend on the location of the origin?
16. Is it possible for a vehicle to travel around a curve without accelerating? Explain.
17. A baseball is thrown with an initial velocity of $(10\mathbf{i} + 15\mathbf{j})$ m/s. When it reaches the top of its trajectory, what are (a) its velocity and (b) its acceleration? Neglect the effect of air resistance.
18. An object moves in a circular path with constant speed v. (a) Is the velocity of the object constant? (b) Is its acceleration constant? Explain.
19. A projectile is fired at some angle to the horizontal with some initial speed v_i, and air resistance is neglected. Is the projectile a freely falling body? What is its acceleration in the vertical direction? What is its acceleration in the horizontal direction?
20. A projectile is fired at an angle of 30° from the horizontal with some initial speed. Firing at what other projectile angle results in the same range if the initial speed is the same in both cases? Neglect air resistance.
21. A projectile is fired on the Earth with some initial velocity. Another projectile is fired on the Moon with the same initial velocity. If air resistance is neglected, which projectile has the greater range? Which reaches the greater altitude? (Note that the free-fall acceleration on the Moon is about 1.6 m/s^2.)
22. As a projectile moves through its parabolic trajectory, which of these quantities, if any, remain constant: (a) speed, (b) acceleration, (c) horizontal component of velocity, (d) vertical component of velocity?
23. A passenger on a train that is moving with constant velocity drops a spoon. What is the acceleration of the spoon relative to (a) the train and (b) the Earth?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 4.1 The Displacement, Velocity, and Acceleration Vectors

WEB **1.** A motorist drives south at 20.0 m/s for 3.00 min, then turns west and travels at 25.0 m/s for 2.00 min, and finally travels northwest at 30.0 m/s for 1.00 min. For this 6.00-min trip, find (a) the total vector displacement, (b) the average speed, and (c) the average velocity. Use a coordinate system in which east is the positive x axis.

2. Suppose that the position vector for a particle is given as $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, with $x = at + b$ and $y = ct^2 + d$, where $a = 1.00$ m/s, $b = 1.00$ m, $c = 0.125$ m/s^2, and $d = 1.00$ m. (a) Calculate the average velocity during the time interval from $t = 2.00$ s to $t = 4.00$ s. (b) Determine the velocity and the speed at $t = 2.00$ s.

3. A golf ball is hit off a tee at the edge of a cliff. Its x and y coordinates versus time are given by the following expressions:

$$x = (18.0 \text{ m/s})t$$

and

$$y = (4.00 \text{ m/s})t - (4.90 \text{ m/s}^2)t^2$$

(a) Write a vector expression for the ball's position as a function of time, using the unit vectors **i** and **j**. By taking derivatives of your results, write expressions for (b) the velocity vector as a function of time and (c) the acceleration vector as a function of time. Now use unit vector notation to write expressions for (d) the position, (e) the velocity, and (f) the acceleration of the ball, all at $t = 3.00$ s.

4. The coordinates of an object moving in the xy plane vary with time according to the equations

$$x = -(5.00 \text{ m}) \sin \omega t$$

and

$$y = (4.00 \text{ m}) - (5.00 \text{ m})\cos \omega t$$

where t is in seconds and ω has units of seconds^{-1}. (a) Determine the components of velocity and components of acceleration at $t = 0$. (b) Write expressions for the position vector, the velocity vector, and the acceleration vector at any time $t > 0$. (c) Describe the path of the object on an xy graph.

Section 4.2 Two-Dimensional Motion with Constant Acceleration

5. At $t = 0$, a particle moving in the xy plane with constant acceleration has a velocity of $\mathbf{v}_i = (3.00\mathbf{i} - 2.00\mathbf{j})$ m/s when it is at the origin. At $t = 3.00$ s, the particle's velocity is $\mathbf{v} = (9.00\mathbf{i} + 7.00\mathbf{j})$ m/s. Find (a) the acceleration of the particle and (b) its coordinates at any time t.

6. The vector position of a particle varies in time according to the expression $\mathbf{r} = (3.00\mathbf{i} - 6.00t^2\mathbf{j})$ m. (a) Find expressions for the velocity and acceleration as functions of time. (b) Determine the particle's position and velocity at $t = 1.00$ s.

7. A fish swimming in a horizontal plane has velocity $\mathbf{v}_i = (4.00\mathbf{i} + 1.00\mathbf{j})$ m/s at a point in the ocean whose displacement from a certain rock is $\mathbf{r}_i = (10.0\mathbf{i} - 4.00\mathbf{j})$ m. After the fish swims with constant acceleration for 20.0 s, its velocity is $\mathbf{v} = (20.0\mathbf{i} - 5.00\mathbf{j})$ m/s. (a) What are the components of the acceleration? (b) What is the direction of the acceleration with respect to the unit vector **i**? (c) Where is the fish at $t = 25.0$ s if it maintains its original acceleration and in what direction is it moving?

8. A particle initially located at the origin has an acceleration of $\mathbf{a} = 3.00\mathbf{j}$ m/s^2 and an initial velocity of $\mathbf{v}_i = 5.00\mathbf{i}$ m/s. Find (a) the vector position and velocity at any time t and (b) the coordinates and speed of the particle at $t = 2.00$ s.

Section 4.3 Projectile Motion

(Neglect air resistance in all problems and take $g = 9.80$ m/s^2.)

WEB **9.** In a local bar, a customer slides an empty beer mug down the counter for a refill. The bartender is momentarily distracted and does not see the mug, which slides off the counter and strikes the floor 1.40 m from the base of the counter. If the height of the counter is 0.860 m, (a) with what velocity did the mug leave the counter and (b) what was the direction of the mug's velocity just before it hit the floor?

10. In a local bar, a customer slides an empty beer mug down the counter for a refill. The bartender is momentarily distracted and does not see the mug, which slides off the counter and strikes the floor at distance d from the base of the counter. If the height of the counter is h, (a) with what velocity did the mug leave the counter and (b) what was the direction of the mug's velocity just before it hit the floor?

11. One strategy in a snowball fight is to throw a first snowball at a high angle over level ground. While your opponent is watching the first one, you throw a second one at a low angle and timed to arrive at your opponent before or at the same time as the first one. Assume both snowballs are thrown with a speed of 25.0 m/s. The first one is thrown at an angle of 70.0° with respect to the horizontal. (a) At what angle should the second (low-angle) snowball be thrown if it is to land at the same point as the first? (b) How many seconds later should the second snowball be thrown if it is to land at the same time as the first?

12. A tennis player standing 12.6 m from the net hits the ball at 3.00° above the horizontal. To clear the net, the ball must rise at least 0.330 m. If the ball just clears the net at the apex of its trajectory, how fast was the ball moving when it left the racket?

13. An artillery shell is fired with an initial velocity of 300 m/s at 55.0° above the horizontal. It explodes on a mountainside 42.0 s after firing. What are the x and y coordinates of the shell where it explodes, relative to its firing point?

14. An astronaut on a strange planet finds that she can jump a maximum horizontal distance of 15.0 m if her initial speed is 3.00 m/s. What is the free-fall acceleration on the planet?

15. A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection? Give your answer to three significant figures.

16. A ball is tossed from an upper-story window of a building. The ball is given an initial velocity of 8.00 m/s at an angle of 20.0° below the horizontal. It strikes the ground 3.00 s later. (a) How far horizontally from the base of the building does the ball strike the ground? (b) Find the height from which the ball was thrown. (c) How long does it take the ball to reach a point 10.0 m below the level of launching?

17. A cannon with a muzzle speed of 1 000 m/s is used to start an avalanche on a mountain slope. The target is 2 000 m from the cannon horizontally and 800 m above the cannon. At what angle, above the horizontal, should the cannon be fired?

18. Consider a projectile that is launched from the origin of an xy coordinate system with speed v_i at initial angle θ_i above the horizontal. Note that at the apex of its trajectory the projectile is moving horizontally, so that the slope of its path is zero. Use the expression for the trajectory given in Equation 4.12 to find the x coordinate that corresponds to the maximum height. Use this x coordinate and the symmetry of the trajectory to determine the horizontal range of the projectile.

WEB **19.** A placekicker must kick a football from a point 36.0 m (about 40 yards) from the goal, and half the crowd hopes the ball will clear the crossbar, which is 3.05 m high. When kicked, the ball leaves the ground with a speed of 20.0 m/s at an angle of 53.0° to the horizontal. (a) By how much does the ball clear or fall short of clearing the crossbar? (b) Does the ball approach the crossbar while still rising or while falling?

20. A firefighter 50.0 m away from a burning building directs a stream of water from a fire hose at an angle of 30.0° above the horizontal, as in Figure P4.20. If the speed of the stream is 40.0 m/s, at what height will the water strike the building?

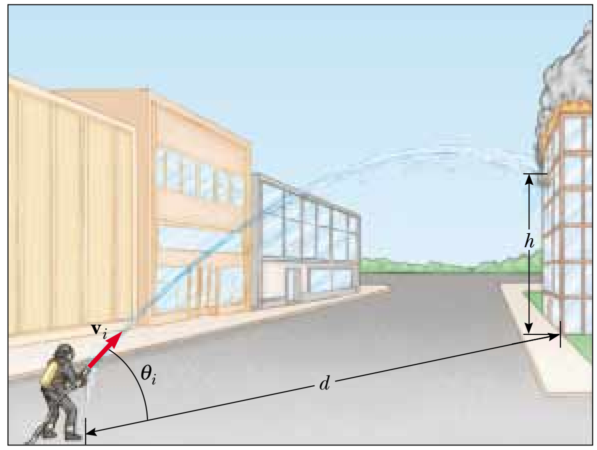


Figure P4.20 Problems 20 and 21. *(Frederick McKinney/FPG International)*

21. A firefighter a distance d from a burning building directs a stream of water from a fire hose at angle θ_i above the horizontal as in Figure P4.20. If the initial speed of the stream is v_i, at what height h does the water strike the building?

22. A soccer player kicks a rock horizontally off a cliff 40.0 m high into a pool of water. If the player hears the sound of the splash 3.00 s later, what was the initial speed given to the rock? Assume the speed of sound in air to be 343 m/s.

23. A basketball star covers 2.80 m horizontally in a jump to dunk the ball (Fig. P4.23). His motion through space can be modeled as that of a particle at a point called his center of mass (which we shall define in Chapter 9). His center of mass is at elevation 1.02 m when he leaves the floor. It reaches a maximum height of 1.85 m above the floor and is at elevation 0.900 m when he touches down again. Determine (a) his time of flight (his "hang time"), (b) his horizontal and (c) vertical velocity components at the instant of takeoff, and (d) his takeoff angle. (e) For comparison, determine the hang time of a whitetail deer making a jump with center-of-mass elevations $y_i = 1.20$ m, $y_{max} = 2.50$ m, $y_f = 0.700$ m.

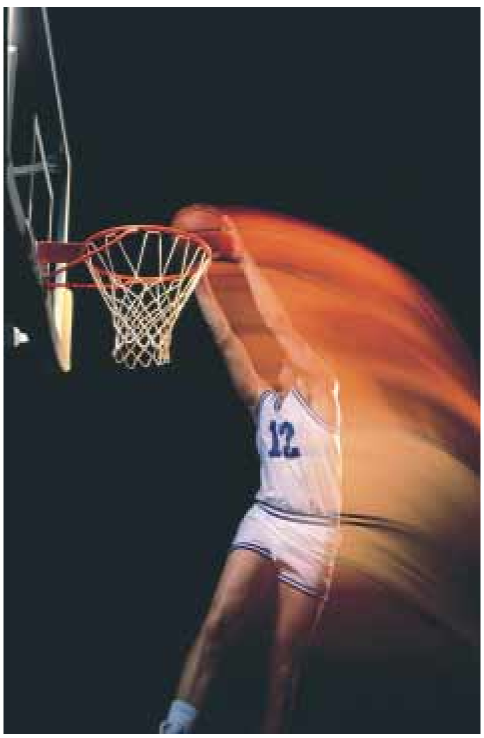


Figure P4.23 *(Top, Ron Chapple/FPG International; bottom, Bill Lea/Dembinsky Photo Associates)*

Section 4.4 Uniform Circular Motion

24. The orbit of the Moon about the Earth is approximately circular, with a mean radius of 3.84×10^8 m. It takes 27.3 days for the Moon to complete one revolution about the Earth. Find (a) the mean orbital speed of the Moon and (b) its centripetal acceleration.

WEB 25. The athlete shown in Figure P4.25 rotates a 1.00-kg discus along a circular path of radius 1.06 m. The maximum speed of the discus is 20.0 m/s. Determine the magnitude of the maximum radial acceleration of the discus.

Figure P4.25 *(Sam Sargent/Liaison International)*

26. From information on the endsheets of this book, compute, for a point located on the surface of the Earth at the equator, the radial acceleration due to the rotation of the Earth about its axis.

27. A tire 0.500 m in radius rotates at a constant rate of 200 rev/min. Find the speed and acceleration of a small stone lodged in the tread of the tire (on its outer edge). (*Hint:* In one revolution, the stone travels a distance equal to the circumference of its path, $2\pi r$.)

28. During liftoff, Space Shuttle astronauts typically feel accelerations up to $1.4g$, where $g = 9.80$ m/s^2. In their training, astronauts ride in a device where they experience such an acceleration as a centripetal acceleration. Specifically, the astronaut is fastened securely at the end of a mechanical arm that then turns at constant speed in a horizontal circle. Determine the rotation rate, in revolutions per second, required to give an astronaut a centripetal acceleration of $1.40g$ while the astronaut moves in a circle of radius 10.0 m.

29. Young David who slew Goliath experimented with slings before tackling the giant. He found that he could revolve a sling of length 0.600 m at the rate of 8.00 rev/s. If he increased the length to 0.900 m, he could revolve the sling only 6.00 times per second. (a) Which rate of rotation gives the greater speed for the stone at the end of the sling? (b) What is the centripetal acceleration of the stone at 8.00 rev/s? (c) What is the centripetal acceleration at 6.00 rev/s?

30. The astronaut orbiting the Earth in Figure P4.30 is preparing to dock with a Westar VI satellite. The satellite is in a circular orbit 600 km above the Earth's surface, where the free-fall acceleration is 8.21 m/s^2. The radius of the Earth is 6 400 km. Determine the speed of the satellite and the time required to complete one orbit around the Earth.

Figure P4.30 *(Courtesy of NASA)*

Section 4.5 Tangential and Radial Acceleration

31. A train slows down as it rounds a sharp horizontal curve, slowing from 90.0 km/h to 50.0 km/h in the 15.0 s that it takes to round the curve. The radius of the curve is 150 m. Compute the acceleration at the moment the train speed reaches 50.0 km/h. Assume that the train slows down at a uniform rate during the 15.0-s interval.

32. An automobile whose speed is increasing at a rate of 0.600 m/s^2 travels along a circular road of radius 20.0 m. When the instantaneous speed of the automobile is 4.00 m/s, find (a) the tangential acceleration component, (b) the radial acceleration component, and (c) the magnitude and direction of the total acceleration.

33. Figure P4.33 shows the total acceleration and velocity of a particle moving clockwise in a circle of radius 2.50 m at a given instant of time. At this instant, find (a) the radial acceleration, (b) the speed of the particle, and (c) its tangential acceleration.

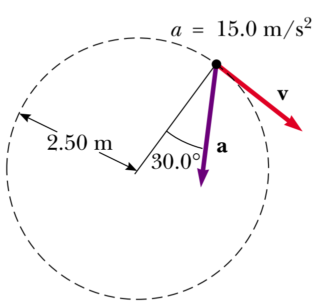


Figure P4.33

34. A student attaches a ball to the end of a string 0.600 m in length and then swings the ball in a vertical circle. The speed of the ball is 4.30 m/s at its highest point and 6.50 m/s at its lowest point. Find the acceleration of the ball when the string is vertical and the ball is at (a) its highest point and (b) its lowest point.

35. A ball swings in a vertical circle at the end of a rope 1.50 m long. When the ball is 36.9° past the lowest point and on its way up, its total acceleration is $(-22.5\mathbf{i} + 20.2\mathbf{j}) \text{ m/s}^2$. At that instant, (a) sketch a vector diagram showing the components of this acceleration, (b) determine the magnitude of its radial acceleration, and (c) determine the speed and velocity of the ball.

Section 4.6 Relative Velocity and Relative Acceleration

36. Heather in her Corvette accelerates at the rate of $(3.00\mathbf{i} - 2.00\mathbf{j}) \text{ m/s}^2$, while Jill in her Jaguar accelerates at $(1.00\mathbf{i} + 3.00\mathbf{j}) \text{ m/s}^2$. They both start from rest at the origin of an *xy* coordinate system. After 5.00 s, (a) what is Heather's speed with respect to Jill, (b) how far apart are they, and (c) what is Heather's acceleration relative to Jill?

37. A river has a steady speed of 0.500 m/s. A student swims upstream a distance of 1.00 km and swims back to the starting point. If the student can swim at a speed of 1.20 m/s in still water, how long does the trip take? Compare this with the time the trip would take if the water were still.

38. How long does it take an automobile traveling in the left lane at 60.0 km/h to pull alongside a car traveling in the right lane at 40.0 km/h if the cars' front bumpers are initially 100 m apart?

39. The pilot of an airplane notes that the compass indicates a heading due west. The airplane's speed relative to the air is 150 km/h. If there is a wind of 30.0 km/h toward the north, find the velocity of the airplane relative to the ground.

40. Two swimmers, Alan and Beth, start at the same point in a stream that flows with a speed v. Both move at the same speed c $(c > v)$ relative to the stream. Alan swims downstream a distance L and then upstream the same distance. Beth swims such that her motion relative to the ground is perpendicular to the banks of the stream. She swims a distance L in this direction and then back. The result of the motions of Alan and Beth is that they both return to the starting point. Which swimmer returns first? (*Note:* First guess at the answer.)

41. A child in danger of drowning in a river is being carried downstream by a current that has a speed of 2.50 km/h. The child is 0.600 km from shore and 0.800 km upstream of a boat landing when a rescue boat sets out. (a) If the boat proceeds at its maximum speed of 20.0 km/h relative to the water, what heading relative to the shore should the pilot take? (b) What angle does the boat velocity make with the shore? (c) How long does it take the boat to reach the child?

42. A bolt drops from the ceiling of a train car that is accelerating northward at a rate of 2.50 m/s^2. What is the acceleration of the bolt relative to (a) the train car and (b) the Earth?

43. A science student is riding on a flatcar of a train traveling along a straight horizontal track at a constant speed of 10.0 m/s. The student throws a ball into the air along a path that he judges to make an initial angle of 60.0° with the horizontal and to be in line with the track. The student's professor, who is standing on the ground nearby, observes the ball to rise vertically. How high does she see the ball rise?

ADDITIONAL PROBLEMS

44. A ball is thrown with an initial speed v_i at an angle θ_i with the horizontal. The horizontal range of the ball is R, and the ball reaches a maximum height $R/6$. In terms of R and g, find (a) the time the ball is in motion, (b) the ball's speed at the peak of its path, (c) the initial vertical component of its velocity, (d) its initial speed, and (e) the angle θ_i. (f) Suppose the ball is thrown at the same initial speed found in part (d) but at the angle appropriate for reaching the maximum height. Find this height. (g) Suppose the ball is thrown at the same initial speed but at the angle necessary for maximum range. Find this range.

45. As some molten metal splashes, one droplet flies off to the east with initial speed v_i at angle θ_i above the horizontal, and another droplet flies off to the west with the same speed at the same angle above the horizontal, as in Figure P4.45. In terms of v_i and θ_i, find the distance between the droplets as a function of time.

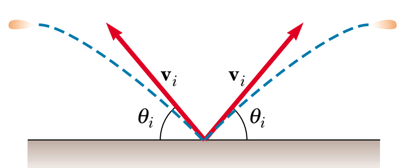


Figure P4.45

46. A ball on the end of a string is whirled around in a horizontal circle of radius 0.300 m. The plane of the circle is 1.20 m above the ground. The string breaks and the ball lands 2.00 m (horizontally) away from the point on the ground directly beneath the ball's location when the string breaks. Find the radial acceleration of the ball during its circular motion.

47. A projectile is fired up an incline (incline angle ϕ) with an initial speed v_i at an angle θ_i with respect to the horizontal ($\theta_i > \phi$), as shown in Figure P4.47. (a) Show that the projectile travels a distance d up the incline, where

$$d = \frac{2v_i^2 \cos\theta_i \sin(\theta_i - \phi)}{g\cos^2\phi}$$

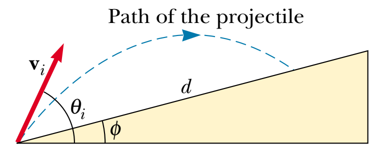


Figure P4.47

(b) For what value of θ_i is d a maximum, and what is that maximum value of d?

48. A student decides to measure the muzzle velocity of the pellets from his BB gun. He points the gun horizontally. On a vertical wall a distance x away from the gun, a target is placed. The shots hit the target a vertical distance y below the gun. (a) Show that the vertical displacement component of the pellets when traveling through the air is given by $y = Ax^2$, where A is a constant. (b) Express the constant A in terms of the initial velocity and the free-fall acceleration. (c) If $x = 3.00$ m and $y = 0.210$ m, what is the initial speed of the pellets?

49. A home run is hit in such a way that the baseball just clears a wall 21.0 m high, located 130 m from home plate. The ball is hit at an angle of 35.0° to the horizontal, and air resistance is negligible. Find (a) the initial speed of the ball, (b) the time it takes the ball to reach the wall, and (c) the velocity components and the speed of the ball when it reaches the wall. (Assume the ball is hit at a height of 1.00 m above the ground.)

50. An astronaut standing on the Moon fires a gun so that the bullet leaves the barrel initially moving in a horizontal direction. (a) What must be the muzzle speed of the bullet so that it travels completely around the Moon and returns to its original location? (b) How long does this trip around the Moon take? Assume that the free-fall acceleration on the Moon is one-sixth that on the Earth.

51. A pendulum of length 1.00 m swings in a vertical plane (Fig. 4.19). When the pendulum is in the two horizontal positions $\theta = 90°$ and $\theta = 270°$, its speed is 5.00 m/s. (a) Find the magnitude of the radial acceleration and tangential acceleration for these positions. (b) Draw a vector diagram to determine the direction of the total acceleration for these two positions. (c) Calculate the magnitude and direction of the total acceleration.

52. A basketball player who is 2.00 m tall is standing on the floor 10.0 m from the basket, as in Figure P4.52. If he shoots the ball at a 40.0° angle with the horizontal, at what initial speed must he throw so that it goes through the hoop without striking the backboard? The basket height is 3.05 m.

53. A particle has velocity components

$$v_x = +4 \text{ m/s} \qquad v_y = -(6 \text{ m/s}^2)t + 4 \text{ m/s}$$

Calculate the speed of the particle and the direction $\theta = \tan^{-1}(v_y/v_x)$ of the velocity vector at $t = 2.00$ s.

54. When baseball players throw the ball in from the outfield, they usually allow it to take one bounce before it reaches the infielder on the theory that the ball arrives

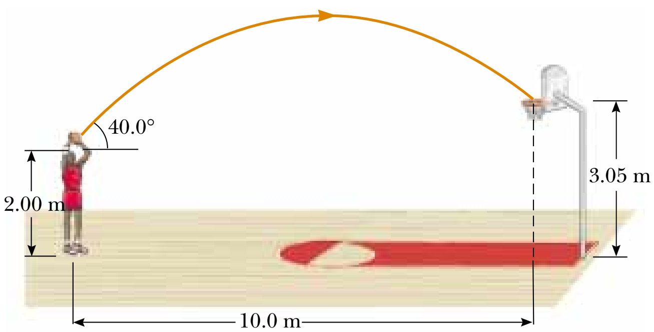


Figure P4.52

sooner that way. Suppose that the angle at which a bounced ball leaves the ground is the same as the angle at which the outfielder launched it, as in Figure P4.54, but that the ball's speed after the bounce is one half of what it was before the bounce. (a) Assuming the ball is always thrown with the same initial speed, at what angle θ should the ball be thrown in order to go the same distance D with one bounce (blue path) as a ball thrown upward at 45.0° with no bounce (green path)? (b) Determine the ratio of the times for the one-bounce and no-bounce throws.

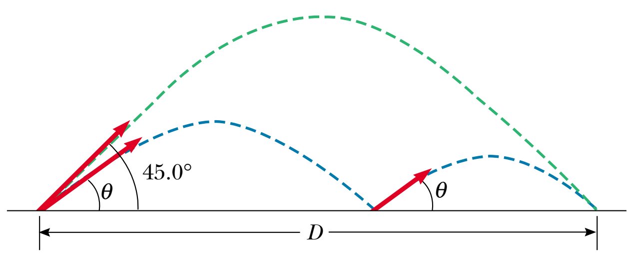


Figure P4.54

55. A boy can throw a ball a maximum horizontal distance of 40.0 m on a level field. How far can he throw the same ball vertically upward? Assume that his muscles give the ball the same speed in each case.

56. A boy can throw a ball a maximum horizontal distance of R on a level field. How far can he throw the same ball vertically upward? Assume that his muscles give the ball the same speed in each case.

57. A stone at the end of a sling is whirled in a vertical circle of radius 1.20 m at a constant speed $v_i = 1.50$ m/s as in Figure P4.57. The center of the string is 1.50 m above the ground. What is the range of the stone if it is released when the sling is inclined at 30.0° with the horizontal (a) at A? (b) at B? What is the acceleration of the stone (c) just before it is released at A? (d) just after it is released at A?

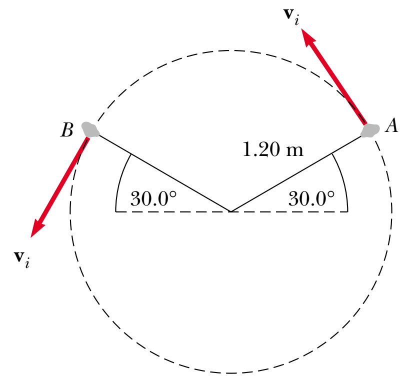


Figure P4.57

58. A quarterback throws a football straight toward a receiver with an initial speed of 20.0 m/s, at an angle of 30.0° above the horizontal. At that instant, the receiver is 20.0 m from the quarterback. In what direction and with what constant speed should the receiver run to catch the football at the level at which it was thrown?

59. A bomber is flying horizontally over level terrain, with a speed of 275 m/s relative to the ground, at an altitude of 3 000 m. Neglect the effects of air resistance. (a) How far will a bomb travel horizontally between its release from the plane and its impact on the ground? (b) If the plane maintains its original course and speed, where will it be when the bomb hits the ground? (c) At what angle from the vertical should the telescopic bombsight be set so that the bomb will hit the target seen in the sight at the time of release?

60. A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\mathbf{v}_i$ as in Figure P4.60. (a) What must be its minimum initial speed if the ball is never to hit the rock after it is kicked? (b) With this initial speed, how far from the base of the rock does the ball hit the ground?

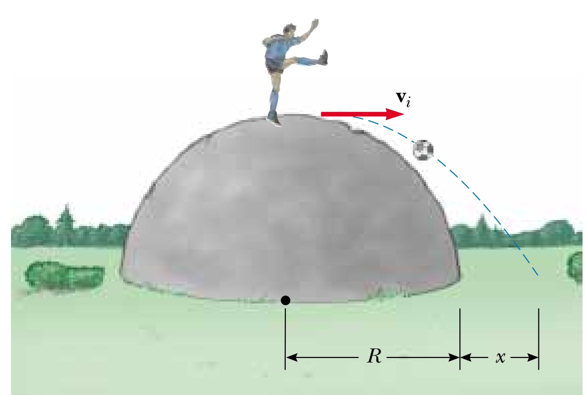


Figure P4.60

61. A hawk is flying horizontally at 10.0 m/s in a straight line, 200 m above the ground. A mouse it has been carrying struggles free from its grasp. The hawk continues on its path at the same speed for 2.00 s before attempting to retrieve its prey. To accomplish the retrieval, it dives in a straight line at constant speed and recaptures the mouse 3.00 m above the ground. (a) Assuming no air resistance, find the diving speed of the hawk. (b) What angle did the hawk make with the horizontal during its descent? (c) For how long did the mouse "enjoy" free fall?

62. A truck loaded with cannonball watermelons stops suddenly to avoid running over the edge of a washed-out bridge (Fig. P4.62). The quick stop causes a number of melons to fly off the truck. One melon rolls over the edge with an initial speed $v_i = 10.0$ m/s in the horizontal direction. A cross-section of the bank has the shape of the bottom half of a parabola with its vertex at the edge of the road, and with the equation $y^2 = 16x$, where x and y are measured in meters. What are the x and y coordinates of the melon when it splatters on the bank?

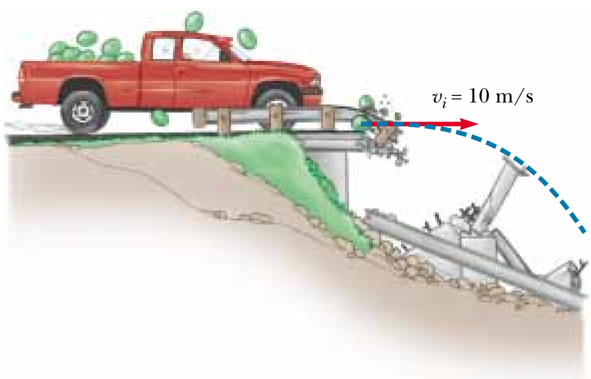


Figure P4.62

63. A catapult launches a rocket at an angle of 53.0° above the horizontal with an initial speed of 100 m/s. The rocket engine immediately starts a burn, and for 3.00 s the rocket moves along its initial line of motion with an acceleration of 30.0 m/s^2. Then its engine fails, and the rocket proceeds to move in free fall. Find (a) the maximum altitude reached by the rocket, (b) its total time of flight, and (c) its horizontal range.

64. A river flows with a uniform velocity **v**. A person in a motorboat travels 1.00 km upstream, at which time she passes a log floating by. Always with the same throttle setting, the boater continues to travel upstream for another 60.0 min and then returns downstream to her starting point, which she reaches just as the same log does. Find the velocity of the river. (*Hint:* The time of travel of the boat after it meets the log equals the time of travel of the log.)

WEB 65. A car is parked on a steep incline overlooking the ocean, where the incline makes an angle of 37.0° below the horizontal. The negligent driver leaves the car in neutral, and the parking brakes are defective. The car rolls from rest down the incline with a constant acceleration of 4.00 m/s^2, traveling 50.0 m to the edge of a vertical cliff. The cliff is 30.0 m above the ocean. Find (a) the speed of the car when it reaches the edge of the cliff and the time it takes to get there, (b) the velocity of the car when it lands in the ocean, (c) the total time the car is in motion, and (d) the position of the car when it lands in the ocean, relative to the base of the cliff.

66. The determined coyote is out once more to try to capture the elusive roadrunner. The coyote wears a pair of Acme jet-powered roller skates, which provide a constant horizontal acceleration of 15.0 m/s^2 (Fig. P4.66). The coyote starts off at rest 70.0 m from the edge of a cliff at the instant the roadrunner zips past him in the direction of the cliff. (a) If the roadrunner moves with constant speed, determine the minimum speed he must have to reach the cliff before the coyote. At the brink of the cliff, the roadrunner escapes by making a sudden turn, while the coyote continues straight ahead. (b) If the cliff is 100 m above the floor of a canyon, determine where the coyote lands in the canyon (assume his skates remain horizontal and continue to operate when he is in "flight"). (c) Determine the components of the coyote's impact velocity.

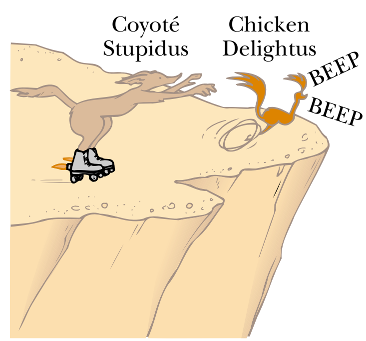


Figure P4.66

67. A skier leaves the ramp of a ski jump with a velocity of 10.0 m/s, 15.0° above the horizontal, as in Figure P4.67. The slope is inclined at 50.0°, and air resistance is negligible. Find (a) the distance from the ramp to where the jumper lands and (b) the velocity components just before the landing. (How do you think the results might be affected if air resistance were included? Note that jumpers lean forward in the shape of an airfoil, with their hands at their sides, to increase their distance. Why does this work?)

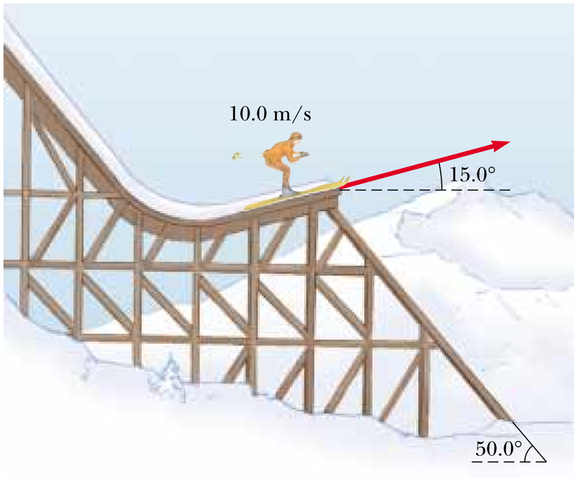


Figure P4.67

68. Two soccer players, Mary and Jane, begin running from nearly the same point at the same time. Mary runs in an easterly direction at 4.00 m/s, while Jane takes off in a direction 60.0° north of east at 5.40 m/s. (a) How long is it before they are 25.0 m apart? (b) What is the velocity of Jane relative to Mary? (c) How far apart are they after 4.00 s?

69. Do not hurt yourself; do not strike your hand against anything. Within these limitations, describe what you do to give your hand a large acceleration. Compute an order-of-magnitude estimate of this acceleration, stating the quantities you measure or estimate and their values.

70. An enemy ship is on the western side of a mountain island, as shown in Figure P4.70. The enemy ship can maneuver to within 2 500 m of the 1 800-m-high mountain peak and can shoot projectiles with an initial speed of 250 m/s. If the eastern shoreline is horizontally 300 m from the peak, what are the distances from the eastern shore at which a ship can be safe from the bombardment of the enemy ship?

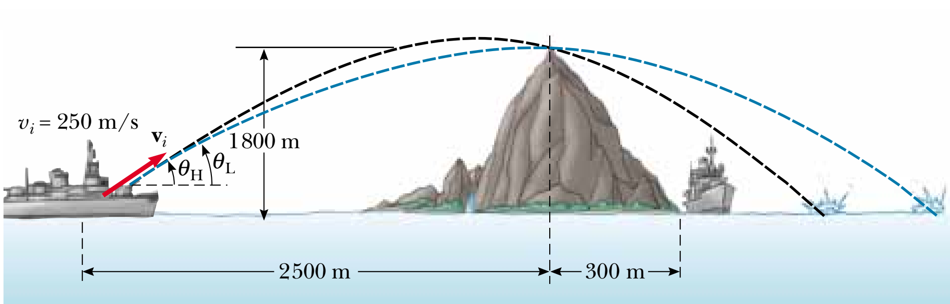


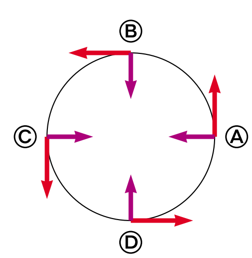
Figure P4.70

Answers to Quick Quizzes

4.1 (a) Because acceleration occurs whenever the velocity changes in any way—with an increase or decrease in speed, a change in direction, or both—the brake pedal can also be considered an accelerator because it causes the car to slow down. The steering wheel is also an accelerator because it changes the direction of the velocity vector. (b) When the car is moving with constant speed, the gas pedal is not causing an acceleration; it is an accelerator only when it causes a change in the speedometer reading.

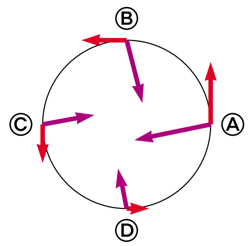
4.2 (a) At only one point—the peak of the trajectory—are the velocity and acceleration vectors perpendicular to each other. (b) If the object is thrown straight up or down, **v** and **a** are parallel to each other throughout the downward motion. Otherwise, the velocity and acceleration vectors are never parallel to each other. (c) The greater the maximum height, the longer it takes the projectile to reach that altitude and then fall back down from it. So, as the angle increases from 0° to 90°, the time of flight increases. Therefore, the 15° angle gives the shortest time of flight, and the 75° angle gives the longest.

4.3 (a) Because the object is moving with a constant speed, the velocity vector is always the same length; because the motion is circular, this vector is always tangent to the circle. The only acceleration is that which changes the direction of the velocity vector; it points radially inward.



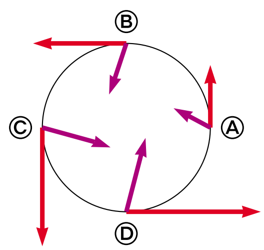
(a)

(b) Now there is a component of the acceleration vector that is tangent to the circle and points in the direction opposite the velocity. As a result, the acceleration vector does not point toward the center. The object is slowing down, and so the velocity vectors become shorter and shorter.



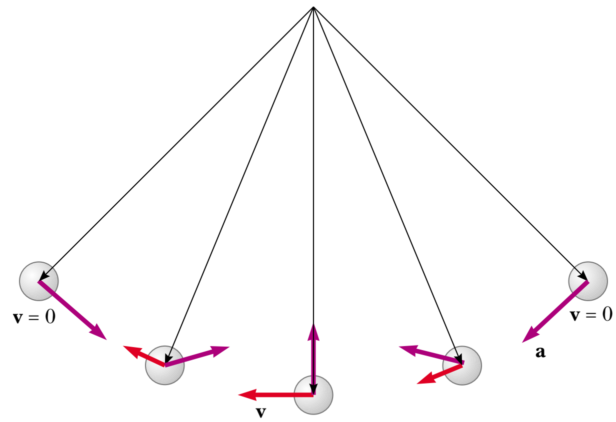
(b)

(c) Now the tangential component of the acceleration points in the same direction as the velocity. The object is speeding up, and so the velocity vectors become longer and longer.



(c)

4.4 The motion diagram is as shown below. Note that each position vector points from the pivot point at the center of the circle to the position of the ball.



4.5 (a) The passenger sees the coffee pouring nearly vertically into the cup, just as if she were standing on the ground pouring it. (b) The stationary observer sees the coffee moving in a parabolic path with a constant horizontal velocity of 60 mi/h (=88 ft/s) and a downward acceleration of $-g$. If it takes the coffee 0.10 s to reach the cup, the stationary observer sees the coffee moving 8.8 ft horizontally before it hits the cup! (c) If the car slows suddenly, the coffee reaches the place where the cup *would have been* had there been no change in velocity and continues falling because the cup has not yet reached that location. If the car rapidly speeds up, the coffee falls behind the cup. If the car accelerates sideways, the coffee again ends up somewhere other than the cup.

PUZZLER

The *Spirit of Akron* is an airship that is more than 60 m long. When it is parked at an airport, one person can easily support it overhead using a single hand. Nonetheless, it is impossible for even a very strong adult to move the ship abruptly. What property of this huge airship makes it very difficult to cause any sudden changes in its motion? *(Courtesy of Edward E. Ogden)*

web

For more information about the airship, visit **http://www.goodyear.com/us/blimp/index.html**

chapter

5 The Laws of Motion

Chapter Outline

In Chapters 2 and 4, we described motion in terms of displacement, velocity, and acceleration without considering what might cause that motion. What might cause one particle to remain at rest and another particle to accelerate? In this chapter, we investigate what causes changes in motion. The two main factors we need to consider are the forces acting on an object and the mass of the object. We discuss the three basic laws of motion, which deal with forces and masses and were formulated more than three centuries ago by Isaac Newton. Once we understand these laws, we can answer such questions as "What mechanism changes motion?" and "Why do some objects accelerate more than others?"

5.1 THE CONCEPT OF FORCE

Everyone has a basic understanding of the concept of force from everyday experience. When you push your empty dinner plate away, you exert a force on it. Similarly, you exert a force on a ball when you throw or kick it. In these examples, the word *force* is associated with muscular activity and some change in the velocity of an object. Forces do not always cause motion, however. For example, as you sit reading this book, the force of gravity acts on your body and yet you remain stationary. As a second example, you can push (in other words, exert a force) on a large boulder and not be able to move it.

What force (if any) causes the Moon to orbit the Earth? Newton answered this and related questions by stating that forces are what cause any change in the velocity of an object. Therefore, if an object moves with uniform motion (constant velocity), no force is required for the motion to be maintained. The Moon's velocity is not constant because it moves in a nearly circular orbit around the Earth. We now know that this change in velocity is caused by the force exerted on the Moon by the Earth. Because only a force can cause a change in velocity, we can think of force as *that which causes a body to accelerate.* In this chapter, we are concerned with the relationship between the force exerted on an object and the acceleration of that object.

A body accelerates because of an external force

What happens when several forces act simultaneously on an object? In this case, the object accelerates only if the net force acting on it is not equal to zero. The **net force** acting on an object is defined as the vector sum of all forces acting on the object. (We sometimes refer to the net force as the *total force,* the *resultant force,* or the *unbalanced force.*) **If the net force exerted on an object is zero, then the acceleration of the object is zero and its velocity remains constant.** That is, if the net force acting on the object is zero, then the object either remains at rest or continues to move with constant velocity. When the velocity of an object is constant (including the case in which the object remains at rest), the object is said to be in **equilibrium.**

Definition of equilibrium

When a coiled spring is pulled, as in Figure 5.1a, the spring stretches. When a stationary cart is pulled suffcently hard that friction is overcome, as in Figure 5.1b, the cart moves. When a football is kicked, as in Figure 5.1c, it is both deformed and set in motion. These situations are all examples of a class of forces called *contact forces.* That is, they involve physical contact between two objects. Other examples of contact forces are the force exerted by gas molecules on the walls of a container and the force exerted by your feet on the floor.

Another class of forces, known as *field forces,* do not involve physical contact between two objects but instead act through empty space. The force of gravitational attraction between two objects, illustrated in Figure 5.1d, is an example of this class of force. This gravitational force keeps objects bound to the Earth. The plan-

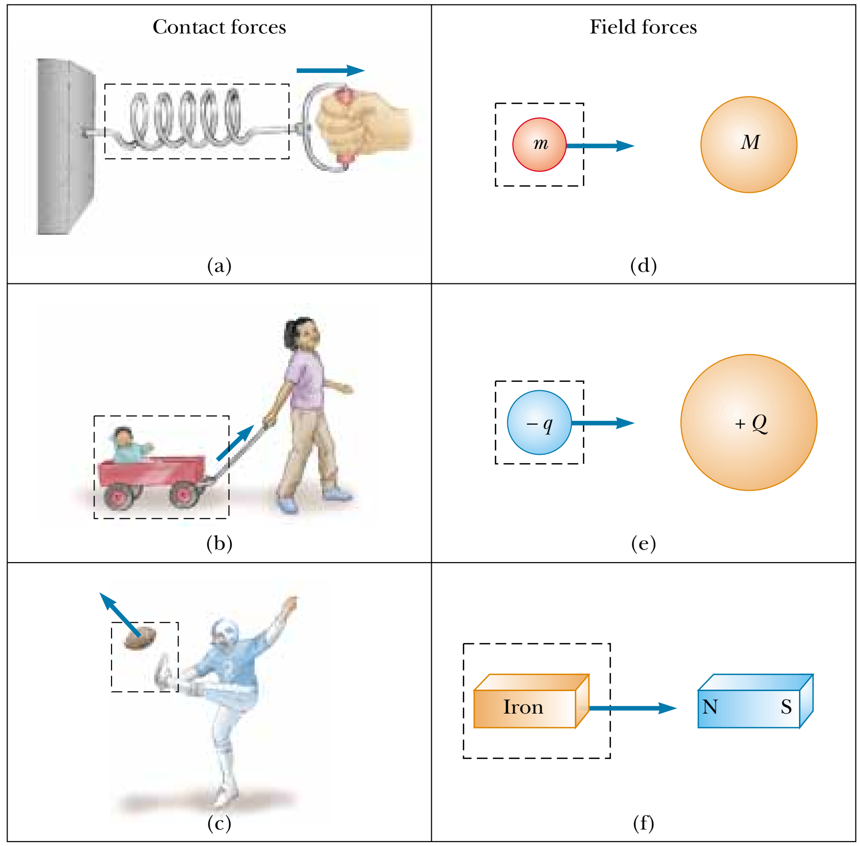


Figure 5.1 Some examples of applied forces. In each case a force is exerted on the object within the boxed area. Some agent in the environment external to the boxed area exerts a force on the object.

ets of our Solar System are bound to the Sun by the action of gravitational forces. Another common example of a field force is the electric force that one electric charge exerts on another, as shown in Figure 5.1e. These charges might be those of the electron and proton that form a hydrogen atom. A third example of a field force is the force a bar magnet exerts on a piece of iron, as shown in Figure 5.1f. The forces holding an atomic nucleus together also are field forces but are very short in range. They are the dominating interaction for particle separations of the order of 10^{-15} m.

Early scientists, including Newton, were uneasy with the idea that a force can act between two disconnected objects. To overcome this conceptual problem, Michael Faraday (1791–1867) introduced the concept of a *field*. According to this approach, when object 1 is placed at some point P near object 2, we say that object 1 interacts with object 2 by virtue of the gravitational field that exists at P. The gravitational field at P is created by object 2. Likewise, a gravitational field created by object 1 exists at the position of object 2. In fact, all objects create a gravitational field in the space around themselves.

The distinction between contact forces and field forces is not as sharp as you may have been led to believe by the previous discussion. When examined at the atomic level, all the forces we classify as contact forces turn out to be caused by electric (field) forces of the type illustrated in Figure 5.1e. Nevertheless, in developing models for macroscopic phenomena, it is convenient to use both classifications of forces. The only known *fundamental* forces in nature are all field forces: (1) gravitational forces between objects, (2) electromagnetic forces between electric charges, (3) strong nuclear forces between subatomic particles, and (4) weak nuclear forces that arise in certain radioactive decay processes. In classical physics, we are concerned only with gravitational and electromagnetic forces.

Measuring the Strength of a Force

It is convenient to use the deformation of a spring to measure force. Suppose we apply a vertical force to a spring scale that has a fixed upper end, as shown in Figure 5.2a. The spring elongates when the force is applied, and a pointer on the scale reads the value of the applied force. We can calibrate the spring by defining the unit force $\mathbf{F}_1$ as the force that produces a pointer reading of 1.00 cm. (Because force is a vector quantity, we use the bold-faced symbol $\mathbf{F}$.) If we now apply a different downward force $\mathbf{F}_2$ whose magnitude is 2 units, as seen in Figure 5.2b, the pointer moves to 2.00 cm. Figure 5.2c shows that the combined effect of the two collinear forces is the sum of the effects of the individual forces.

Now suppose the two forces are applied simultaneously with $\mathbf{F}_1$ downward and $\mathbf{F}_2$ horizontal, as illustrated in Figure 5.2d. In this case, the pointer reads $\sqrt{5\text{ cm}^2} = 2.24$ cm. The single force $\mathbf{F}$ that would produce this same reading is the sum of the two vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, as described in Figure 5.2d. That is, $|\mathbf{F}| = \sqrt{F_1{}^2 + F_2{}^2} = 2.24$ units, and its direction is $\theta = \tan^{-1}(-0.500) = -26.6°$. **Because forces are vector quantities, you must use the rules of vector addition to obtain the net force acting on an object.**

QuickLab

Find a tennis ball, two drinking straws, and a friend. Place the ball on a table. You and your friend can each apply a force to the ball by blowing through the straws (held horizontally a few centimeters above the table) so that the air rushing out strikes the ball. Try a variety of configurations: Blow in opposite directions against the ball, blow in the same direction, blow at right angles to each other, and so forth. Can you verify the vector nature of the forces?

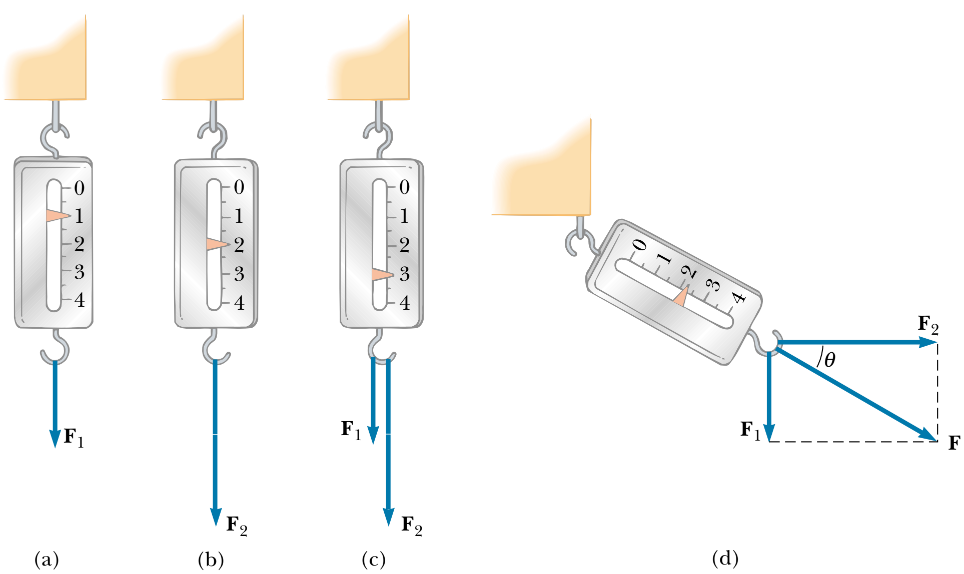


Figure 5.2 The vector nature of a force is tested with a spring scale. (a) A downward force $\mathbf{F}_1$ elongates the spring 1 cm. (b) A downward force $\mathbf{F}_2$ elongates the spring 2 cm. (c) When $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied simultaneously, the spring elongates by 3 cm. (d) When $\mathbf{F}_1$ is downward and $\mathbf{F}_2$ is horizontal, the combination of the two forces elongates the spring $\sqrt{1^2 + 2^2}$ cm $= \sqrt{5}$ cm.

5.2 NEWTON'S FIRST LAW AND INERTIAL FRAMES

4.2

Before we state Newton's first law, consider the following simple experiment. Suppose a book is lying on a table. Obviously, the book remains at rest. Now imagine that you push the book with a horizontal force great enough to overcome the force of friction between book and table. (This force you exert, the force of friction, and any other forces exerted on the book by other objects are referred to as *external forces.*) You can keep the book in motion with constant velocity by applying a force that is just equal in magnitude to the force of friction and acts in the opposite direction. If you then push harder so that the magnitude of your applied force exceeds the magnitude of the force of friction, the book accelerates. If you stop pushing, the book stops after moving a short distance because the force of friction retards its motion. Suppose you now push the book across a smooth, highly waxed floor. The book again comes to rest after you stop pushing but not as quickly as before. Now imagine a floor so highly polished that friction is absent; in this case, the book, once set in motion, moves until it hits a wall.

QuickLab

Use a drinking straw to impart a strong, short-duration burst of air against a tennis ball as it rolls along a tabletop. Make the force perpendicular to the ball's path. What happens to the ball's motion? What is different if you apply a continuous force (constant magnitude and direction) that is directed along the direction of motion?

Before about 1600, scientists felt that the natural state of matter was the state of rest. Galileo was the first to take a different approach to motion and the natural state of matter. He devised thought experiments, such as the one we just discussed for a book on a frictionless surface, and concluded that it is not the nature of an object to stop once set in motion: rather, it is its nature to *resist changes in its motion.* In his words, "Any velocity once imparted to a moving body will be rigidly maintained as long as the external causes of retardation are removed."

This new approach to motion was later formalized by Newton in a form that has come to be known as **Newton's first law of motion:**

Newton's first law

In the absence of external forces, an object at rest remains at rest and an object in motion continues in motion with a constant velocity (that is, with a constant speed in a straight line).

In simpler terms, we can say that **when no force acts on an object, the acceleration of the object is zero.** If nothing acts to change the object's motion, then its velocity does not change. From the first law, we conclude that any *isolated object* (one that does not interact with its environment) is either at rest or moving with constant velocity. The tendency of an object to resist any attempt to change its velocity is called the **inertia** of the object. Figure 5.3 shows one dramatic example of a consequence of Newton's first law.

Definition of inertia

Another example of uniform (constant-velocity) motion on a nearly frictionless surface is the motion of a light disk on a film of air (the lubricant), as shown in Figure 5.4. If the disk is given an initial velocity, it coasts a great distance before stopping.

Finally, consider a spaceship traveling in space and far removed from any planets or other matter. The spaceship requires some propulsion system to change its velocity. However, if the propulsion system is turned off when the spaceship reaches a velocity **v**, the ship coasts at that constant velocity and the astronauts get a free ride (that is, no propulsion system is required to keep them moving at the velocity **v**).

Inertial Frames

As we saw in Section 4.6, a moving object can be observed from any number of reference frames. Newton's first law, sometimes called the *law of inertia,* defines a special set of reference frames called *inertial frames.* **An inertial frame of reference**

Definition of inertial frame

Figure 5.3 Unless a net external force acts on it, an object at rest remains at rest and an object in motion continues in motion with constant velocity. In this case, the wall of the building did not exert a force on the moving train that was large enough to stop it.

is one that is not accelerating. Because Newton's first law deals only with objects that are not accelerating, it holds only in inertial frames. Any reference frame that moves with constant velocity relative to an inertial frame is itself an inertial frame. (The Galilean transformations given by Equations 4.20 and 4.21 relate positions and velocities between two inertial frames.)

A reference frame that moves with constant velocity relative to the distant stars is the best approximation of an inertial frame, and for our purposes we can consider planet Earth as being such a frame. The Earth is not really an inertial frame because of its orbital motion around the Sun and its rotational motion about its own axis. As the Earth travels in its nearly circular orbit around the Sun, it experiences an acceleration of about $4.4 \times 10^{-3}\ \mathrm{m/s^2}$ directed toward the Sun. In addition, because the Earth rotates about its own axis once every 24 h, a point on the equator experiences an additional acceleration of $3.37 \times 10^{-2}\ \mathrm{m/s^2}$ directed toward the center of the Earth. However, these accelerations are small compared with g and can often be neglected. For this reason, we assume that the Earth is an inertial frame, as is any other frame attached to it.

If an object is moving with constant velocity, an observer in one inertial frame (say, one at rest relative to the object) claims that the acceleration of the object and the resultant force acting on it are zero. An observer in *any other* inertial frame also finds that $\mathbf{a} = 0$ and $\Sigma\mathbf{F} = 0$ for the object. According to the first law, a body at rest and one moving with constant velocity are equivalent. A passenger in a car moving along a straight road at a constant speed of 100 km/h can easily pour coffee into a cup. But if the driver steps on the gas or brake pedal or turns the steering wheel while the coffee is being poured, the car accelerates and it is no longer an inertial frame. The laws of motion do not work as expected, and the coffee ends up in the passenger's lap!

Isaac Newton English physicist and mathematician (1642–1727) Isaac Newton was one of the most brilliant scientists in history. Before the age of 30, he formulated the basic concepts and laws of mechanics, discovered the law of universal gravitation, and invented the mathematical methods of calculus. As a consequence of his theories, Newton was able to explain the motions of the planets, the ebb and flow of the tides, and many special features of the motions of the Moon and the Earth. He also interpreted many fundamental observations concerning the nature of light. His contributions to physical theories dominated scientific thought for two centuries and remain important today. *(Giraudon/Art Resource)*

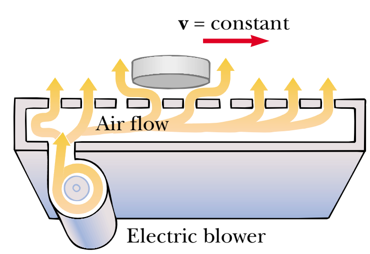


Figure 5.4 Air hockey takes advantage of Newton's first law to make the game more exciting.

Quick Quiz 5.1

True or false: (a) It is possible to have motion in the absence of a force. (b) It is possible to have force in the absence of motion.

5.3 MASS

4.3 Imagine playing catch with either a basketball or a bowling ball. Which ball is more likely to keep moving when you try to catch it? Which ball has the greater tendency to remain motionless when you try to throw it? Because the bowling ball is more resistant to changes in its velocity, we say it has greater inertia than the basketball. As noted in the preceding section, inertia is a measure of how an object responds to an external force.

Definition of mass

Mass is that property of an object that specifies how much inertia the object has, and as we learned in Section 1.1, the SI unit of mass is the kilogram. The greater the mass of an object, the less that object accelerates under the action of an applied force. For example, if a given force acting on a 3-kg mass produces an acceleration of 4 m/s^2, then the same force applied to a 6-kg mass produces an acceleration of 2 m/s^2.

To describe mass quantitatively, we begin by comparing the accelerations a given force produces on different objects. Suppose a force acting on an object of mass m_1 produces an acceleration $\mathbf{a}_1$, and the *same force* acting on an object of mass m_2 produces an acceleration $\mathbf{a}_2$. The ratio of the two masses is defined as the *inverse* ratio of the magnitudes of the accelerations produced by the force:

$$\frac{m_1}{m_2} \equiv \frac{a_2}{a_1} \tag{5.1}$$

If one object has a known mass, the mass of the other object can be obtained from acceleration measurements.

Mass is an inherent property of an object and is independent of the object's surroundings and of the method used to measure it. Also, **mass is a scalar quantity** and thus obeys the rules of ordinary arithmetic. That is, several masses can be combined in simple numerical fashion. For example, if you combine a 3-kg mass with a 5-kg mass, their total mass is 8 kg. We can verify this result experimentally by comparing the acceleration that a known force gives to several objects separately with the acceleration that the same force gives to the same objects combined as one unit.

Mass and weight are different quantities

Mass should not be confused with weight. **Mass and weight are two different quantities.** As we see later in this chapter, the weight of an object is equal to the magnitude of the gravitational force exerted on the object and varies with location. For example, a person who weighs 180 lb on the Earth weighs only about 30 lb on the Moon. On the other hand, the mass of a body is the same everywhere: an object having a mass of 2 kg on the Earth also has a mass of 2 kg on the Moon.

5.4 NEWTON'S SECOND LAW

4.4 Newton's first law explains what happens to an object when no forces act on it. It either remains at rest or moves in a straight line with constant speed. Newton's second law answers the question of what happens to an object that has a nonzero resultant force acting on it.

Imagine pushing a block of ice across a frictionless horizontal surface. When you exert some horizontal force **F**, the block moves with some acceleration **a**. If you apply a force twice as great, the acceleration doubles. If you increase the applied force to 3**F**, the acceleration triples, and so on. From such observations, we conclude that **the acceleration of an object is directly proportional to the resultant force acting on it.**

The acceleration of an object also depends on its mass, as stated in the preceding section. We can understand this by considering the following experiment. If you apply a force **F** to a block of ice on a frictionless surface, then the block undergoes some acceleration **a**. If the mass of the block is doubled, then the same applied force produces an acceleration **a**/2. If the mass is tripled, then the same applied force produces an acceleration **a**/3, and so on. According to this observation, we conclude that **the magnitude of the acceleration of an object is inversely proportional to its mass.**

These observations are summarized in **Newton's second law:**

> The acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.

Newton's second law

Thus, we can relate mass and force through the following mathematical statement of Newton's second law:[1]

$$\sum \mathbf{F} = m\mathbf{a} \tag{5.2}$$

Note that this equation is a vector expression and hence is equivalent to three component equations:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z \tag{5.3}$$

Newton's second law—component form

Quick Quiz 5.2

Is there any relationship between the net force acting on an object and the direction in which the object moves?

Unit of Force

The SI unit of force is the **newton,** which is defined as the force that, when acting on a 1-kg mass, produces an acceleration of 1 m/s^2. From this definition and Newton's second law, we see that the newton can be expressed in terms of the following fundamental units of mass, length, and time:

$$1 \text{ N} \equiv 1 \text{ kg}\cdot\text{m/s}^2 \tag{5.4}$$

Definition of newton

In the British engineering system, the unit of force is the **pound,** which is defined as the force that, when acting on a 1-slug mass,[2] produces an acceleration of 1 ft/s^2:

$$1 \text{ lb} \equiv 1 \text{ slug}\cdot\text{ft/s}^2 \tag{5.5}$$

A convenient approximation is that 1 N $\approx \frac{1}{4}$ lb.

[1] Equation 5.2 is valid only when the speed of the object is much less than the speed of light. We treat the relativistic situation in Chapter 39.

[2] The *slug* is the unit of mass in the British engineering system and is that system's counterpart of the SI unit the *kilogram*. Because most of the calculations in our study of classical mechanics are in SI units, the slug is seldom used in this text.

TABLE 5.1 Units of Force, Mass, and Acceleration[a]

System of Units	Mass	Acceleration	Force
SI	kg	m/s^2	$N = kg \cdot m/s^2$
British engineering	slug	ft/s^2	$lb = slug \cdot ft/s^2$

[a] 1 N = 0.225 lb.

The units of force, mass, and acceleration are summarized in Table 5.1.

We can now understand how a single person can hold up an airship but is not able to change its motion abruptly, as stated at the beginning of the chapter. The mass of the blimp is greater than 6 800 kg. In order to make this large mass accelerate appreciably, a very large force is required—certainly one much greater than a human can provide.

EXAMPLE 5.1 An Accelerating Hockey Puck

A hockey puck having a mass of 0.30 kg slides on the horizontal, frictionless surface of an ice rink. Two forces act on the puck, as shown in Figure 5.5. The force $\mathbf{F}_1$ has a magnitude of 5.0 N, and the force $\mathbf{F}_2$ has a magnitude of 8.0 N. Determine both the magnitude and the direction of the puck's acceleration.

Solution The resultant force in the x direction is

$$\sum F_x = F_{1x} + F_{2x} = F_1 \cos(-20°) + F_2 \cos 60°$$
$$= (5.0 \text{ N})(0.940) + (8.0 \text{ N})(0.500) = 8.7 \text{ N}$$

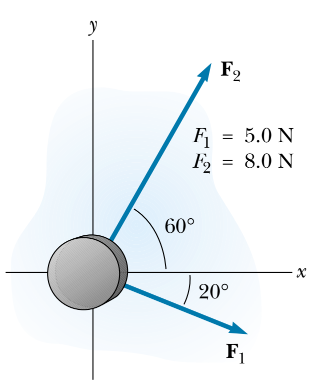


Figure 5.5 A hockey puck moving on a frictionless surface accelerates in the direction of the resultant force $\mathbf{F}_1 + \mathbf{F}_2$.

The resultant force in the y direction is

$$\sum F_y = F_{1y} + F_{2y} = F_1 \sin(-20°) + F_2 \sin 60°$$
$$= (5.0 \text{ N})(-0.342) + (8.0 \text{ N})(0.866) = 5.2 \text{ N}$$

Now we use Newton's second law in component form to find the x and y components of acceleration:

$$a_x = \frac{\sum F_x}{m} = \frac{8.7 \text{ N}}{0.30 \text{ kg}} = 29 \text{ m/s}^2$$

$$a_y = \frac{\sum F_y}{m} = \frac{5.2 \text{ N}}{0.30 \text{ kg}} = 17 \text{ m/s}^2$$

The acceleration has a magnitude of

$$a = \sqrt{(29)^2 + (17)^2} \text{ m/s}^2 = \boxed{34 \text{ m/s}^2}$$

and its direction relative to the positive x axis is

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{17}{29}\right) = \boxed{30°}$$

We can graphically add the vectors in Figure 5.5 to check the reasonableness of our answer. Because the acceleration vector is along the direction of the resultant force, a drawing showing the resultant force helps us check the validity of the answer.

Exercise Determine the components of a third force that, when applied to the puck, causes it to have zero acceleration.

Answer $F_{3x} = -8.7$ N, $F_{3y} = -5.2$ N.

5.5 THE FORCE OF GRAVITY AND WEIGHT

We are well aware that all objects are attracted to the Earth. The attractive force exerted by the Earth on an object is called the **force of gravity** $\mathbf{F}_g$. This force is directed toward the center of the Earth,[3] and its magnitude is called the **weight** of the object.

Definition of weight

We saw in Section 2.6 that a freely falling object experiences an acceleration $\mathbf{g}$ acting toward the center of the Earth. Applying Newton's second law $\Sigma\mathbf{F} = m\mathbf{a}$ to a freely falling object of mass m, with $\mathbf{a} = \mathbf{g}$ and $\Sigma\mathbf{F} = \mathbf{F}_g$, we obtain

$$\mathbf{F}_g = m\mathbf{g} \tag{5.6}$$

Thus, the weight of an object, being defined as the magnitude of $\mathbf{F}_g$, is mg. (You should not confuse the italicized symbol g for gravitational acceleration with the nonitalicized symbol g used as the abbreviation for "gram.")

QuickLab

Drop a pen and your textbook simultaneously from the same height and watch as they fall. How can they have the same acceleration when their weights are so different?

Because it depends on g, weight varies with geographic location. Hence, weight, unlike mass, is not an inherent property of an object. Because g decreases with increasing distance from the center of the Earth, bodies weigh less at higher altitudes than at sea level. For example, a 1 000-kg palette of bricks used in the construction of the Empire State Building in New York City weighed about 1 N less by the time it was lifted from sidewalk level to the top of the building. As another example, suppose an object has a mass of 70.0 kg. Its weight in a location where $g = 9.80\ \text{m/s}^2$ is $F_g = mg = 686$ N (about 150 lb). At the top of a mountain, however, where $g = 9.77\ \text{m/s}^2$, its weight is only 684 N. Therefore, if you want to lose weight without going on a diet, climb a mountain or weigh yourself at 30 000 ft during an airplane flight!

Because weight $= F_g = mg$, we can compare the masses of two objects by measuring their weights on a spring scale. At a given location, the ratio of the weights of two objects equals the ratio of their masses.

The life-support unit strapped to the back of astronaut Edwin Aldrin weighed 300 lb on the Earth. During his training, a 50-lb mock-up was used. Although this effectively simulated the reduced weight the unit would have on the Moon, it did not correctly mimic the unchanging mass. It was just as difficult to accelerate the unit (perhaps by jumping or twisting suddenly) on the Moon as on the Earth.

[3] This statement ignores the fact that the mass distribution of the Earth is not perfectly spherical.

Conceptual Example 5.2 How Much Do You Weigh in an Elevator?

You have most likely had the experience of standing in an elevator that accelerates upward as it moves toward a higher floor. In this case, you feel heavier. In fact, if you are standing on a bathroom scale at the time, the scale measures a force magnitude that is greater than your weight. Thus, you have tactile and measured evidence that leads you to believe you are heavier in this situation. *Are* you heavier?

Solution No, your weight is unchanged. To provide the acceleration upward, the floor or scale must exert on your feet an upward force that is greater in magnitude than your weight. It is this greater force that you feel, which you interpret as feeling heavier. The scale reads this upward force, not your weight, and so its reading increases.

Quick Quiz 5.3

A baseball of mass m is thrown upward with some initial speed. If air resistance is neglected, what forces are acting on the ball when it reaches (a) half its maximum height and (b) its maximum height?

5.6 NEWTON'S THIRD LAW

4.5 If you press against a corner of this textbook with your fingertip, the book pushes back and makes a small dent in your skin. If you push harder, the book does the same and the dent in your skin gets a little larger. This simple experiment illustrates a general principle of critical importance known as **Newton's third law:**

Newton's third law

If two objects interact, the force $\mathbf{F}_{12}$ exerted by object 1 on object 2 is equal in magnitude to and opposite in direction to the force $\mathbf{F}_{21}$ exerted by object 2 on object 1:

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \quad (5.7)$$

This law, which is illustrated in Figure 5.6a, states that a force that affects the motion of an object must come from a second, *external,* object. The external object, in turn, is subject to an equal-magnitude but oppositely directed force exerted on it.

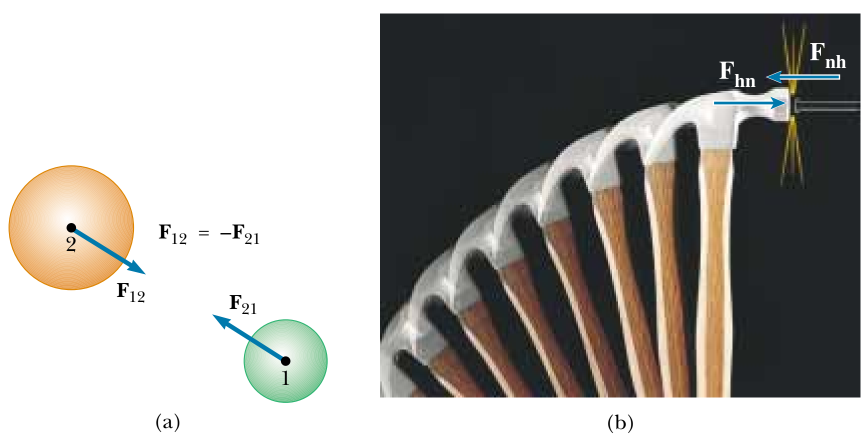
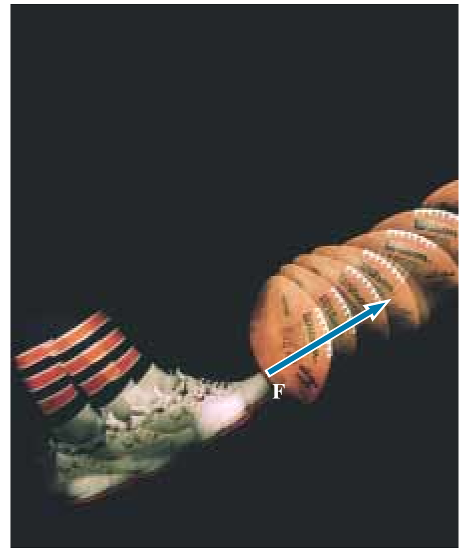


Figure 5.6 Newton's third law. (a) The force $\mathbf{F}_{12}$ exerted by object 1 on object 2 is equal in magnitude to and opposite in direction to the force $\mathbf{F}_{21}$ exerted by object 2 on object 1. (b) The force $\mathbf{F}_{hn}$ exerted by the hammer on the nail is equal to and opposite the force $\mathbf{F}_{nh}$ exerted by the nail on the hammer.

This is equivalent to stating that **a single isolated force cannot exist.** The force that object 1 exerts on object 2 is sometimes called the *action force,* while the force object 2 exerts on object 1 is called the *reaction force.* In reality, either force can be labeled the action or the reaction force. **The action force is equal in magnitude to the reaction force and opposite in direction. In all cases, the action and reaction forces act on different objects.** For example, the force acting on a freely falling projectile is $\mathbf{F}_g = m\mathbf{g}$, which is the force of gravity exerted by the Earth on the projectile. The reaction to this force is the force exerted by the projectile on the Earth, $\mathbf{F}_g' = -\mathbf{F}_g$. The reaction force $\mathbf{F}_g'$ accelerates the Earth toward the projectile just as the action force $\mathbf{F}_g$ accelerates the projectile toward the Earth. However, because the Earth has such a great mass, its acceleration due to this reaction force is negligibly small.

Another example of Newton's third law is shown in Figure 5.6b. The force exerted by the hammer on the nail (the action force $\mathbf{F}_{\text{hn}}$) is equal in magnitude and opposite in direction to the force exerted by the nail on the hammer (the reaction force $\mathbf{F}_{\text{nh}}$). It is this latter force that causes the hammer to stop its rapid forward motion when it strikes the nail.

You experience Newton's third law directly whenever you slam your fist against a wall or kick a football. You should be able to identify the action and reaction forces in these cases.



Compression of a football as the force exerted by a player's foot sets the ball in motion.

Quick Quiz 5.4

A person steps from a boat toward a dock. Unfortunately, he forgot to tie the boat to the dock, and the boat scoots away as he steps from it. Analyze this situation in terms of Newton's third law.

The force of gravity $\mathbf{F}_g$ was defined as the attractive force the Earth exerts on an object. If the object is a TV at rest on a table, as shown in Figure 5.7a, why does the TV not accelerate in the direction of $\mathbf{F}_g$? The TV does not accelerate because the table holds it up. What is happening is that the table exerts on the TV an upward force $\mathbf{n}$ called the **normal force**.[4] The normal force is a contact force that prevents the TV from falling through the table and can have any magnitude needed to balance the downward force $\mathbf{F}_g$, up to the point of breaking the table. If someone stacks books on the TV, the normal force exerted by the table on the TV increases. If someone lifts up on the TV, the normal force exerted by the table on the TV decreases. (The normal force becomes zero if the TV is raised off the table.)

Definition of normal force

The two forces in an action–reaction pair **always act on different objects.** For the hammer-and-nail situation shown in Figure 5.6b, one force of the pair acts on the hammer and the other acts on the nail. For the unfortunate person stepping out of the boat in Quick Quiz 5.4, one force of the pair acts on the person, and the other acts on the boat.

For the TV in Figure 5.7, the force of gravity $\mathbf{F}_g$ and the normal force $\mathbf{n}$ are *not* an action–reaction pair because they act on the same body—the TV. The two reaction forces in this situation—$\mathbf{F}_g'$ and $\mathbf{n}'$—are exerted on objects other than the TV. Because the reaction to $\mathbf{F}_g$ is the force $\mathbf{F}_g'$ exerted by the TV on the Earth and the reaction to $\mathbf{n}$ is the force $\mathbf{n}'$ exerted by the TV on the table, we conclude that

$$\mathbf{F}_g = -\mathbf{F}_g' \qquad \text{and} \qquad \mathbf{n} = -\mathbf{n}'$$

[4] *Normal* in this context means *perpendicular.*

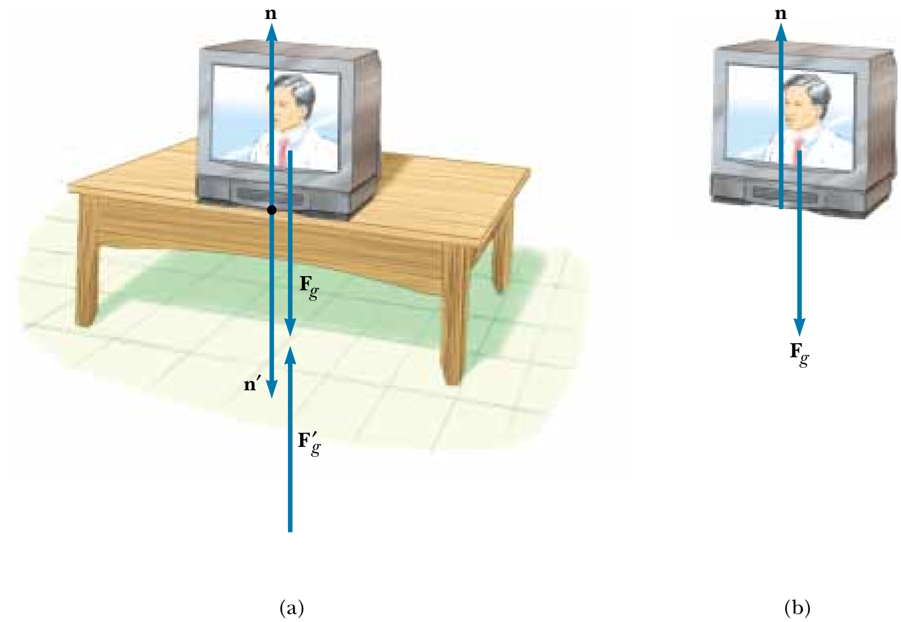


Figure 5.7 When a TV is at rest on a table, the forces acting on the TV are the normal force **n** and the force of gravity $\mathbf{F}_g$, as illustrated in part (b). The reaction to **n** is the force **n**′ exerted by the TV on the table. The reaction to $\mathbf{F}_g$ is the force $\mathbf{F}'_g$ exerted by the TV on the Earth.

The forces **n** and **n**′ have the same magnitude, which is the same as that of $\mathbf{F}_g$ until the table breaks. From the second law, we see that, because the TV is in equilibrium ($\mathbf{a} = 0$), it follows[5] that $F_g = n = mg$.

Quick Quiz 5.5

If a fly collides with the windshield of a fast-moving bus, (a) which experiences the greater impact force: the fly or the bus, or is the same force experienced by both? (b) Which experiences the greater acceleration: the fly or the bus, or is the same acceleration experienced by both?

Conceptual Example 5.3 You Push Me and I'll Push You

A large man and a small boy stand facing each other on frictionless ice. They put their hands together and push against each other so that they move apart. (a) Who moves away with the higher speed?

Solution This situation is similar to what we saw in Quick Quiz 5.5. According to Newton's third law, the force exerted by the man on the boy and the force exerted by the boy on the man are an action–reaction pair, and so they must be equal in magnitude. (A bathroom scale placed between their hands would read the same, regardless of which way it faced.) Therefore, the boy, having the lesser mass, experiences the greater acceleration. Both individuals accelerate for the same amount of time, but the greater acceleration of the boy over this time interval results in his moving away from the interaction with the higher speed.

(b) Who moves farther while their hands are in contact?

Solution Because the boy has the greater acceleration, he moves farther during the interval in which the hands are in contact.

[5] Technically, we should write this equation in the component form $F_{gy} = n_y = mg_y$. This component notation is cumbersome, however, and so in situations in which a vector is parallel to a coordinate axis, we usually do not include the subscript for that axis because there is no other component.

5.7 SOME APPLICATIONS OF NEWTON'S LAWS

4.6

In this section we apply Newton's laws to objects that are either in equilibrium ($\mathbf{a} = 0$) or accelerating along a straight line under the action of constant external forces. We assume that the objects behave as particles so that we need not worry about rotational motion. We also neglect the effects of friction in those problems involving motion; this is equivalent to stating that the surfaces are *frictionless.* Finally, we usually neglect the mass of any ropes involved. In this approximation, the magnitude of the force exerted at any point along a rope is the same at all points along the rope. In problem statements, the synonymous terms *light, lightweight,* and *of negligible mass* are used to indicate that a mass is to be ignored when you work the problems.

When we apply Newton's laws to an object, we are interested only in external forces that act on the object. For example, in Figure 5.7 the only external forces acting on the TV are $\mathbf{n}$ and $\mathbf{F}_g$. The reactions to these forces, $\mathbf{n}'$ and $\mathbf{F}_g'$, act on the table and on the Earth, respectively, and therefore do not appear in Newton's second law applied to the TV.

Tension

When a rope attached to an object is pulling on the object, the rope exerts a force $\mathbf{T}$ on the object, and the magnitude of that force is called the **tension** in the rope. Because it is the magnitude of a vector quantity, tension is a scalar quantity.

Consider a crate being pulled to the right on a frictionless, horizontal surface, as shown in Figure 5.8a. Suppose you are asked to find the acceleration of the crate and the force the floor exerts on it. First, note that the horizontal force being applied to the crate acts through the rope. Use the symbol $\mathbf{T}$ to denote the force exerted by the rope on the crate. The magnitude of $\mathbf{T}$ is equal to the tension in the rope. A dotted circle is drawn around the crate in Figure 5.8a to remind you that you are interested only in the forces acting on the crate. These are illustrated in Figure 5.8b. In addition to the force $\mathbf{T}$, this force diagram for the crate includes the force of gravity $\mathbf{F}_g$ and the normal force $\mathbf{n}$ exerted by the floor on the crate. Such a force diagram, referred to as a **free-body diagram,** shows all external forces acting on the object. The construction of a correct free-body diagram is an important step in applying Newton's laws. The *reactions* to the forces we have listed—namely, the force exerted by the crate on the rope, the force exerted by the crate on the Earth, and the force exerted by the crate on the floor—are *not* included in the free-body diagram because they act on *other* bodies and not on the crate.

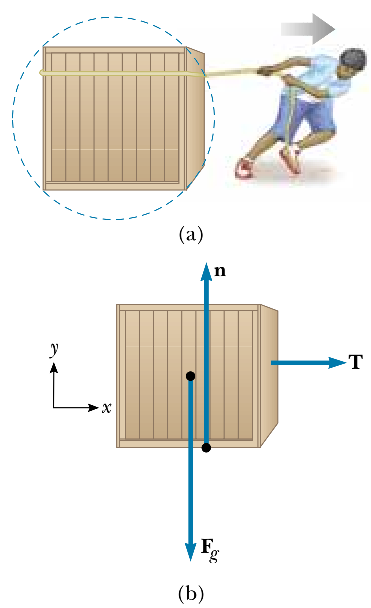


Figure 5.8 (a) A crate being pulled to the right on a frictionless surface. (b) The free-body diagram representing the external forces acting on the crate.

We can now apply Newton's second law in component form to the crate. The only force acting in the x direction is $\mathbf{T}$. Applying $\Sigma F_x = ma_x$ to the horizontal motion gives

$$\sum F_x = T = ma_x \qquad \text{or} \qquad a_x = \frac{T}{m}$$

No acceleration occurs in the y direction. Applying $\Sigma F_y = ma_y$ with $a_y = 0$ yields

$$n + (-F_g) = 0 \qquad \text{or} \qquad n = F_g$$

That is, the normal force has the same magnitude as the force of gravity but is in the opposite direction.

If $\mathbf{T}$ is a constant force, then the acceleration $a_x = T/m$ also is constant. Hence, the constant-acceleration equations of kinematics from Chapter 2 can be used to obtain the crate's displacement Δx and velocity v_x as functions of time. Be-

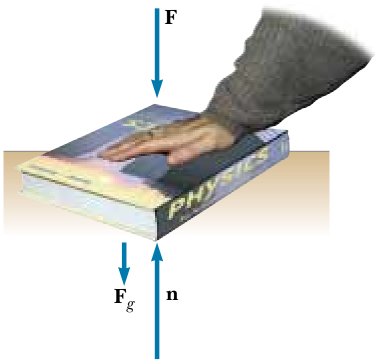


Figure 5.9 When one object pushes downward on another object with a force **F**, the normal force **n** is greater than the force of gravity: $n = F_g + F$.

cause $a_x = T/m =$ constant, Equations 2.8 and 2.11 can be written as

$$v_{xf} = v_{xi} + \left(\frac{T}{m}\right)t$$

$$\Delta x = v_{xi}t + \frac{1}{2}\left(\frac{T}{m}\right)t^2$$

In the situation just described, the magnitude of the normal force **n** is equal to the magnitude of $\mathbf{F}_g$, but this is not always the case. For example, suppose a book is lying on a table and you push down on the book with a force **F**, as shown in Figure 5.9. Because the book is at rest and therefore not accelerating, $\Sigma F_y = 0$, which gives $n - F_g - F = 0$, or $n = F_g + F$. Other examples in which $n \neq F_g$ are presented later.

Consider a lamp suspended from a light chain fastened to the ceiling, as in Figure 5.10a. The free-body diagram for the lamp (Figure 5.10b) shows that the forces acting on the lamp are the downward force of gravity $\mathbf{F}_g$ and the upward force **T** exerted by the chain. If we apply the second law to the lamp, noting that $\mathbf{a} = 0$, we see that because there are no forces in the *x* direction, $\Sigma F_x = 0$ provides no helpful information. The condition $\Sigma F_y = ma_y = 0$ gives

$$\sum F_y = T - F_g = 0 \qquad \text{or} \qquad T = F_g$$

Again, note that **T** and $\mathbf{F}_g$ are *not* an action–reaction pair because they act on the same object—the lamp. The reaction force to **T** is **T**′, the downward force exerted by the lamp on the chain, as shown in Figure 5.10c. The ceiling exerts on the chain a force **T**″ that is equal in magnitude to the magnitude of **T**′ and points in the opposite direction.

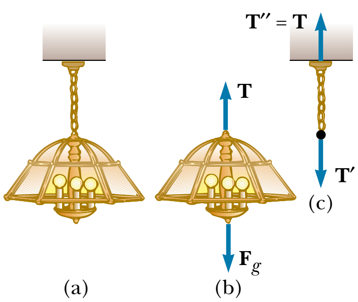


Figure 5.10 (a) A lamp suspended from a ceiling by a chain of negligible mass. (b) The forces acting on the lamp are the force of gravity $\mathbf{F}_g$ and the force exerted by the chain **T**. (c) The forces acting on the chain are the force exerted by the lamp **T**′ and the force exerted by the ceiling **T**″.

Problem-Solving Hints

Applying Newton's Laws

The following procedure is recommended when dealing with problems involving Newton's laws:

- Draw a simple, neat diagram of the system.
- Isolate the object whose motion is being analyzed. Draw a free-body diagram for this object. For systems containing more than one object, draw *separate* free-body diagrams for each object. *Do not* include in the free-body diagram forces exerted by the object on its surroundings. Establish convenient coordinate axes for each object and find the components of the forces along these axes.
- Apply Newton's second law, $\Sigma\mathbf{F} = m\mathbf{a}$, in component form. Check your dimensions to make sure that all terms have units of force.
- Solve the component equations for the unknowns. Remember that you must have as many independent equations as you have unknowns to obtain a complete solution.
- Make sure your results are consistent with the free-body diagram. Also check the predictions of your solutions for extreme values of the variables. By doing so, you can often detect errors in your results.

EXAMPLE 5.4 A Traffic Light at Rest

A traffic light weighing 125 N hangs from a cable tied to two other cables fastened to a support. The upper cables make angles of 37.0° and 53.0° with the horizontal. Find the tension in the three cables.

Solution Figure 5.11a shows the type of drawing we might make of this situation. We then construct two free-body diagrams—one for the traffic light, shown in Figure 5.11b, and one for the knot that holds the three cables together, as seen in Figure 5.11c. This knot is a convenient object to choose because all the forces we are interested in act through it. Because the acceleration of the system is zero, we know that the net force on the light and the net force on the knot are both zero.

In Figure 5.11b the force $\mathbf{T}_3$ exerted by the vertical cable supports the light, and so $T_3 = F_g =$ 125 N. Next, we choose the coordinate axes shown in Figure 5.11c and resolve the forces acting on the knot into their components:

Force	*x* Component	*y* Component
$\mathbf{T}_1$	$-T_1 \cos 37.0°$	$T_1 \sin 37.0°$
$\mathbf{T}_2$	$T_2 \cos 53.0°$	$T_2 \sin 53.0°$
$\mathbf{T}_3$	0	-125 N

Knowing that the knot is in equilibrium ($\mathbf{a} = 0$) allows us to write

$$(1) \quad \sum F_x = -T_1 \cos 37.0° + T_2 \cos 53.0° = 0$$

$$(2) \quad \sum F_y = T_1 \sin 37.0° + T_2 \sin 53.0° + (-125\text{ N}) = 0$$

From (1) we see that the horizontal components of $\mathbf{T}_1$ and $\mathbf{T}_2$ must be equal in magnitude, and from (2) we see that the sum of the vertical components of $\mathbf{T}_1$ and $\mathbf{T}_2$ must balance the weight of the light. We solve (1) for T_2 in terms of T_1 to obtain

$$T_2 = T_1\left(\frac{\cos 37.0°}{\cos 53.0°}\right) = 1.33T_1$$

This value for T_2 is substituted into (2) to yield

$$T_1 \sin 37.0° + (1.33T_1)(\sin 53.0°) - 125\text{ N} = 0$$

$$T_1 = 75.1\text{ N}$$

$$T_2 = 1.33T_1 = 99.9\text{ N}$$

This problem is important because it combines what we have learned about vectors with the new topic of forces. The general approach taken here is very powerful, and we will repeat it many times.

Exercise In what situation does $T_1 = T_2$?

Answer When the two cables attached to the support make equal angles with the horizontal.

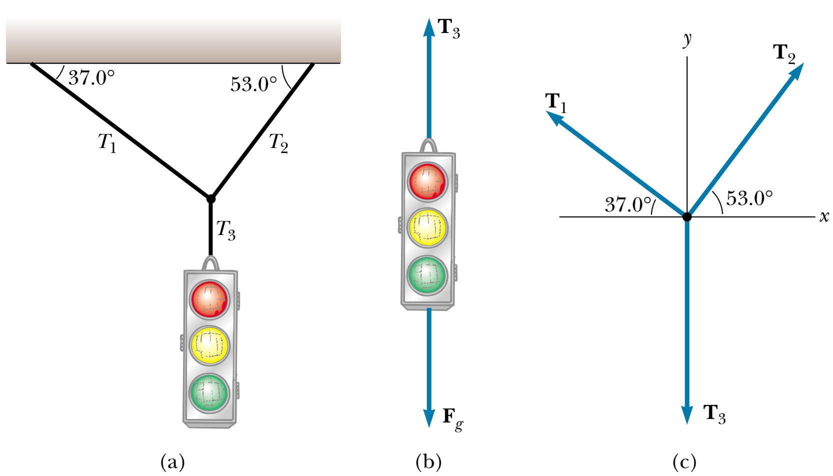


Figure 5.11 (a) A traffic light suspended by cables. (b) Free-body diagram for the traffic light. (c) Free-body diagram for the knot where the three cables are joined.

CONCEPTUAL EXAMPLE 5.5 Forces Between Cars in a Train

In a train, the cars are connected by *couplers,* which are under tension as the locomotive pulls the train. As you move down the train from locomotive to caboose, does the tension in the couplers increase, decrease, or stay the same as the train speeds up? When the engineer applies the brakes, the couplers are under compression. How does this compression force vary from locomotive to caboose? (Assume that only the brakes on the wheels of the engine are applied.)

Solution As the train speeds up, the tension decreases from the front of the train to the back. The coupler between the locomotive and the first car must apply enough force to accelerate all of the remaining cars. As you move back along the train, each coupler is accelerating less mass behind it. The last coupler has to accelerate only the caboose, and so it is under the least tension.

When the brakes are applied, the force again decreases from front to back. The coupler connecting the locomotive to the first car must apply a large force to slow down all the remaining cars. The final coupler must apply a force large enough to slow down only the caboose.

EXAMPLE 5.6 Crate on a Frictionless Incline

A crate of mass m is placed on a frictionless inclined plane of angle θ. (a) Determine the acceleration of the crate after it is released.

Solution Because we know the forces acting on the crate, we can use Newton's second law to determine its acceleration. (In other words, we have classified the problem; this gives us a hint as to the approach to take.) We make a sketch as in Figure 5.12a and then construct the free-body diagram for the crate, as shown in Figure 5.12b. The only forces acting on the crate are the normal force $\mathbf{n}$ exerted by the inclined plane, which acts perpendicular to the plane, and the force of gravity $\mathbf{F}_g = m\mathbf{g}$, which acts vertically downward. For problems involving inclined planes, it is convenient to choose the coordinate axes with x downward along the incline and y perpendicular to it, as shown in Figure 5.12b. (It is possible to solve the problem with "standard" horizontal and vertical axes. You may want to try this, just for practice.) Then, we replace the force of gravity by a component of magnitude $mg \sin \theta$ along the positive x axis and by one of magnitude $mg \cos \theta$ along the negative y axis.

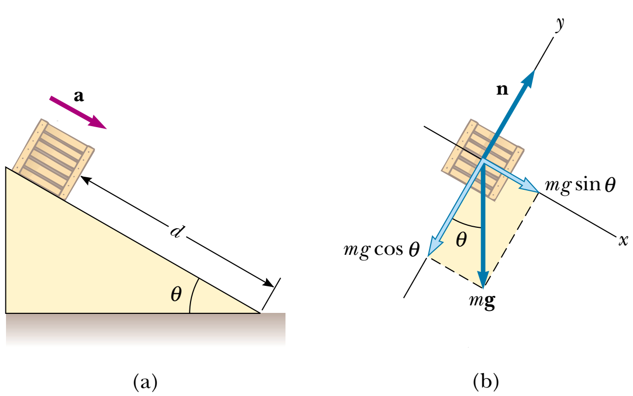


Figure 5.12 (a) A crate of mass m sliding down a frictionless incline. (b) The free-body diagram for the crate. Note that its acceleration along the incline is $g \sin \theta$.

Now we apply Newton's second law in component form, noting that $a_y = 0$:

$$(1) \qquad \sum F_x = mg \sin \theta = ma_x$$

$$(2) \qquad \sum F_y = n - mg \cos \theta = 0$$

Solving (1) for a_x, we see that the acceleration along the incline is caused by the component of $\mathbf{F}_g$ directed down the incline:

$$(3) \qquad a_x = g \sin \theta$$

Note that this acceleration component is independent of the mass of the crate! It depends only on the angle of inclination and on g.

From (2) we conclude that the component of $\mathbf{F}_g$ perpendicular to the incline is balanced by the normal force; that is, $n = mg \cos \theta$. This is one example of a situation in which the normal force is *not* equal in magnitude to the weight of the object.

Special Cases Looking over our results, we see that in the extreme case of $\theta = 90°$, $a_x = g$ and $n = 0$. This condition corresponds to the crate's being in free fall. When $\theta = 0$, $a_x = 0$ and $n = mg$ (its maximum value); in this case, the crate is sitting on a horizontal surface.

(b) Suppose the crate is released from rest at the top of the incline, and the distance from the front edge of the crate to the bottom is d. How long does it take the front edge to reach the bottom, and what is its speed just as it gets there?

Solution Because a_x = constant, we can apply Equation 2.11, $x_f - x_i = v_{xi}t + \frac{1}{2}a_xt^2$, to analyze the crate's motion.

With the displacement $x_f - x_i = d$ and $v_{xi} = 0$, we obtain

$$d = \tfrac{1}{2}a_x t^2$$

$$(4) \qquad t = \sqrt{\frac{2d}{a_x}} = \boxed{\sqrt{\frac{2d}{g\sin\theta}}}$$

Using Equation 2.12, $v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$, with $v_{xi} = 0$, we find that

$$v_{xf}^2 = 2a_x d$$

$$(5) \qquad v_{xf} = \sqrt{2a_x d} = \boxed{\sqrt{2gd\sin\theta}}$$

We see from equations (4) and (5) that the time t needed to reach the bottom and the speed v_{xf}, like acceleration, are independent of the crate's mass. This suggests a simple method you can use to measure g, using an inclined air track: Measure the angle of inclination, some distance traveled by a cart along the incline, and the time needed to travel that distance. The value of g can then be calculated from (4).

EXAMPLE 5.7 One Block Pushes Another

Two blocks of masses m_1 and m_2 are placed in contact with each other on a frictionless horizontal surface. A constant horizontal force $\mathbf{F}$ is applied to the block of mass m_1. (a) Determine the magnitude of the acceleration of the two-block system.

Solution Common sense tells us that both blocks must experience the same acceleration because they remain in contact with each other. Just as in the preceding example, we make a labeled sketch and free-body diagrams, which are shown in Figure 5.13. In Figure 5.13a the dashed line indicates that we treat the two blocks together as a system. Because $\mathbf{F}$ is the only external horizontal force acting on the system (the two blocks), we have

$$\sum F_x(\text{system}) = F = (m_1 + m_2)a_x$$

$$(1) \qquad \boxed{a_x = \frac{F}{m_1 + m_2}}$$

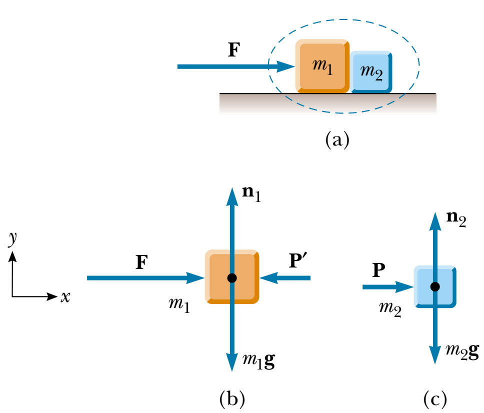


Figure 5.13

Treating the two blocks together as a system simplifies the solution but does not provide information about internal forces.

(b) Determine the magnitude of the contact force between the two blocks.

Solution To solve this part of the problem, we must treat each block separately with its own free-body diagram, as in Figures 5.13b and 5.13c. We denote the contact force by $\mathbf{P}$. From Figure 5.13c, we see that the only horizontal force acting on block 2 is the contact force $\mathbf{P}$ (the force exerted by block 1 on block 2), which is directed to the right. Applying Newton's second law to block 2 gives

$$(2) \qquad \sum F_x = P = m_2 a_x$$

Substituting into (2) the value of a_x given by (1), we obtain

$$(3) \qquad \boxed{P = m_2 a_x = \left(\frac{m_2}{m_1 + m_2}\right)F}$$

From this result, we see that the contact force $\mathbf{P}$ exerted by block 1 on block 2 is *less* than the applied force $\mathbf{F}$. This is consistent with the fact that the force required to accelerate block 2 alone must be less than the force required to produce the same acceleration for the two-block system.

It is instructive to check this expression for P by considering the forces acting on block 1, shown in Figure 5.13b. The horizontal forces acting on this block are the applied force $\mathbf{F}$ to the right and the contact force $\mathbf{P}'$ to the left (the force exerted by block 2 on block 1). From Newton's third law, $\mathbf{P}'$ is the reaction to $\mathbf{P}$, so that $|\mathbf{P}'| = |\mathbf{P}|$. Applying Newton's second law to block 1 produces

$$(4) \qquad \sum F_x = F - P' = F - P = m_1 a_x$$

Substituting into (4) the value of a_x from (1), we obtain

$$P = F - m_1 a_x = F - \frac{m_1 F}{m_1 + m_2} = \left(\frac{m_2}{m_1 + m_2}\right) F$$

This agrees with (3), as it must.

Exercise If $m_1 = 4.00$ kg, $m_2 = 3.00$ kg, and $F = 9.00$ N, find the magnitude of the acceleration of the system and the magnitude of the contact force.

Answer $a_x = 1.29\ \text{m/s}^2$; $P = 3.86$ N.

EXAMPLE 5.8 Weighing a Fish in an Elevator

A person weighs a fish of mass m on a spring scale attached to the ceiling of an elevator, as illustrated in Figure 5.14. Show that if the elevator accelerates either upward or downward, the spring scale gives a reading that is different from the weight of the fish.

Solution The external forces acting on the fish are the downward force of gravity $\mathbf{F}_g = m\mathbf{g}$ and the force $\mathbf{T}$ exerted by the scale. By Newton's third law, the tension T is also the reading of the scale. If the elevator is either at rest or moving at constant velocity, the fish is not accelerating, and so $\Sigma F_y = T - mg = 0$ or $T = mg$ (remember that the scalar mg is the weight of the fish).

If the elevator moves upward with an acceleration $\mathbf{a}$ relative to an observer standing outside the elevator in an inertial frame (see Fig. 5.14a), Newton's second law applied to the fish gives the net force on the fish:

$$(1) \qquad \sum F_y = T - mg = ma_y$$

where we have chosen upward as the positive direction. Thus, we conclude from (1) that the scale reading T is greater than the weight mg if $\mathbf{a}$ is upward, so that a_y is positive, and that the reading is less than mg if $\mathbf{a}$ is downward, so that a_y is negative.

For example, if the weight of the fish is 40.0 N and $\mathbf{a}$ is upward, so that $a_y = +2.00\ \text{m/s}^2$, the scale reading from (1) is

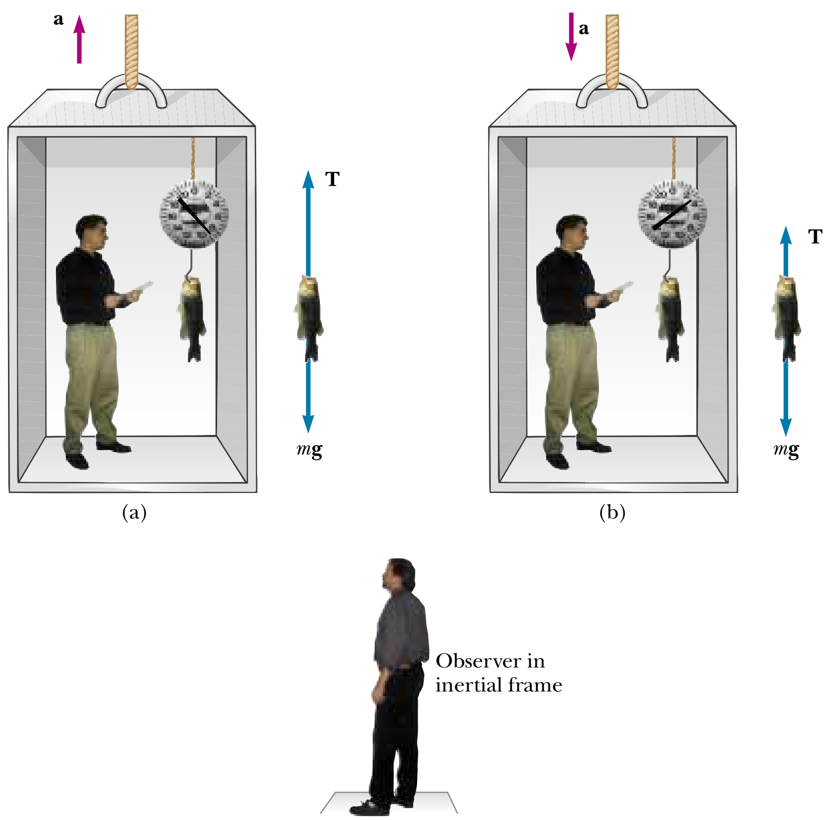


Figure 5.14 Apparent weight versus true weight. (a) When the elevator accelerates upward, the spring scale reads a value greater than the weight of the fish. (b) When the elevator accelerates downward, the spring scale reads a value less than the weight of the fish.

$$(2) \qquad T = ma_y + mg = mg\left(\frac{a_y}{g} + 1\right)$$

$$= (40.0\text{ N})\left(\frac{2.00\text{ m/s}^2}{9.80\text{ m/s}^2} + 1\right)$$

$$= \boxed{48.2\text{ N}}$$

If **a** is downward so that $a_y = -2.00\text{ m/s}^2$, then (2) gives us

$$T = mg\left(\frac{a_y}{g} + 1\right) = (40.0\text{ N})\left(\frac{-2.00\text{ m/s}^2}{9.80\text{ m/s}^2} + 1\right)$$

$$= \boxed{31.8\text{ N}}$$

Hence, if you buy a fish by weight in an elevator, make sure the fish is weighed while the elevator is either at rest or accelerating downward! Furthermore, note that from the information given here one cannot determine the direction of motion of the elevator.

Special Cases If the elevator cable breaks, the elevator falls freely and $a_y = -g$. We see from (2) that the scale reading T is zero in this case; that is, the fish appears to be weightless. If the elevator accelerates downward with an acceleration greater than g, the fish (along with the person in the elevator) eventually hits the ceiling because the acceleration of fish and person is still that of a freely falling object relative to an outside observer.

EXAMPLE 5.9 Atwood's Machine

When two objects of unequal mass are hung vertically over a frictionless pulley of negligible mass, as shown in Figure 5.15a, the arrangement is called an *Atwood machine.* The device is sometimes used in the laboratory to measure the free-fall acceleration. Determine the magnitude of the acceleration of the two objects and the tension in the lightweight cord.

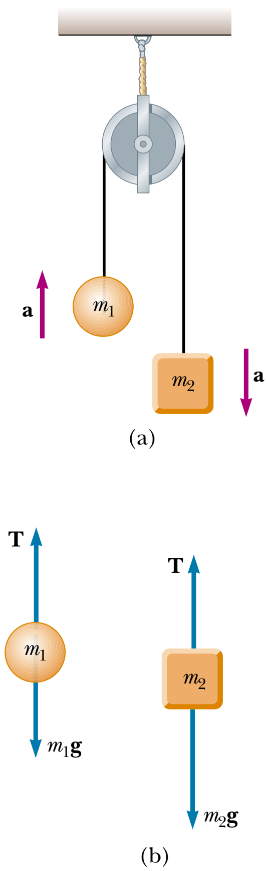


Figure 5.15 Atwood's machine. (a) Two objects ($m_2 > m_1$) connected by a cord of negligible mass strung over a frictionless pulley. (b) Free-body diagrams for the two objects.

Solution If we were to define our system as being made up of both objects, as we did in Example 5.7, we would have to determine an *internal* force (tension in the cord). We must define two systems here—one for each object—and apply Newton's second law to each. The free-body diagrams for the two objects are shown in Figure 5.15b. Two forces act on each object: the upward force **T** exerted by the cord and the downward force of gravity.

We need to be very careful with signs in problems such as this, in which a string or rope passes over a pulley or some other structure that causes the string or rope to bend. In Figure 5.15a, notice that if object 1 accelerates upward, then object 2 accelerates downward. Thus, for consistency with signs, if we define the upward direction as positive for object 1, we must define the downward direction as positive for object 2. With this sign convention, both objects accelerate in the same direction as defined by the choice of sign. With this sign convention applied to the forces, the y component of the net force exerted on object 1 is $T - m_1g$, and the y component of the net force exerted on object 2 is $m_2g - T$. Because the objects are connected by a cord, their accelerations must be equal in magnitude. (Otherwise the cord would stretch or break as the distance between the objects increased.) If we assume $m_2 > m_1$, then object 1 must accelerate upward and object 2 downward.

When Newton's second law is applied to object 1, we obtain

$$(1) \qquad \sum F_y = T - m_1g = m_1a_y$$

Similarly, for object 2 we find

$$(2) \qquad \sum F_y = m_2g - T = m_2a_y$$

When (2) is added to (1), T drops out and we get

$$-m_1g + m_2g = m_1a_y + m_2a_y$$

$$(3) \qquad a_y = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

When (3) is substituted into (1), we obtain

$$(4) \qquad T = \left(\frac{2m_1m_2}{m_1 + m_2}\right)g$$

The result for the acceleration in (3) can be interpreted as the ratio of the unbalanced force on the system $(m_2g - m_1g)$ to the total mass of the system $(m_1 + m_2)$, as expected from Newton's second law.

Special Cases When $m_1 = m_2$, then $a_y = 0$ and $T = m_1g$, as we would expect for this balanced case. If $m_2 >> m_1$, then $a_y \approx g$ (a freely falling body) and $T \approx 2m_1g$.

Exercise Find the magnitude of the acceleration and the string tension for an Atwood machine in which $m_1 = 2.00$ kg and $m_2 = 4.00$ kg.

Answer $a_y = 3.27\ \text{m/s}^2$, $T = 26.1$ N.

EXAMPLE 5.10 Acceleration of Two Objects Connected by a Cord

A ball of mass m_1 and a block of mass m_2 are attached by a lightweight cord that passes over a frictionless pulley of negligible mass, as shown in Figure 5.16a. The block lies on a frictionless incline of angle θ. Find the magnitude of the acceleration of the two objects and the tension in the cord.

Solution Because the objects are connected by a cord (which we assume does not stretch), their accelerations have the same magnitude. The free-body diagrams are shown in Figures 5.16b and 5.16c. Applying Newton's second law in component form to the ball, with the choice of the upward direction as positive, yields

$$(1) \qquad \sum F_x = 0$$

$$(2) \qquad \sum F_y = T - m_1g = m_1a_y = m_1a$$

Note that in order for the ball to accelerate upward, it is necessary that $T > m_1g$. In (2) we have replaced a_y with a because the acceleration has only a y component.

For the block it is convenient to choose the positive x' axis along the incline, as shown in Figure 5.16c. Here we choose the positive direction to be down the incline, in the $+x'$ direction. Applying Newton's second law in component form to the block gives

$$(3) \qquad \sum F_{x'} = m_2g\sin\theta - T = m_2a_{x'} = m_2a$$

$$(4) \qquad \sum F_{y'} = n - m_2g\cos\theta = 0$$

In (3) we have replaced $a_{x'}$ with a because that is the acceleration's only component. In other words, the two objects have accelerations of the same magnitude a, which is what we are trying to find. Equations (1) and (4) provide no information regarding the acceleration. However, if we solve (2) for T and then substitute this value for T into (3) and solve for a, we obtain

$$(5) \qquad a = \frac{m_2g\sin\theta - m_1g}{m_1 + m_2}$$

When this value for a is substituted into (2), we find

$$(6) \qquad T = \frac{m_1m_2g(\sin\theta + 1)}{m_1 + m_2}$$

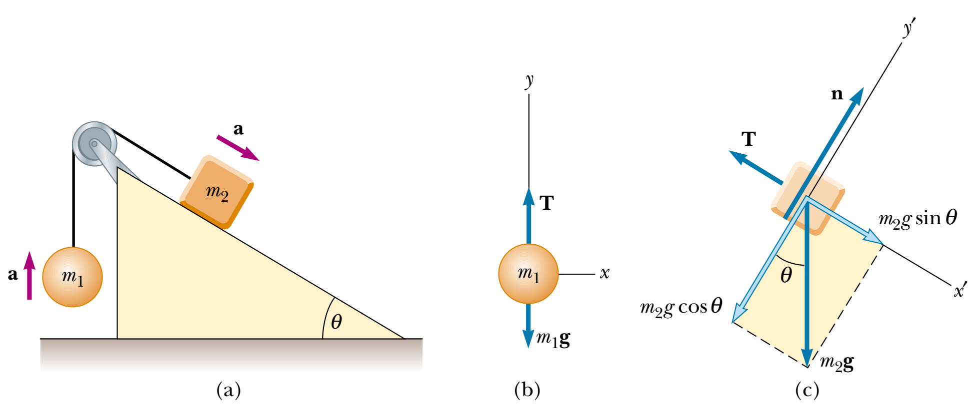


Figure 5.16 (a) Two objects connected by a lightweight cord strung over a frictionless pulley. (b) Free-body diagram for the ball. (c) Free-body diagram for the block. (The incline is frictionless.)

Note that the block accelerates down the incline only if $m_2 \sin\theta > m_1$ (that is, if **a** is in the direction we assumed). If $m_1 > m_2 \sin\theta$, then the acceleration is up the incline for the block and downward for the ball. Also note that the result for the acceleration (5) can be interpreted as the resultant force acting on the system divided by the total mass of the system; this is consistent with Newton's second law. Finally, if $\theta = 90°$, then the results for a and T are identical to those of Example 5.9.

Exercise If $m_1 = 10.0$ kg, $m_2 = 5.00$ kg, and $\theta = 45.0°$, find the acceleration of each object.

Answer $a = -4.22 \text{ m/s}^2$, where the negative sign indicates that the block accelerates up the incline and the ball accelerates downward.

5.8 FORCES OF FRICTION

When a body is in motion either on a surface or in a viscous medium such as air or water, there is resistance to the motion because the body interacts with its surroundings. We call such resistance a **force of friction**. Forces of friction are very important in our everyday lives. They allow us to walk or run and are necessary for the motion of wheeled vehicles.

Have you ever tried to move a heavy desk across a rough floor? You push harder and harder until all of a sudden the desk seems to "break free" and subsequently moves relatively easily. It takes a greater force to start the desk moving than it does to keep it going once it has started sliding. To understand why this happens, consider a book on a table, as shown in Figure 5.17a. If we apply an external horizontal force **F** to the book, acting to the right, the book remains stationary if **F** is not too great. The force that counteracts **F** and keeps the book from moving acts to the left and is called the **frictional force f.**

Force of static friction

As long as the book is not moving, $f = F$. Because the book is stationary, we call this frictional force the **force of static friction $\mathbf{f}_s$**. Experiments show that this force arises from contacting points that protrude beyond the general level of the surfaces in contact, even for surfaces that are apparently very smooth, as shown in the magnified view in Figure 5.17a. (If the surfaces are clean and smooth at the atomic level, they are likely to weld together when contact is made.) The frictional force arises in part from one peak's physically blocking the motion of a peak from the opposing surface, and in part from chemical bonding of opposing points as they come into contact. If the surfaces are rough, bouncing is likely to occur, further complicating the analysis. Although the details of friction are quite complex at the atomic level, this force ultimately involves an electrical interaction between atoms or molecules.

If we increase the magnitude of **F**, as shown in Figure 5.17b, the magnitude of $\mathbf{f}_s$ increases along with it, keeping the book in place. The force $\mathbf{f}_s$ cannot increase indefinitely, however. Eventually the surfaces in contact can no longer supply sufficient frictional force to counteract **F**, and the book accelerates. When it is on the verge of moving, f_s is a maximum, as shown in Figure 5.17c. When F exceeds $f_{s,\text{max}}$, the book accelerates to the right. Once the book is in motion, the retarding frictional force becomes less than $f_{s,\text{max}}$ (see Fig. 5.17c). When the book is in motion, we call the retarding force the **force of kinetic friction $\mathbf{f}_k$**. If $F = f_k$, then the book moves to the right with constant speed. If $F > f_k$, then there is an unbalanced force $F - f_k$ in the positive x direction, and this force accelerates the book to the right. If the applied force **F** is removed, then the frictional force $\mathbf{f}_k$ acting to the left accelerates the book in the negative x direction and eventually brings it to rest.

Force of kinetic friction

Experimentally, we find that, to a good approximation, both $f_{s,\text{max}}$ and f_k are proportional to the normal force acting on the book. The following empirical laws of friction summarize the experimental observations:

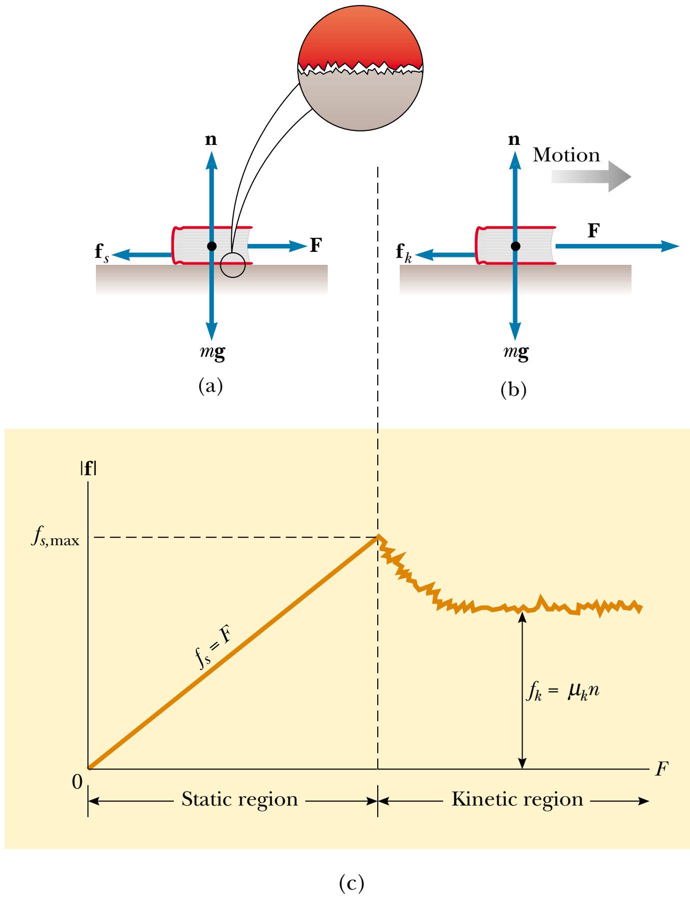


Figure 5.17 The direction of the force of friction **f** between a book and a rough surface is opposite the direction of the applied force **F**. Because the two surfaces are both rough, contact is made only at a few points, as illustrated in the "magnified" view. (a) The magnitude of the force of static friction equals the magnitude of the applied force. (b) When the magnitude of the applied force exceeds the magnitude of the force of kinetic friction, the book accelerates to the right. (c) A graph of frictional force versus applied force. Note that $f_{s,\max} > f_k$.

- The direction of the force of static friction between any two surfaces in contact with each other is opposite the direction of relative motion and can have values

$$f_s \leq \mu_s n \tag{5.8}$$

  where the dimensionless constant μ_s is called the **coefficient of static friction** and n is the magnitude of the normal force. The equality in Equation 5.8 holds when one object is on the verge of moving, that is, when $f_s = f_{s,\max} = \mu_s n$. The inequality holds when the applied force is less than $\mu_s n$.
- The direction of the force of kinetic friction acting on an object is opposite the direction of the object's sliding motion relative to the surface applying the frictional force and is given by

$$f_k = \mu_k n \tag{5.9}$$

  where μ_k is the **coefficient of kinetic friction.**
- The values of μ_k and μ_s depend on the nature of the surfaces, but μ_k is generally less than μ_s. Typical values range from around 0.03 to 1.0. Table 5.2 lists some reported values.

TABLE 5.2 Coefficients of Friction[a]

	μ_s	μ_k
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Rubber on concrete	1.0	0.8
Wood on wood	0.25–0.5	0.2
Glass on glass	0.94	0.4
Waxed wood on wet snow	0.14	0.1
Waxed wood on dry snow	—	0.04
Metal on metal (lubricated)	0.15	0.06
Ice on ice	0.1	0.03
Teflon on Teflon	0.04	0.04
Synovial joints in humans	0.01	0.003

[a] All values are approximate. In some cases, the coefficient of friction can exceed 1.0.

- The coefficients of friction are nearly independent of the area of contact between the surfaces. To understand why, we must examine the difference between the *apparent contact area,* which is the area we see with our eyes, and the *real contact area,* represented by two irregular surfaces touching, as shown in the magnified view in Figure 5.17a. It seems that increasing the apparent contact area does not increase the real contact area. When we increase the apparent area (without changing anything else), there is less force per unit area driving the jagged points together. This decrease in force counteracts the effect of having more points involved.

If you would like to learn more about this subject, read the article "Friction at the Atomic Scale" by J. Krim in the October 1996 issue of *Scientific American.*

Although the coefficient of kinetic friction can vary with speed, we shall usually neglect any such variations in this text. We can easily demonstrate the approximate nature of the equations by trying to get a block to slip down an incline at constant speed. Especially at low speeds, the motion is likely to be characterized by alternate episodes of sticking and movement.

QuickLab

Can you apply the ideas of Example 5.12 to determine the coefficients of static and kinetic friction between the cover of your book and a quarter? What should happen to those coefficients if you make the measurements between your book and *two* quarters taped one on top of the other?

Quick Quiz 5.6

A crate is sitting in the center of a flatbed truck. The truck accelerates to the right, and the crate moves with it, not sliding at all. What is the direction of the frictional force exerted by the truck on the crate? (a) To the left. (b) To the right. (c) No frictional force because the crate is not sliding.

CONCEPTUAL EXAMPLE 5.11 Why Does the Sled Accelerate?

A horse pulls a sled along a level, snow-covered road, causing the sled to accelerate, as shown in Figure 5.18a. Newton's third law states that the sled exerts an equal and opposite force on the horse. In view of this, how can the sled accelerate? Under what condition does the system (horse plus sled) move with constant velocity?

Solution It is important to remember that the forces described in Newton's third law act on different objects—the horse exerts a force on the sled, and the sled exerts an equal-magnitude and oppositely directed force on the horse. Because we are interested only in the motion of the sled, we do not consider the forces it exerts on the horse. When deter-

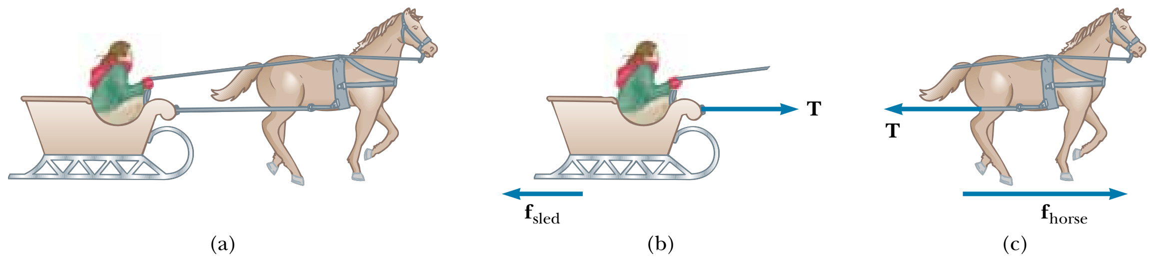


Figure 5.18

mining the motion of an object, you must add only the forces on that object. The horizontal forces exerted on the sled are the forward force **T** exerted by the horse and the backward force of friction $\mathbf{f}_{\text{sled}}$ between sled and snow (see Fig. 5.18b). When the forward force exceeds the backward force, the sled accelerates to the right.

The force that accelerates the system (horse plus sled) is the frictional force $\mathbf{f}_{\text{horse}}$ exerted by the Earth on the horse's feet. The horizontal forces exerted on the horse are the forward force $\mathbf{f}_{\text{horse}}$ exerted by the Earth and the backward tension force **T** exerted by the sled (Fig. 5.18c). The resultant of these two forces causes the horse to accelerate. When $\mathbf{f}_{\text{horse}}$ balances $\mathbf{f}_{\text{sled}}$, the system moves with constant velocity.

Exercise Are the normal force exerted by the snow on the horse and the gravitational force exerted by the Earth on the horse a third-law pair?

Answer No, because they act on the same object. Third-law force pairs are equal in magnitude and opposite in direction, and the forces act on *different* objects.

EXAMPLE 5.12 Experimental Determination of μ_s and μ_k

The following is a simple method of measuring coefficients of friction: Suppose a block is placed on a rough surface inclined relative to the horizontal, as shown in Figure 5.19. The incline angle is increased until the block starts to move. Let us show that by measuring the critical angle θ_c at which this slipping just occurs, we can obtain μ_s.

Solution The only forces acting on the block are the force of gravity $m\mathbf{g}$, the normal force $\mathbf{n}$, and the force of static friction $\mathbf{f}_s$. These forces balance when the block is on the verge of slipping but has not yet moved. When we take x to be parallel to the plane and y perpendicular to it, Newton's second law applied to the block for this balanced situation gives

Static case: (1) $\sum F_x = mg\sin\theta - f_s = ma_x = 0$

(2) $\sum F_y = n - mg\cos\theta = ma_y = 0$

We can eliminate mg by substituting $mg = n/\cos\theta$ from (2) into (1) to get

$$(3)\qquad f_s = mg\sin\theta = \left(\frac{n}{\cos\theta}\right)\sin\theta = n\tan\theta$$

When the incline is at the critical angle θ_c, we know that $f_s = f_{s,\max} = \mu_s n$, and so at this angle, (3) becomes

$$\mu_s n = n\tan\theta_c$$

Static case: $\mu_s = \tan\theta_c$

For example, if the block just slips at $\theta_c = 20°$, then we find that $\mu_s = \tan 20° = 0.364$.

Once the block starts to move at $\theta \geq \theta_c$, it accelerates down the incline and the force of friction is $f_k = \mu_k n$. However, if θ is reduced to a value less than θ_c, it may be possible to find an angle θ_c' such that the block moves down the incline with constant speed ($a_x = 0$). In this case, using (1) and (2) with f_s replaced by f_k gives

Kinetic case: $\mu_k = \tan\theta_c'$

where $\theta_c' < \theta_c$.

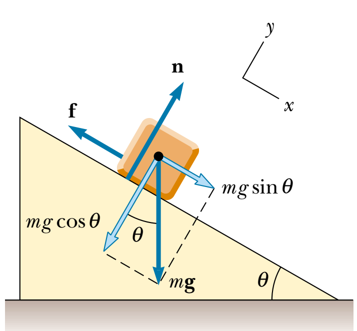


Figure 5.19 The external forces exerted on a block lying on a rough incline are the force of gravity $m\mathbf{g}$, the normal force $\mathbf{n}$, and the force of friction $\mathbf{f}$. For convenience, the force of gravity is resolved into a component along the incline $mg\sin\theta$ and a component perpendicular to the incline $mg\cos\theta$.

EXAMPLE 5.13 The Sliding Hockey Puck

A hockey puck on a frozen pond is given an initial speed of 20.0 m/s. If the puck always remains on the ice and slides 115 m before coming to rest, determine the coefficient of kinetic friction between the puck and ice.

Solution The forces acting on the puck after it is in motion are shown in Figure 5.20. If we assume that the force of kinetic friction f_k remains constant, then this force produces a uniform acceleration of the puck in the direction opposite its velocity, causing the puck to slow down. First, we find this acceleration in terms of the coefficient of kinetic friction, using Newton's second law. Knowing the acceleration of the puck and the distance it travels, we can then use the equations of kinematics to find the coefficient of kinetic friction.

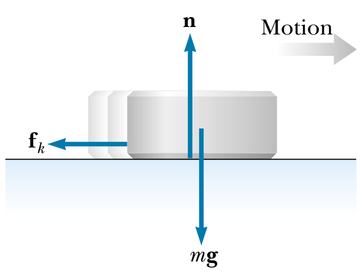


Figure 5.20 *After* the puck is given an initial velocity to the right, the only external forces acting on it are the force of gravity $m\mathbf{g}$, the normal force $\mathbf{n}$, and the force of kinetic friction $\mathbf{f}_k$.

Defining rightward and upward as our positive directions, we apply Newton's second law in component form to the puck and obtain

$$(1) \qquad \sum F_x = -f_k = ma_x$$

$$(2) \qquad \sum F_y = n - mg = 0 \qquad (a_y = 0)$$

But $f_k = \mu_k n$, and from (2) we see that $n = mg$. Therefore, (1) becomes

$$-\mu_k n = -\mu_k mg = ma_x$$

$$a_x = -\mu_k g$$

The negative sign means the acceleration is to the left; this means that the puck is slowing down. The acceleration is independent of the mass of the puck and is constant because we assume that μ_k remains constant.

Because the acceleration is constant, we can use Equation 2.12, $v_{xf}{}^2 = v_{xi}{}^2 + 2a_x(x_f - x_i)$, with $x_i = 0$ and $v_{xf} = 0$:

$$v_{xi}{}^2 + 2ax_f = v_{xi}{}^2 - 2\mu_k g x_f = 0$$

$$\mu_k = \frac{v_{xi}{}^2}{2gx_f}$$

$$\mu_k = \frac{(20.0\text{ m/s})^2}{2(9.80\text{ m/s}^2)(115\text{ m})} = \boxed{0.177}$$

Note that μ_k is dimensionless.

EXAMPLE 5.14 Acceleration of Two Connected Objects When Friction Is Present

A block of mass m_1 on a rough, horizontal surface is connected to a ball of mass m_2 by a lightweight cord over a lightweight, frictionless pulley, as shown in Figure 5.21a. A force of magnitude F at an angle θ with the horizontal is applied to the block as shown. The coefficient of kinetic friction between the block and surface is μ_k. Determine the magnitude of the acceleration of the two objects.

Solution We start by drawing free-body diagrams for the two objects, as shown in Figures 5.21b and 5.21c. (Are you beginning to see the similarities in all these examples?) Next, we apply Newton's second law in component form to each object and use Equation 5.9, $f_k = \mu_k n$. Then we can solve for the acceleration in terms of the parameters given.

The applied force $\mathbf{F}$ has x and y components $F\cos\theta$ and $F\sin\theta$, respectively. Applying Newton's second law to both objects and assuming the motion of the block is to the right, we obtain

Motion of block:

$$(1) \qquad \sum F_x = F\cos\theta - f_k - T = m_1 a_x = m_1 a$$

$$(2) \qquad \sum F_y = n + F\sin\theta - m_1 g = m_1 a_y = 0$$

Motion of ball:

$$\sum F_x = m_2 a_x = 0$$

$$(3) \qquad \sum F_y = T - m_2 g = m_2 a_y = m_2 a$$

Note that because the two objects are connected, we can equate the magnitudes of the x component of the acceleration of the block and the y component of the acceleration of the ball. From Equation 5.9 we know that $f_k = \mu_k n$, and from (2) we know that $n = m_1 g - F\sin\theta$ (note that in this case n is *not* equal to $m_1 g$); therefore,

$$(4) \qquad f_k = \mu_k(m_1 g - F\sin\theta)$$

That is, the frictional force is reduced because of the positive

y component of **F**. Substituting (4) and the value of T from (3) into (1) gives

$$F\cos\theta - \mu_k(m_1 g - F\sin\theta) - m_2(a+g) = m_1 a$$

Solving for a, we obtain

$$(5) \qquad a = \frac{F(\cos\theta + \mu_k\sin\theta) - g(m_2 + \mu_k m_1)}{m_1 + m_2}$$

Note that the acceleration of the block can be either to the right or to the left,[6] depending on the sign of the numerator in (5). If the motion is to the left, then we must reverse the sign of f_k in (1) because the force of kinetic friction must oppose the motion. In this case, the value of a is the same as in (5), with μ_k replaced by $-\mu_k$.

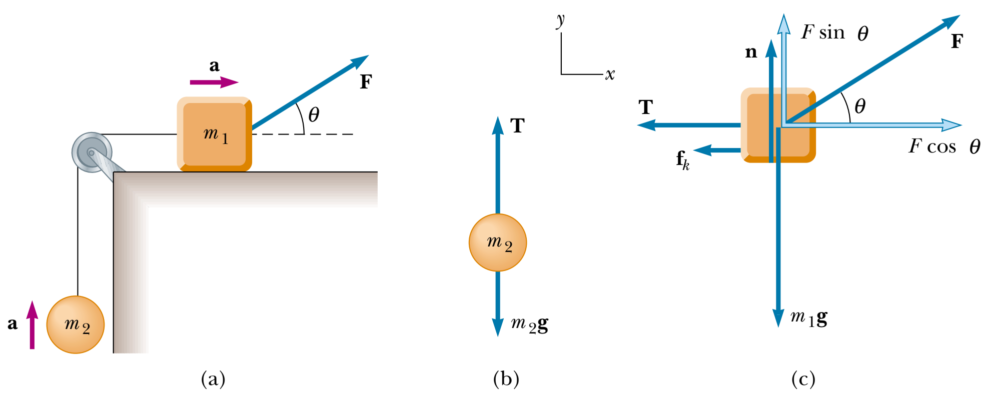


Figure 5.21 (a) The external force **F** applied as shown can cause the block to accelerate to the right. (b) and (c) The free-body diagrams, under the assumption that the block accelerates to the right and the ball accelerates upward. The magnitude of the force of kinetic friction in this case is given by $f_k = \mu_k n = \mu_k(m_1 g - F\sin\theta)$.

APPLICATION Automobile Antilock Braking Systems (ABS)

If an automobile tire is rolling and not slipping on a road surface, then the maximum frictional force that the road can exert on the tire is the force of static friction $\mu_s n$. One must use static friction in this situation because at the point of contact between the tire and the road, no sliding of one surface over the other occurs if the tire is not skidding. However, if the tire starts to skid, the frictional force exerted on it is reduced to the force of kinetic friction $\mu_k n$. Thus, to maximize the frictional force and minimize stopping distance, the wheels must maintain pure rolling motion and not skid. An additional benefit of maintaining wheel rotation is that directional control is not lost as it is in skidding.

Unfortunately, in emergency situations drivers typically press down as hard as they can on the brake pedal, "locking the brakes." This stops the wheels from rotating, ensuring a skid and reducing the frictional force from the static to the kinetic case. To address this problem, automotive engineers have developed antilock braking systems (ABS) that very briefly release the brakes when a wheel is just about to stop turning. This maintains rolling contact between the tire and the pavement. When the brakes are released momentarily, the stopping distance is greater than it would be if the brakes were being applied continuously. However, through the use of computer control, the "brake-off" time is kept to a minimum. As a result, the stopping distance is much less than what it would be if the wheels were to skid.

Let us model the stopping of a car by examining real data. In a recent issue of *AutoWeek*,[7] the braking performance for a Toyota Corolla was measured. These data correspond to the braking force acquired by a highly trained, professional driver. We begin by assuming constant acceleration. (Why do we need to make this assumption?) The magazine provided the initial speed and stopping distance in non-SI units. After converting these values to SI we use $v_{xf}^2 = v_{xi}^2 + 2a_x x$ to deter-

[6] Equation 5 shows that when $\mu_k m_1 > m_2$, there is a range of values of F for which no motion occurs at a given angle θ.

[7] *AutoWeek* magazine, 48:22–23, 1998.

mine the acceleration at different speeds. These do not vary greatly, and so our assumption of constant acceleration is reasonable.

Initial Speed		Stopping Distance		Acceleration
(mi/h)	(m/s)	(ft)	(m)	(m/s^2)
30	13.4	34	10.4	-8.67
60	26.8	143	43.6	-8.25
80	35.8	251	76.5	-8.36

We take an average value of acceleration of $-8.4\ m/s^2$, which is approximately $0.86g$. We then calculate the coefficient of friction from $\Sigma F = \mu_s mg = ma$; this gives $\mu_s = 0.86$ for the Toyota. This is lower than the rubber-to-concrete value given in Table 5.2. Can you think of any reasons for this?

Let us now estimate the stopping distance of the car if the wheels were skidding. Examining Table 5.2 again, we see that the difference between the coefficients of static and kinetic friction for rubber against concrete is about 0.2. Let us therefore assume that our coefficients differ by the same amount, so that $\mu_k \approx 0.66$. This allows us to calculate estimated stopping distances for the case in which the wheels are locked and the car skids across the pavement. The results illustrate the advantage of not allowing the wheels to skid.

Initial Speed (mi/h)	Stopping Distance no skid (m)	Stopping distance skidding (m)
30	10.4	13.9
60	43.6	55.5
80	76.5	98.9

An ABS keeps the wheels rotating, with the result that the higher coefficient of static friction is maintained between the tires and road. This approximates the technique of a professional driver who is able to maintain the wheels at the point of maximum frictional force. Let us estimate the ABS performance by assuming that the magnitude of the acceleration is not quite as good as that achieved by the professional driver but instead is reduced by 5%.

We now plot in Figure 5.22 vehicle speed versus distance from where the brakes were applied (at an initial speed of 80 mi/h = 37.5 m/s) for the three cases of amateur driver, professional driver, and estimated ABS performance (amateur driver). We find that a markedly shorter distance is necessary for stopping without locking the wheels and skidding and a satisfactory value of stopping distance when the ABS computer maintains tire rotation.

The purpose of the ABS is to help typical drivers (whose tendency is to lock the wheels in an emergency) to better maintain control of their automobiles and minimize stopping distance.

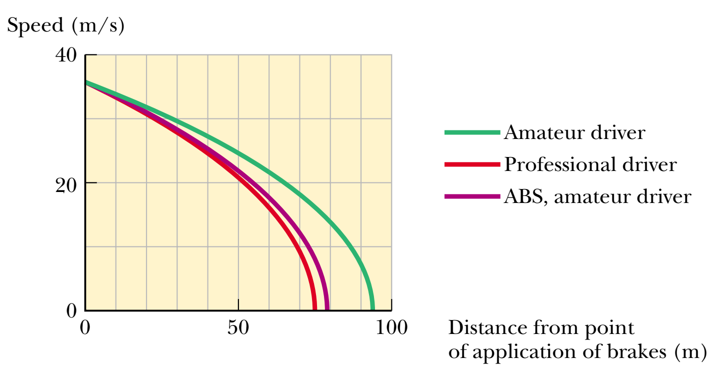


Figure 5.22 This plot of vehicle speed versus distance from where the brakes were applied shows that an antilock braking system (ABS) approaches the performance of a trained professional driver.

SUMMARY

Newton's first law states that, in the absence of an external force, a body at rest remains at rest and a body in uniform motion in a straight line maintains that motion. An **inertial frame** is one that is not accelerating.

Newton's second law states that the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass. The net force acting on an object equals the product of its mass and its acceleration: $\Sigma \mathbf{F} = m\mathbf{a}$. You should be able to apply the x and y component forms of this equation to describe the acceleration of any object acting under the influence of speci-

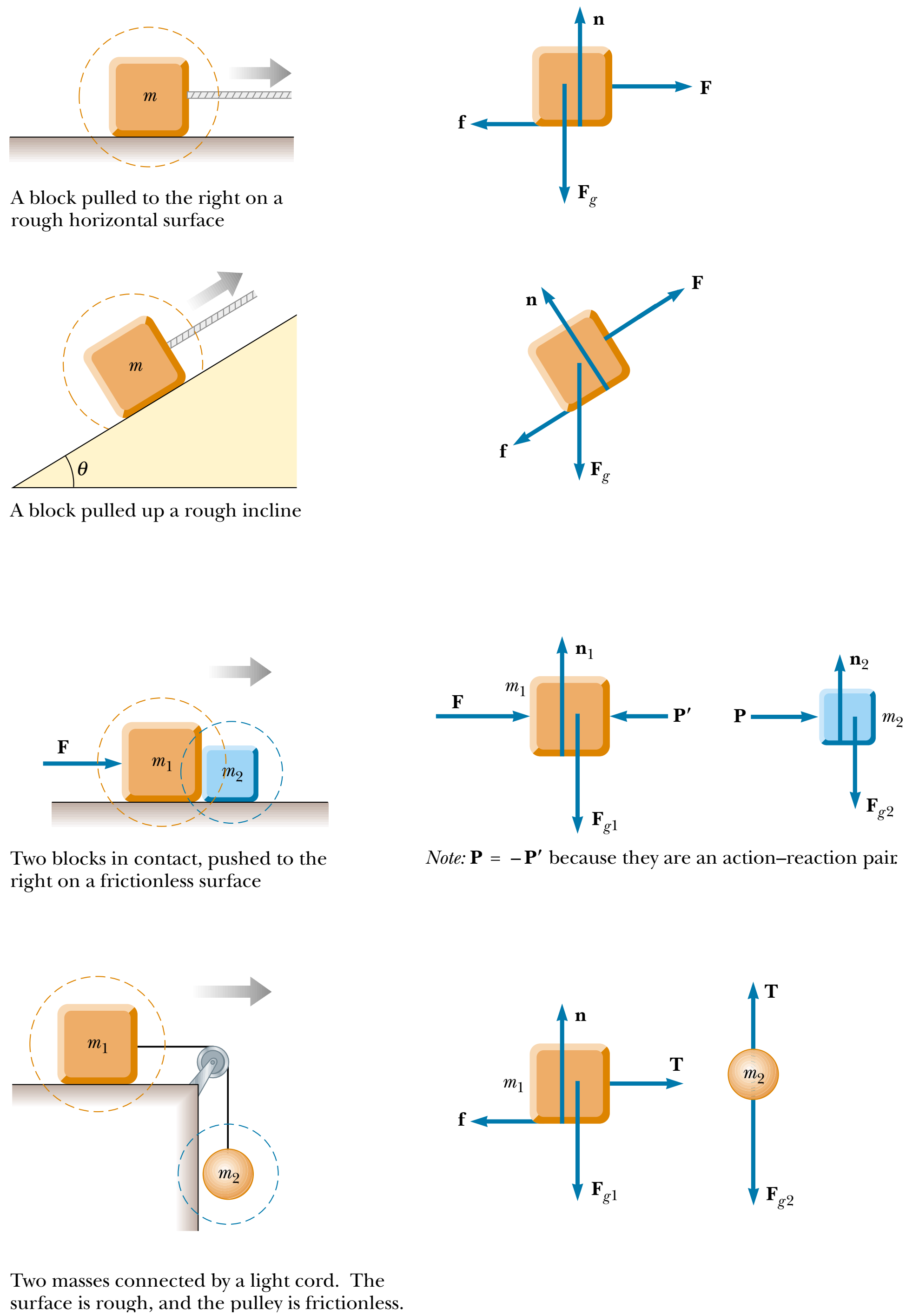


Figure 5.23 Various systems *(left)* and the corresponding free-body diagrams *(right)*.

fied forces. If the object is either stationary or moving with constant velocity, then the forces must vectorially cancel each other.

The **force of gravity** exerted on an object is equal to the product of its mass (a scalar quantity) and the free-fall acceleration: $\mathbf{F}_g = m\mathbf{g}$. The **weight** of an object is the magnitude of the force of gravity acting on the object.

Newton's third law states that if two objects interact, then the force exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force exerted by object 2 on object 1. Thus, an isolated force cannot exist in nature. Make sure you can identify third-law pairs and the two objects upon which they act.

The **maximum force of static friction** $\mathbf{f}_{s,\max}$ between an object and a surface is proportional to the normal force acting on the object. In general, $f_s \leq \mu_s n$, where μ_s is the **coefficient of static friction** and n is the magnitude of the normal force. When an object slides over a surface, the direction of the **force of kinetic friction** $\mathbf{f}_k$ is opposite the direction of sliding motion and is also proportional to the magnitude of the normal force. The magnitude of this force is given by $f_k = \mu_k n$, where μ_k is the **coefficient of kinetic friction.**

More on Free-Body Diagrams

To be successful in applying Newton's second law to a system, you must be able to recognize all the forces acting on the system. That is, you must be able to construct the correct free-body diagram. The importance of constructing the free-body diagram cannot be overemphasized. In Figure 5.23 a number of systems are presented together with their free-body diagrams. You should examine these carefully and then construct free-body diagrams for other systems described in the end-of-chapter problems. When a system contains more than one element, it is important that you construct a separate free-body diagram for *each* element.

As usual, $\mathbf{F}$ denotes some applied force, $\mathbf{F}_g = m\mathbf{g}$ is the force of gravity, $\mathbf{n}$ denotes a normal force, $\mathbf{f}$ is the force of friction, and $\mathbf{T}$ is the force whose magnitude is the tension exerted on an object.

QUESTIONS

1. A passenger sitting in the rear of a bus claims that he was injured when the driver slammed on the brakes, causing a suitcase to come flying toward the passenger from the front of the bus. If you were the judge in this case, what disposition would you make? Why?
2. A space explorer is in a spaceship moving through space far from any planet or star. She notices a large rock, taken as a specimen from an alien planet, floating around the cabin of the spaceship. Should she push it gently toward a storage compartment or kick it toward the compartment? Why?
3. A massive metal object on a rough metal surface may undergo contact welding to that surface. Discuss how this affects the frictional force between object and surface.
4. The observer in the elevator of Example 5.8 would claim that the weight of the fish is T, the scale reading. This claim is obviously wrong. Why does this observation differ from that of a person in an inertial frame outside the elevator?
5. Identify the action–reaction pairs in the following situations: a man takes a step; a snowball hits a woman in the back; a baseball player catches a ball; a gust of wind strikes a window.
6. A ball is held in a person's hand. (a) Identify all the external forces acting on the ball and the reaction to each. (b) If the ball is dropped, what force is exerted on it while it is falling? Identify the reaction force in this case. (Neglect air resistance.)
7. If a car is traveling westward with a constant speed of 20 m/s, what is the resultant force acting on it?
8. "When the locomotive in Figure 5.3 broke through the wall of the train station, the force exerted by the locomotive on the wall was greater than the force the wall could exert on the locomotive." Is this statement true or in need of correction? Explain your answer.
9. A rubber ball is dropped onto the floor. What force causes the ball to bounce?
10. What is wrong with the statement, "Because the car is at rest, no forces are acting on it"? How would you correct this statement?

11. Suppose you are driving a car along a highway at a high speed. Why should you avoid slamming on your brakes if you want to stop in the shortest distance? That is, why should you keep the wheels turning as you brake?

12. If you have ever taken a ride in an elevator of a high-rise building, you may have experienced a nauseating sensation of "heaviness" and "lightness" depending on the direction of the acceleration. Explain these sensations. Are we truly weightless in free-fall?

13. The driver of a speeding empty truck slams on the brakes and skids to a stop through a distance d. (a) If the truck carried a heavy load such that its mass were doubled, what would be its skidding distance? (b) If the initial speed of the truck is halved, what would be its skidding distance?

14. In an attempt to define Newton's third law, a student states that the action and reaction forces are equal in magnitude and opposite in direction to each other. If this is the case, how can there ever be a net force on an object?

15. What force causes (a) a propeller-driven airplane to move? (b) a rocket? (c) a person walking?

16. Suppose a large and spirited Freshman team is beating the Sophomores in a tug-of-war contest. The center of the rope being tugged is gradually accelerating toward the Freshman team. State the relationship between the strengths of these two forces: First, the force the Freshmen exert on the Sophomores; and second, the force the Sophomores exert on the Freshmen.

17. If you push on a heavy box that is at rest, you must exert some force to start its motion. However, once the box is sliding, you can apply a smaller force to maintain that motion. Why?

18. A weight lifter stands on a bathroom scale. He pumps a barbell up and down. What happens to the reading on the scale as this is done? Suppose he is strong enough to actually *throw* the barbell upward. How does the reading on the scale vary now?

19. As a rocket is fired from a launching pad, its speed *and* acceleration increase with time as its engines continue to operate. Explain why this occurs even though the force of the engines exerted on the rocket remains constant.

20. In the motion picture *It Happened One Night* (Columbia Pictures, 1934), Clark Gable is standing inside a stationary bus in front of Claudette Colbert, who is seated. The bus suddenly starts moving forward, and Clark falls into Claudette's lap. Why did this happen?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Sections 5.1 through 5.6

1. A force **F** applied to an object of mass m_1 produces an acceleration of 3.00 m/s^2. The same force applied to a second object of mass m_2 produces an acceleration of 1.00 m/s^2. (a) What is the value of the ratio m_1/m_2? (b) If m_1 and m_2 are combined, find their acceleration under the action of the force **F**.

2. A force of 10.0 N acts on a body of mass 2.00 kg. What are (a) the body's acceleration, (b) its weight in newtons, and (c) its acceleration if the force is doubled?

3. A 3.00-kg mass undergoes an acceleration given by $\mathbf{a} = (2.00\mathbf{i} + 5.00\mathbf{j})$ m/s^2. Find the resultant force $\Sigma\mathbf{F}$ and its magnitude.

4. A heavy freight train has a mass of 15 000 metric tons. If the locomotive can pull with a force of 750 000 N, how long does it take to increase the speed from 0 to 80.0 km/h?

5. A 5.00-g bullet leaves the muzzle of a rifle with a speed of 320 m/s. The expanding gases behind it exert what force on the bullet while it is traveling down the barrel of the rifle, 0.820 m long? Assume constant acceleration and negligible friction.

6. After uniformly accelerating his arm for 0.090 0 s, a pitcher releases a baseball of weight 1.40 N with a velocity of 32.0 m/s horizontally forward. If the ball starts from rest, (a) through what distance does the ball accelerate before its release? (b) What force does the pitcher exert on the ball?

7. After uniformly accelerating his arm for a time t, a pitcher releases a baseball of weight $-F_g\mathbf{j}$ with a velocity $v\mathbf{i}$. If the ball starts from rest, (a) through what distance does the ball accelerate before its release? (b) What force does the pitcher exert on the ball?

8. Define one pound as the weight of an object of mass 0.453 592 37 kg at a location where the acceleration due to gravity is 32.174 0 ft/s^2. Express the pound as one quantity with one SI unit.

WEB **9.** A 4.00-kg object has a velocity of 3.00**i** m/s at one instant. Eight seconds later, its velocity has increased to $(8.00\mathbf{i} + 10.0\mathbf{j})$ m/s. Assuming the object was subject to a constant total force, find (a) the components of the force and (b) its magnitude.

10. The average speed of a nitrogen molecule in air is about 6.70×10^2 m/s, and its mass is 4.68×10^{-26} kg. (a) If it takes 3.00×10^{-13} s for a nitrogen molecule to hit a wall and rebound with the same speed but moving in the opposite direction, what is the average acceleration of the molecule during this time interval? (b) What average force does the molecule exert on the wall?

11. An electron of mass 9.11×10^{-31} kg has an initial speed of 3.00×10^5 m/s. It travels in a straight line, and its speed increases to 7.00×10^5 m/s in a distance of 5.00 cm. Assuming its acceleration is constant, (a) determine the force exerted on the electron and (b) compare this force with the weight of the electron, which we neglected.

12. A woman weighs 120 lb. Determine (a) her weight in newtons and (b) her mass in kilograms.

13. If a man weighs 900 N on the Earth, what would he weigh on Jupiter, where the acceleration due to gravity is 25.9 m/s^2?

14. The distinction between mass and weight was discovered after Jean Richer transported pendulum clocks from Paris to French Guiana in 1671. He found that they ran slower there quite systematically. The effect was reversed when the clocks returned to Paris. How much weight would you personally lose in traveling from Paris, where $g = 9.809\,5$ m/s^2, to Cayenne, where $g = 9.780\,8$ m/s^2? (We shall consider how the free-fall acceleration influences the period of a pendulum in Section 13.4.)

15. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a 5.00-kg mass. If $F_1 = 20.0$ N and $F_2 = 15.0$ N, find the accelerations in (a) and (b) of Figure P5.15.

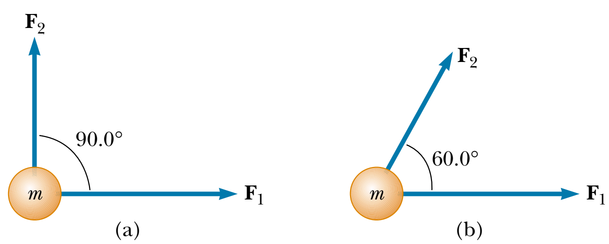


Figure P5.15

16. Besides its weight, a 2.80-kg object is subjected to one other constant force. The object starts from rest and in 1.20 s experiences a displacement of (4.20 m)$\mathbf{i}$ − (3.30 m)$\mathbf{j}$, where the direction of $\mathbf{j}$ is the upward vertical direction. Determine the other force.

17. You stand on the seat of a chair and then hop off. (a) During the time you are in flight down to the floor, the Earth is lurching up toward you with an acceleration of what order of magnitude? In your solution explain your logic. Visualize the Earth as a perfectly solid object. (b) The Earth moves up through a distance of what order of magnitude?

18. Forces of 10.0 N north, 20.0 N east, and 15.0 N south are simultaneously applied to a 4.00-kg mass as it rests on an air table. Obtain the object's acceleration.

19. A boat moves through the water with two horizontal forces acting on it. One is a 2000-N forward push caused by the motor; the other is a constant 1800-N resistive force caused by the water. (a) What is the acceleration of the 1 000-kg boat? (b) If it starts from rest, how far will it move in 10.0 s? (c) What will be its speed at the end of this time?

20. Three forces, given by $\mathbf{F}_1 = (-2.00\mathbf{i} + 2.00\mathbf{j})$ N, $\mathbf{F}_2 = (5.00\mathbf{i} - 3.00\mathbf{j})$ N, and $\mathbf{F}_3 = (-45.0\mathbf{i})$ N, act on an object to give it an acceleration of magnitude 3.75 m/s^2. (a) What is the direction of the acceleration? (b) What is the mass of the object? (c) If the object is initially at rest, what is its speed after 10.0 s? (d) What are the velocity components of the object after 10.0 s?

21. A 15.0-lb block rests on the floor. (a) What force does the floor exert on the block? (b) If a rope is tied to the block and run vertically over a pulley, and the other end is attached to a free-hanging 10.0-lb weight, what is the force exerted by the floor on the 15.0-lb block? (c) If we replace the 10.0-lb weight in part (b) with a 20.0-lb weight, what is the force exerted by the floor on the 15.0-lb block?

Section 5.7 Some Applications of Newton's Laws

22. A 3.00-kg mass is moving in a plane, with its x and y coordinates given by $x = 5t^2 - 1$ and $y = 3t^3 + 2$, where x and y are in meters and t is in seconds. Find the magnitude of the net force acting on this mass at $t = 2.00$ s.

23. The distance between two telephone poles is 50.0 m. When a 1.00-kg bird lands on the telephone wire midway between the poles, the wire sags 0.200 m. Draw a free-body diagram of the bird. How much tension does the bird produce in the wire? Ignore the weight of the wire.

24. A bag of cement of weight 325 N hangs from three wires as shown in Figure P5.24. Two of the wires make angles $\theta_1 = 60.0°$ and $\theta_2 = 25.0°$ with the horizontal. If the system is in equilibrium, find the tensions T_1, T_2, and T_3 in the wires.

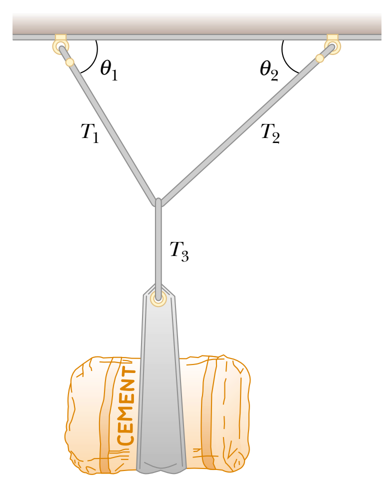


Figure P5.24 Problems 24 and 25.

25. A bag of cement of weight F_g hangs from three wires as shown in Figure P5.24. Two of the wires make angles θ_1 and θ_2 with the horizontal. If the system is in equilibrium, show that the tension in the left-hand wire is

$$T_1 = F_g \cos \theta_2 / \sin(\theta_1 + \theta_2)$$

26. You are a judge in a children's kite-flying contest, and two children will win prizes for the kites that pull most strongly and least strongly on their strings. To measure string tensions, you borrow a weight hanger, some slotted weights, and a protractor from your physics teacher and use the following protocol, illustrated in Figure P5.26: Wait for a child to get her kite well-controlled, hook the hanger onto the kite string about 30 cm from her hand, pile on weights until that section of string is horizontal, record the mass required, and record the angle between the horizontal and the string running up to the kite. (a) Explain how this method works. As you construct your explanation, imagine that the children's parents ask you about your method, that they might make false assumptions about your ability without concrete evidence, and that your explanation is an opportunity to give them confidence in your evaluation technique. (b) Find the string tension if the mass required to make the string horizontal is 132 g and the angle of the kite string is 46.3°.

Figure P5.26

27. The systems shown in Figure P5.27 are in equilibrium. If the spring scales are calibrated in newtons, what do they read? (Neglect the masses of the pulleys and strings, and assume the incline is frictionless.)

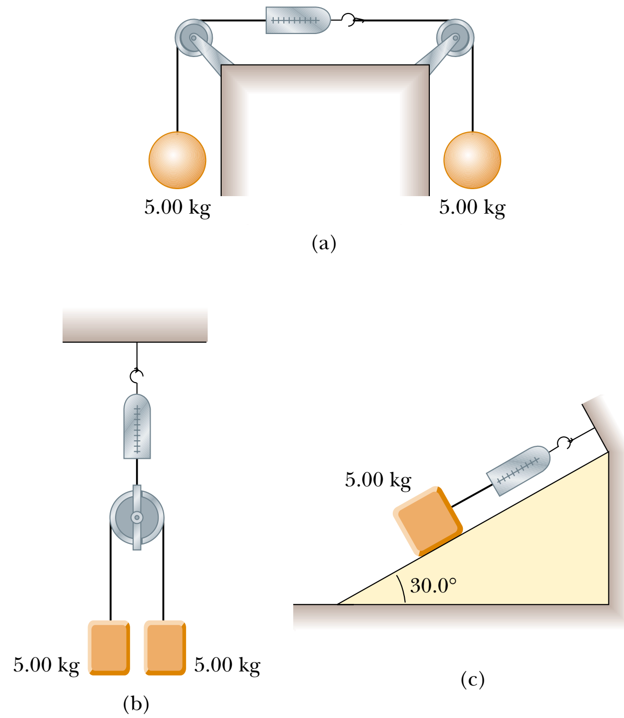


Figure P5.27

28. A fire helicopter carries a 620-kg bucket of water at the end of a cable 20.0 m long. As the aircraft flies back from a fire at a constant speed of 40.0 m/s, the cable makes an angle of 40.0° with respect to the vertical. (a) Determine the force of air resistance on the bucket. (b) After filling the bucket with sea water, the pilot returns to the fire at the same speed with the bucket now making an angle of 7.00° with the vertical. What is the mass of the water in the bucket?

WEB 29. A 1.00-kg mass is observed to accelerate at 10.0 m/s² in a direction 30.0° north of east (Fig. P5.29). The force $\mathbf{F}_2$ acting on the mass has a magnitude of 5.00 N and is directed north. Determine the magnitude and direction of the force $\mathbf{F}_1$ acting on the mass.

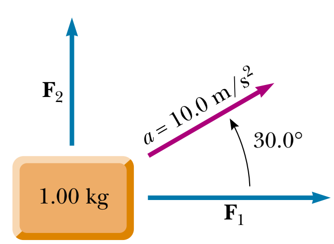


Figure P5.29

30. A simple accelerometer is constructed by suspending a mass m from a string of length L that is tied to the top of a cart. As the cart is accelerated the string-mass system makes a constant angle θ with the vertical. (a) Assuming that the string mass is negligible compared with m, derive an expression for the cart's acceleration in terms of θ and show that it is independent of

the mass m and the length L. (b) Determine the acceleration of the cart when $\theta = 23.0°$.

31. Two people pull as hard as they can on ropes attached to a boat that has a mass of 200 kg. If they pull in the same direction, the boat has an acceleration of 1.52 m/s^2 to the right. If they pull in opposite directions, the boat has an acceleration of 0.518 m/s^2 to the left. What is the force exerted by each person on the boat? (Disregard any other forces on the boat.)

32. Draw a free-body diagram for a block that slides down a frictionless plane having an inclination of $\theta = 15.0°$ (Fig. P5.32). If the block starts from rest at the top and the length of the incline is 2.00 m, find (a) the acceleration of the block and (b) its speed when it reaches the bottom of the incline.

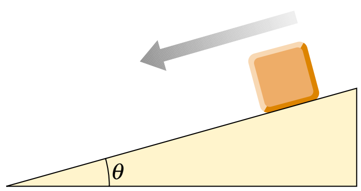


Figure P5.32

WEB **33.** A block is given an initial velocity of 5.00 m/s up a frictionless 20.0° incline. How far up the incline does the block slide before coming to rest?

34. Two masses are connected by a light string that passes over a frictionless pulley, as in Figure P5.34. If the incline is frictionless and if $m_1 = 2.00$ kg, $m_2 = 6.00$ kg, and $\theta = 55.0°$, find (a) the accelerations of the masses, (b) the tension in the string, and (c) the speed of each mass 2.00 s after being released from rest.

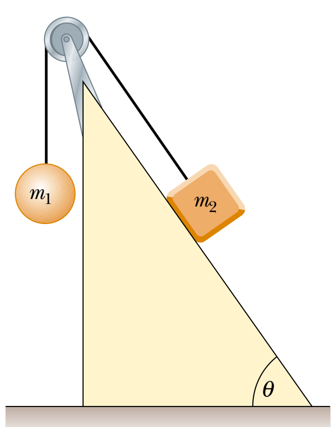


Figure P5.34

35. Two masses m_1 and m_2 situated on a frictionless, horizontal surface are connected by a light string. A force **F** is exerted on one of the masses to the right (Fig. P5.35). Determine the acceleration of the system and the tension T in the string.

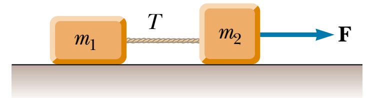


Figure P5.35 Problems 35 and 51.

36. Two masses of 3.00 kg and 5.00 kg are connected by a light string that passes over a frictionless pulley, as was shown in Figure 5.15a. Determine (a) the tension in the string, (b) the acceleration of each mass, and (c) the distance each mass will move in the first second of motion if they start from rest.

37. In the system shown in Figure P5.37, a horizontal force F_x acts on the 8.00-kg mass. The horizontal surface is frictionless. (a) For what values of F_x does the 2.00-kg mass accelerate upward? (b) For what values of F_x is the tension in the cord zero? (c) Plot the acceleration of the 8.00-kg mass versus F_x. Include values of F_x from -100 N to $+100$ N.

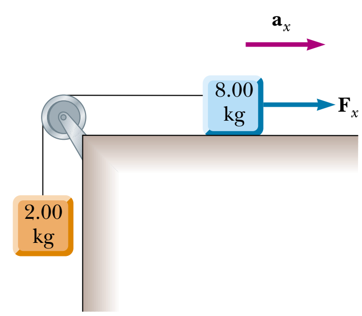


Figure P5.37

38. Mass m_1 on a frictionless horizontal table is connected to mass m_2 by means of a very light pulley P_1 and a light fixed pulley P_2 as shown in Figure P5.38. (a) If a_1 and a_2

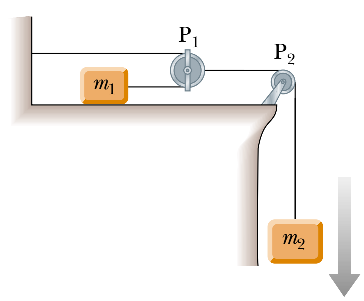


Figure P5.38

are the accelerations of m_1 and m_2, respectively, what is the relationship between these accelerations? Express (b) the tensions in the strings and (c) the accelerations a_1 and a_2 in terms of the masses m_1 and m_2 and g.

39. A 72.0-kg man stands on a spring scale in an elevator. Starting from rest, the elevator ascends, attaining its maximum speed of 1.20 m/s in 0.800 s. It travels with this constant speed for the next 5.00 s. The elevator then undergoes a uniform acceleration in the negative y direction for 1.50 s and comes to rest. What does the spring scale register (a) before the elevator starts to move? (b) during the first 0.800 s? (c) while the elevator is traveling at constant speed? (d) during the time it is slowing down?

Section 5.8 Forces of Friction

40. The coefficient of static friction is 0.800 between the soles of a sprinter's running shoes and the level track surface on which she is running. Determine the maximum acceleration she can achieve. Do you need to know that her mass is 60.0 kg?

41. A 25.0-kg block is initially at rest on a horizontal surface. A horizontal force of 75.0 N is required to set the block in motion. After it is in motion, a horizontal force of 60.0 N is required to keep the block moving with constant speed. Find the coefficients of static and kinetic friction from this information.

42. A racing car accelerates uniformly from 0 to 80.0 mi/h in 8.00 s. The external force that accelerates the car is the frictional force between the tires and the road. If the tires do not slip, determine the minimum coefficient of friction between the tires and the road.

43. A car is traveling at 50.0 mi/h on a horizontal highway. (a) If the coefficient of friction between road and tires on a rainy day is 0.100, what is the minimum distance in which the car will stop? (b) What is the stopping distance when the surface is dry and $\mu_s = 0.600$?

44. A woman at an airport is towing her 20.0-kg suitcase at constant speed by pulling on a strap at an angle of θ above the horizontal (Fig. P5.44). She pulls on the strap with a 35.0-N force, and the frictional force on the suitcase is 20.0 N. Draw a free-body diagram for the suitcase. (a) What angle does the strap make with the horizontal? (b) What normal force does the ground exert on the suitcase?

WEB **45.** A 3.00-kg block starts from rest at the top of a 30.0° incline and slides a distance of 2.00 m down the incline in 1.50 s. Find (a) the magnitude of the acceleration of the block, (b) the coefficient of kinetic friction between block and plane, (c) the frictional force acting on the block, and (d) the speed of the block after it has slid 2.00 m.

46. To determine the coefficients of friction between rubber and various surfaces, a student uses a rubber eraser and an incline. In one experiment the eraser begins to slip down the incline when the angle of inclination is 36.0° and then moves down the incline with constant speed when the angle is reduced to 30.0°. From these data, determine the coefficients of static and kinetic friction for this experiment.

Figure P5.44

47. A boy drags his 60.0-N sled at constant speed up a 15.0° hill. He does so by pulling with a 25.0-N force on a rope attached to the sled. If the rope is inclined at 35.0° to the horizontal, (a) what is the coefficient of kinetic friction between sled and snow? (b) At the top of the hill, he jumps on the sled and slides down the hill. What is the magnitude of his acceleration down the slope?

48. Determine the stopping distance for a skier moving down a slope with friction with an initial speed of 20.0 m/s (Fig. P5.48). Assume $\mu_k = 0.180$ and $\theta = 5.00°$.

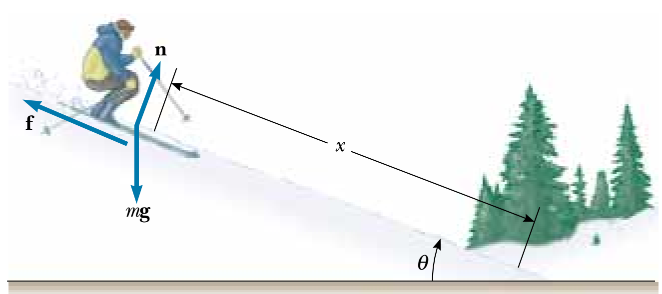


Figure P5.48

49. A 9.00-kg hanging weight is connected by a string over a pulley to a 5.00-kg block that is sliding on a flat table (Fig. P5.49). If the coefficient of kinetic friction is 0.200, find the tension in the string.

50. Three blocks are connected on a table as shown in Figure P5.50. The table is rough and has a coefficient of ki-

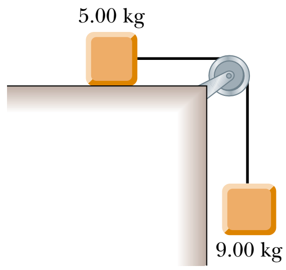


Figure P5.49

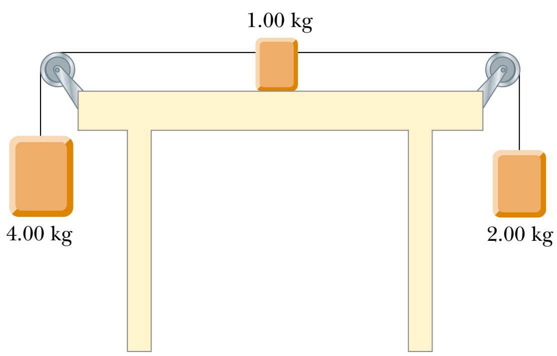


Figure P5.50

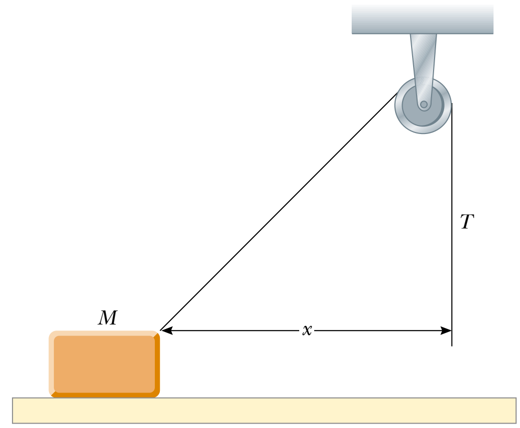


Figure P5.52

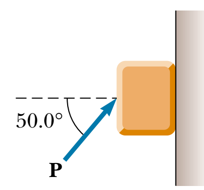


Figure P5.53

netic friction of 0.350. The three masses are 4.00 kg, 1.00 kg, and 2.00 kg, and the pulleys are frictionless. Draw a free-body diagram for each block. (a) Determine the magnitude and direction of the acceleration of each block. (b) Determine the tensions in the two cords.

51. Two blocks connected by a rope of negligible mass are being dragged by a horizontal force **F** (see Fig. P5.35). Suppose that $F = 68.0$ N, $m_1 = 12.0$ kg, $m_2 = 18.0$ kg, and the coefficient of kinetic friction between each block and the surface is 0.100. (a) Draw a free-body diagram for each block. (b) Determine the tension T and the magnitude of the acceleration of the system.

52. A block of mass 2.20 kg is accelerated across a rough surface by a rope passing over a pulley, as shown in Figure P5.52. The tension in the rope is 10.0 N, and the pulley is 10.0 cm above the top of the block. The coefficient of kinetic friction is 0.400. (a) Determine the acceleration of the block when $x = 0.400$ m. (b) Find the value of x at which the acceleration becomes zero.

53. A block of mass 3.00 kg is pushed up against a wall by a force **P** that makes a 50.0° angle with the horizontal as shown in Figure P5.53. The coefficient of static friction between the block and the wall is 0.250. Determine the possible values for the magnitude of **P** that allow the block to remain stationary.

ADDITIONAL PROBLEMS

54. A time-dependent force $\mathbf{F} = (8.00\mathbf{i} - 4.00t\mathbf{j})$ N (where t is in seconds) is applied to a 2.00-kg object initially at rest. (a) At what time will the object be moving with a speed of 15.0 m/s? (b) How far is the object from its initial position when its speed is 15.0 m/s? (c) What is the object's displacement at the time calculated in (a)?

55. An inventive child named Pat wants to reach an apple in a tree without climbing the tree. Sitting in a chair connected to a rope that passes over a frictionless pulley (Fig. P5.55), Pat pulls on the loose end of the rope with such a force that the spring scale reads 250 N. Pat's weight is 320 N, and the chair weighs 160 N. (a) Draw free-body diagrams for Pat and the chair considered as separate systems, and draw another diagram for Pat and the chair considered as one system. (b) Show that the acceleration of the system is *upward* and find its magnitude. (c) Find the force Pat exerts on the chair.

56. Three blocks are in contact with each other on a frictionless, horizontal surface, as in Figure P5.56. A horizontal force **F** is applied to m_1. If $m_1 = 2.00$ kg, $m_2 = 3.00$ kg, $m_3 = 4.00$ kg, and $F = 18.0$ N, draw a separate free-body diagram for each block and find (a) the acceleration of the blocks, (b) the *resultant* force on each block, and (c) the magnitudes of the contact forces between the blocks.

Figure P5.55

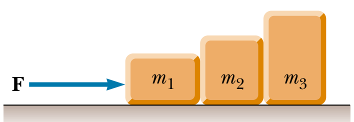


Figure P5.56

57. A high diver of mass 70.0 kg jumps off a board 10.0 m above the water. If his downward motion is stopped 2.00 s after he enters the water, what average upward force did the water exert on him?

58. Consider the three connected objects shown in Figure P5.58. If the inclined plane is frictionless and the system is in equilibrium, find (in terms of m, g, and θ) (a) the mass M and (b) the tensions T_1 and T_2. If the value of M is double the value found in part (a), find (c) the acceleration of each object, and (d) the tensions T_1 and T_2. If the coefficient of static friction between m and $2m$ and the inclined plane is μ_s, and the system is in equilibrium, find (e) the minimum value of M and (f) the maximum value of M. (g) Compare the values of T_2 when M has its minimum and maximum values.

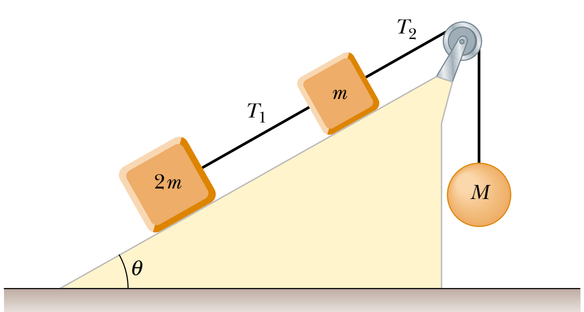


Figure P5.58

WEB **59.** A mass M is held in place by an applied force **F** and a pulley system as shown in Figure P5.59. The pulleys are massless and frictionless. Find (a) the tension in each section of rope, T_1, T_2, T_3, T_4, and T_5 and (b) the magnitude of **F**. (*Hint:* Draw a free-body diagram for each pulley.)

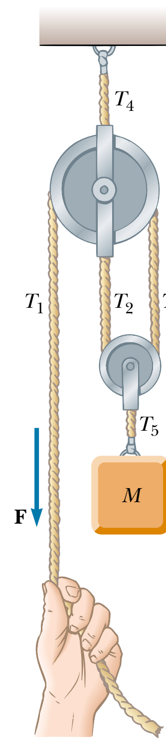


Figure P5.59

60. Two forces, given by $\mathbf{F}_1 = (-6.00\mathbf{i} - 4.00\mathbf{j})$ N and $\mathbf{F}_2 = (-3.00\mathbf{i} + 7.00\mathbf{j})$ N, act on a particle of mass 2.00 kg that is initially at rest at coordinates (−2.00 m, +4.00 m). (a) What are the components of the particle's velocity at $t = 10.0$ s? (b) In what direction is the particle moving at $t = 10.0$ s? (c) What displacement does the particle undergo during the first 10.0 s? (d) What are the coordinates of the particle at $t = 10.0$ s?

61. A crate of weight $\mathbf{F}_g$ is pushed by a force **P** on a horizontal floor. (a) If the coefficient of static friction is μ_s and **P** is directed at an angle θ below the horizontal, show that the minimum value of P that will move the crate is given by

$$P = \mu_s F_g \sec\theta (1 - \mu_s \tan\theta)^{-1}$$

(b) Find the minimum value of P that can produce motion when $\mu_s = 0.400$, $F_g = 100$ N, and $\theta = 0°$, 15.0°, 30.0°, 45.0°, and 60.0°.

62. **Review Problem.** A block of mass $m = 2.00$ kg is released from rest $h = 0.500$ m from the surface of a table, at the top of a $\theta = 30.0°$ incline as shown in Figure P5.62. The frictionless incline is fixed on a table of height $H = 2.00$ m. (a) Determine the acceleration of the block as it slides down the incline. (b) What is the velocity of the block as it leaves the incline? (c) How far from the table will the block hit the floor? (d) How much time has elapsed between when the block is released and when it hits the floor? (e) Does the mass of the block affect any of the above calculations?

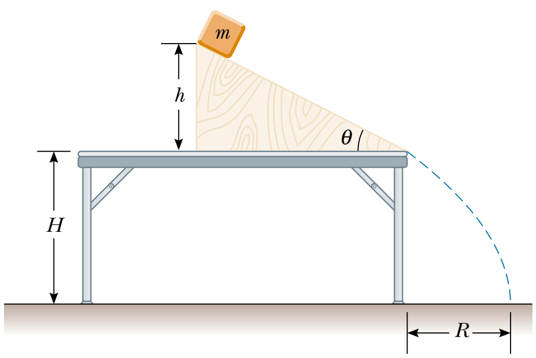


Figure P5.62

63. A 1.30-kg toaster is not plugged in. The coefficient of static friction between the toaster and a horizontal countertop is 0.350. To make the toaster start moving, you carelessly pull on its electric cord. (a) For the cord tension to be as small as possible, you should pull at what angle above the horizontal? (b) With this angle, how large must the tension be?

64. A 2.00-kg aluminum block and a 6.00-kg copper block are connected by a light string over a frictionless pulley. They sit on a steel surface, as shown in Figure P5.64, and $\theta = 30.0°$. Do they start to move once any holding mechanism is released? If so, determine (a) their acceleration and (b) the tension in the string. If not, determine the sum of the magnitudes of the forces of friction acting on the blocks.

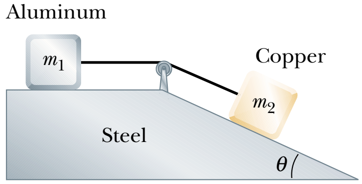


Figure P5.64

65. A block of mass $m = 2.00$ kg rests on the left edge of a block of larger mass $M = 8.00$ kg. The coefficient of kinetic friction between the two blocks is 0.300, and the surface on which the 8.00-kg block rests is frictionless. A constant horizontal force of magnitude $F = 10.0$ N is applied to the 2.00-kg block, setting it in motion as shown in Figure P5.65a. If the length L that the leading edge of the smaller block travels on the larger block is 3.00 m, (a) how long will it take before this block makes it to the right side of the 8.00-kg block, as shown in Figure P5.65b? (*Note:* Both blocks are set in motion when **F** is applied.) (b) How far does the 8.00-kg block move in the process?

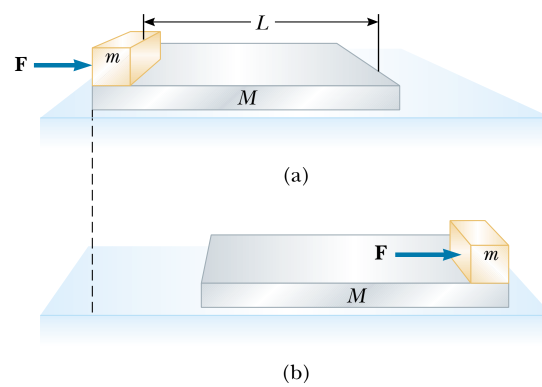


Figure P5.65

66. A student is asked to measure the acceleration of a cart on a "frictionless" inclined plane as seen in Figure P5.32, using an air track, a stopwatch, and a meter stick. The height of the incline is measured to be 1.774 cm, and the total length of the incline is measured to be $d = 127.1$ cm. Hence, the angle of inclination θ is determined from the relation $\sin\theta = 1.774/127.1$. The cart is released from rest at the top of the incline, and its displacement x along the incline is measured versus time, where $x = 0$ refers to the initial position of the cart. For x values of 10.0 cm, 20.0 cm, 35.0 cm, 50.0 cm, 75.0 cm, and 100 cm, the measured times to undergo these displacements (averaged over five runs) are 1.02 s, 1.53 s, 2.01 s, 2.64 s, 3.30 s, and 3.75 s, respectively. Construct a graph of x versus t^2, and perform a linear least-squares fit to the data. Determine the acceleration of the cart from the slope of this graph, and compare it with the value you would get using $a' = g\sin\theta$, where $g = 9.80 \text{ m/s}^2$.

67. A 2.00-kg block is placed on top of a 5.00-kg block as in Figure P5.67. The coefficient of kinetic friction between the 5.00-kg block and the surface is 0.200. A horizontal force **F** is applied to the 5.00-kg block. (a) Draw a free-body diagram for each block. What force accelerates the 2.00-kg block? (b) Calculate the magnitude of the force necessary to pull both blocks to the right with an

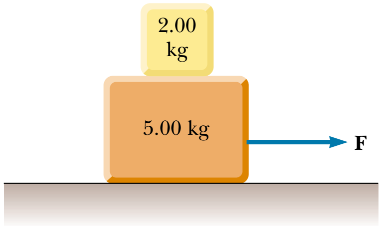


Figure P5.67

acceleration of 3.00 m/s^2. (c) Find the minimum coefficient of static friction between the blocks such that the 2.00-kg block does not slip under an acceleration of 3.00 m/s^2.

68. A 5.00-kg block is placed on top of a 10.0-kg block (Fig. P5.68). A horizontal force of 45.0 N is applied to the 10.0-kg block, and the 5.00-kg block is tied to the wall. The coefficient of kinetic friction between all surfaces is 0.200. (a) Draw a free-body diagram for each block and identify the action–reaction forces between the blocks. (b) Determine the tension in the string and the magnitude of the acceleration of the 10.0-kg block.

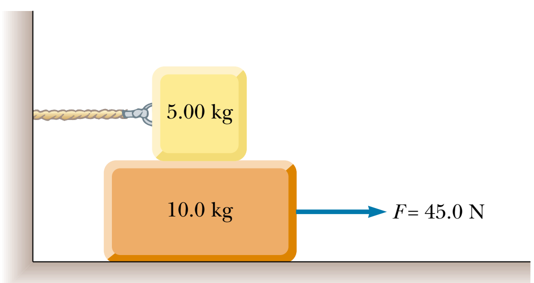


Figure P5.68

69. What horizontal force must be applied to the cart shown in Figure P5.69 so that the blocks remain stationary relative to the cart? Assume all surfaces, wheels, and pulley are frictionless. (*Hint:* Note that the force exerted by the string accelerates m_1.)

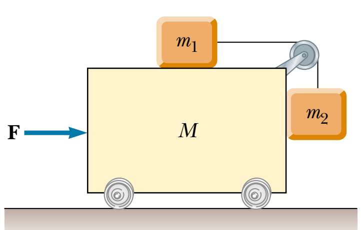


Figure P5.69 Problems 69 and 70.

70. Initially the system of masses shown in Figure P5.69 is held motionless. All surfaces, pulley, and wheels are frictionless. Let the force **F** be zero and assume that m_2 can move only vertically. At the instant after the system of masses is released, find (a) the tension T in the string, (b) the acceleration of m_2, (c) the acceleration of M, and (d) the acceleration of m_1. (*Note:* The pulley accelerates along with the cart.)

71. A block of mass 5.00 kg sits on top of a second block of mass 15.0 kg, which in turn sits on a horizontal table. The coefficients of friction between the two blocks are $\mu_s = 0.300$ and $\mu_k = 0.100$. The coefficients of friction between the lower block and the rough table are $\mu_s = 0.500$ and $\mu_k = 0.400$. You apply a constant horizontal force to the lower block, just large enough to make this block start sliding out from between the upper block and the table. (a) Draw a free-body diagram of each block, naming the forces acting on each. (b) Determine the magnitude of each force on each block at the instant when you have started pushing but motion has not yet started. (c) Determine the acceleration you measure for each block.

72. Two blocks of mass 3.50 kg and 8.00 kg are connected by a string of negligible mass that passes over a frictionless pulley (Fig. P5.72). The inclines are frictionless. Find (a) the magnitude of the acceleration of each block and (b) the tension in the string.

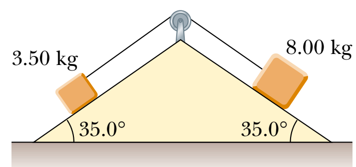


Figure P5.72 Problems 72 and 73.

73. The system shown in Figure P5.72 has an acceleration of magnitude 1.50 m/s^2. Assume the coefficients of kinetic friction between block and incline are the same for both inclines. Find (a) the coefficient of kinetic friction and (b) the tension in the string.

74. In Figure P5.74, a 500-kg horse pulls a sledge of mass 100 kg. The system (horse plus sledge) has a forward acceleration of 1.00 m/s^2 when the frictional force exerted on the sledge is 500 N. Find (a) the tension in the connecting rope and (b) the magnitude and direction of the force of friction exerted on the horse. (c) Verify that the total forces of friction the ground exerts on the system will give the system an acceleration of 1.00 m/s^2.

75. A van accelerates down a hill (Fig. P5.75), going from rest to 30.0 m/s in 6.00 s. During the acceleration, a toy ($m = 0.100$ kg) hangs by a string from the van's ceiling. The acceleration is such that the string remains perpendicular to the ceiling. Determine (a) the angle θ and (b) the tension in the string.

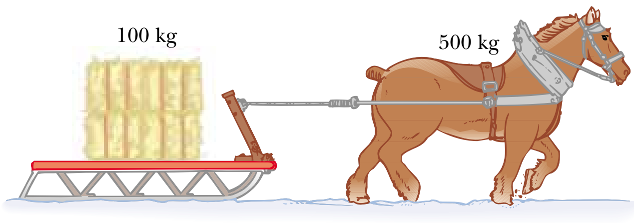


Figure P5.74

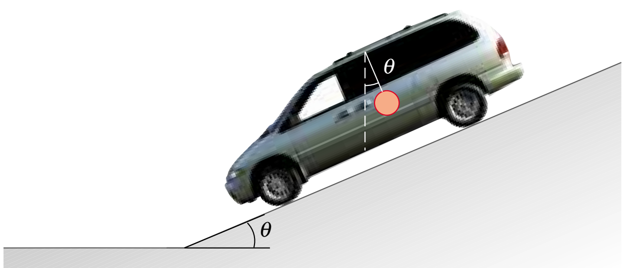


Figure P5.75

76. A mobile is formed by supporting four metal butterflies of equal mass m from a string of length L. The points of support are evenly spaced a distance ℓ apart as shown in Figure P5.76. The string forms an angle θ_1 with the ceiling at each end point. The center section of string is horizontal. (a) Find the tension in each section of string in terms of θ_1, m, and g. (b) Find the angle θ_2, in terms of θ_1, that the sections of string between the outside butterflies and the inside butterflies form with the horizontal. (c) Show that the distance D between the end points of the string is

$$D = \frac{L}{5}\left\{2\cos\theta_1 + 2\cos\left[\tan^{-1}\left(\frac{1}{2}\tan\theta_1\right)\right] + 1\right\}$$

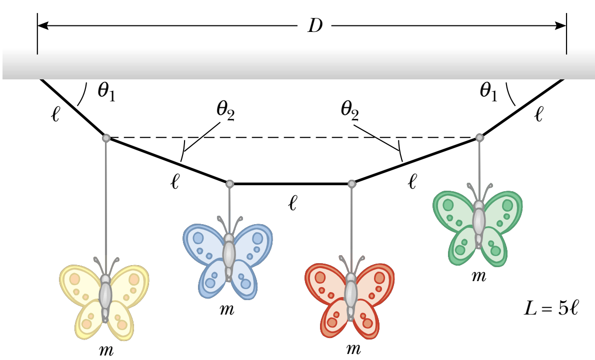


Figure P5.76

77. Before 1960 it was believed that the maximum attainable coefficient of static friction for an automobile tire was less than 1. Then about 1962, three companies independently developed racing tires with coefficients of 1.6. Since then, tires have improved, as illustrated in this problem. According to the 1990 Guinness Book of Records, the fastest time in which a piston-engine car initially at rest has covered a distance of one-quarter mile is 4.96 s. This record was set by Shirley Muldowney in September 1989 (Fig. P5.77). (a) Assuming that the rear wheels nearly lifted the front wheels off the pavement, what minimum value of μ_s is necessary to achieve the record time? (b) Suppose Muldowney were able to double her engine power, keeping other things equal. How would this change affect the elapsed time?

Figure P5.77

78. An 8.40-kg mass slides down a fixed, frictionless inclined plane. Use a computer to determine and tabulate the normal force exerted on the mass and its acceleration for a series of incline angles (measured from the horizontal) ranging from 0 to 90° in 5° increments. Plot a graph of the normal force and the acceleration as functions of the incline angle. In the limiting cases of 0 and 90°, are your results consistent with the known behavior?

Answers to Quick Quizzes

5.1 (a) True. Newton's first law tells us that motion requires no force: An object in motion continues to move at constant velocity in the absence of external forces. (b) True. A stationary object can have several forces acting on it, but if the vector sum of all these external forces is zero, there is no net force and the object remains stationary. It also is possible to have a net force and no motion, but only for an instant. A ball tossed vertically upward stops at the peak of its path for an infinitesimally short time, but the force of gravity is still acting on it. Thus, although $\mathbf{v} = 0$ at the peak, the net force acting on the ball is *not* zero.

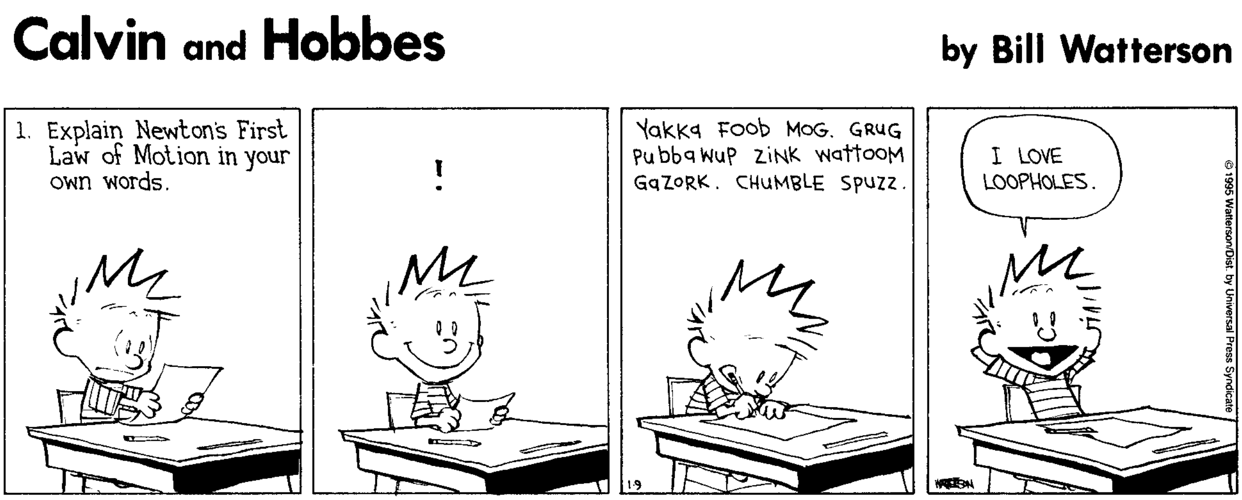
5.2 No. Direction of motion is part of an object's *velocity,* and force determines the direction of acceleration, not that of velocity.

5.3 (a) Force of gravity. (b) Force of gravity. The only external force acting on the ball at *all* points in its trajectory is the downward force of gravity.

5.4 As the person steps out of the boat, he pushes against it with his foot, expecting the boat to push back on him so that he accelerates toward the dock. However, because the boat is untied, the force exerted by the foot causes the boat to scoot away from the dock. As a result, the person is not able to exert a very large force on the boat before it moves out of reach. Therefore, the boat does not exert a very large reaction force on him, and he ends up not being accelerated sufficiently to make it to the dock. Consequently, he falls into the water instead. If a small dog were to jump from the untied boat toward the dock, the force exerted by the boat on the dog would probably be enough to ensure the dog's successful landing because of the dog's small mass.

5.5 (a) The same force is experienced by both. The fly and bus experience forces that are equal in magnitude but opposite in direction. (b) The fly. Because the fly has such a small mass, it undergoes a very large acceleration. The huge mass of the bus means that it more effectively resists any change in its motion.

5.6 (b) The crate accelerates to the right. Because the only horizontal force acting on it is the force of static friction between its bottom surface and the truck bed, that force must also be directed to the right.



PUZZLER

This sky diver is falling at more than 50 m/s (120 mi/h), but once her parachute opens, her downward velocity will be greatly reduced. Why does she slow down rapidly when her chute opens, enabling her to fall safely to the ground? If the chute does not function properly, the sky diver will almost certainly be seriously injured. What force exerted on her limits her maximum speed?

(Guy Savage/Photo Researchers, Inc.)

chapter 6

Circular Motion and Other Applications of Newton's Laws

Chapter Outline

In the preceding chapter we introduced Newton's laws of motion and applied them to situations involving linear motion. Now we discuss motion that is slightly more complicated. For example, we shall apply Newton's laws to objects traveling in circular paths. Also, we shall discuss motion observed from an accelerating frame of reference and motion in a viscous medium. For the most part, this chapter is a series of examples selected to illustrate the application of Newton's laws to a wide variety of circumstances.

6.1 NEWTON'S SECOND LAW APPLIED TO UNIFORM CIRCULAR MOTION

In Section 4.4 we found that a particle moving with uniform speed v in a circular path of radius r experiences an acceleration $\mathbf{a}_r$ that has a magnitude

$$a_r = \frac{v^2}{r}$$

4.7 The acceleration is called the *centripetal acceleration* because $\mathbf{a}_r$ is directed toward the center of the circle. Furthermore, $\mathbf{a}_r$ is *always* perpendicular to $\mathbf{v}$. (If there were a component of acceleration parallel to $\mathbf{v}$, the particle's speed would be changing.)

Consider a ball of mass m that is tied to a string of length r and is being whirled at constant speed in a horizontal circular path, as illustrated in Figure 6.1. Its weight is supported by a low-friction table. Why does the ball move in a circle? Because of its inertia, the tendency of the ball is to move in a straight line; however, the string prevents motion along a straight line by exerting on the ball a force that makes it follow the circular path. This force is directed along the string toward the center of the circle, as shown in Figure 6.1. This force can be any one of our familiar forces causing an object to follow a circular path.

If we apply Newton's second law along the radial direction, we find that the value of the net force causing the centripetal acceleration can be evaluated:

Force causing centripetal acceleration

$$\sum F_r = ma_r = m\frac{v^2}{r} \tag{6.1}$$

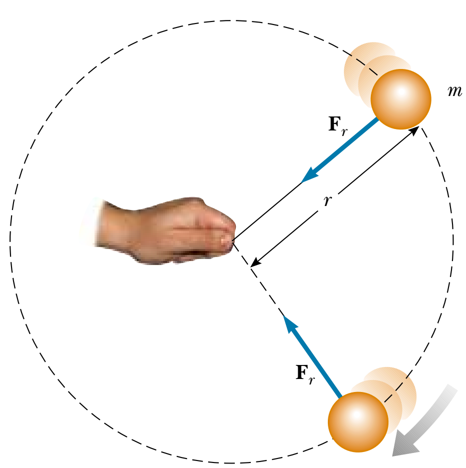


Figure 6.1 Overhead view of a ball moving in a circular path in a horizontal plane. A force $\mathbf{F}_r$ directed toward the center of the circle keeps the ball moving in its circular path.

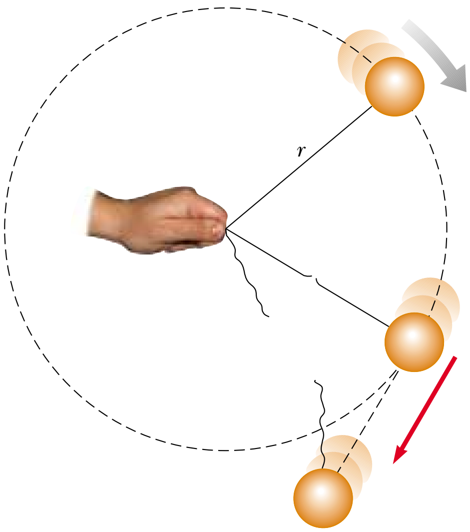


Figure 6.2 When the string breaks, the ball moves in the direction tangent to the circle.

An athlete in the process of throwing the hammer at the 1996 Olympic Games in Atlanta, Georgia. The force exerted by the chain is the force causing the circular motion. Only when the athlete releases the hammer will it move along a straight-line path tangent to the circle.

A force causing a centripetal acceleration acts toward the center of the circular path and causes a change in the direction of the velocity vector. If that force should vanish, the object would no longer move in its circular path; instead, it would move along a straight-line path tangent to the circle. This idea is illustrated in Figure 6.2 for the ball whirling at the end of a string. If the string breaks at some instant, the ball moves along the straight-line path tangent to the circle at the point where the string broke.

Quick Quiz 6.1

Is it possible for a car to move in a circular path in such a way that it has a tangential acceleration but no centripetal acceleration?

CONCEPTUAL EXAMPLE 6.1 Forces That Cause Centripetal Acceleration

The force causing centripetal acceleration is sometimes called a *centripetal force*. We are familiar with a variety of forces in nature—friction, gravity, normal forces, tension, and so forth. Should we add *centripetal* force to this list?

Solution No; centripetal force *should not* be added to this list. This is a pitfall for many students. Giving the force causing circular motion a name—centripetal force—leads many students to consider it a new kind of force rather than a new *role* for force. A common mistake in force diagrams is to draw all the usual forces and then to add another vector for the centripetal force. But it is not a separate force—it is simply one of our familiar forces *acting in the role of a force that causes a circular motion.*

Consider some examples. For the motion of the Earth around the Sun, the centripetal force is *gravity*. For an object sitting on a rotating turntable, the centripetal force is *friction*. For a rock whirled on the end of a string, the centripetal force is the force of *tension* in the string. For an amusement-park patron pressed against the inner wall of a rapidly rotating circular room, the centripetal force is the *normal force* exerted by the wall. What's more, the centripetal force could be a combination of two or more forces. For example, as a Ferris-wheel rider passes through the lowest point, the centripetal force on her is the difference between the normal force exerted by the seat and her weight.

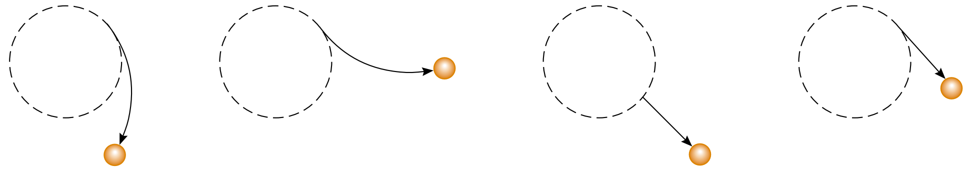


Figure 6.3 A ball that had been moving in a circular path is acted on by various external forces that change its path.

Quick Quiz 6.2

A ball is following the dotted circular path shown in Figure 6.3 under the influence of a force. At a certain instant of time, the force on the ball changes abruptly to a new force, and the ball follows the paths indicated by the solid line with an arrowhead in each of the four parts of the figure. For each part of the figure, describe the magnitude and direction of the force required to make the ball move in the solid path. If the dotted line represents the path of a ball being whirled on the end of a string, which path does the ball follow if the string breaks?

QuickLab

Tie a string to a tennis ball, swing it in a circle, and then, while it is swinging, let go of the string to verify your answer to the last part of Quick Quiz 6.2.

Let us consider some examples of uniform circular motion. In each case, be sure to recognize the external force (or forces) that causes the body to move in its circular path.

EXAMPLE 6.2 How Fast Can It Spin?

A ball of mass 0.500 kg is attached to the end of a cord 1.50 m long. The ball is whirled in a horizontal circle as was shown in Figure 6.1. If the cord can withstand a maximum tension of 50.0 N, what is the maximum speed the ball can attain before the cord breaks? Assume that the string remains horizontal during the motion.

Solution It is difficult to know what might be a reasonable value for the answer. Nonetheless, we know that it cannot be too large, say 100 m/s, because a person cannot make a ball move so quickly. It makes sense that the stronger the cord, the faster the ball can twirl before the cord breaks. Also, we expect a more massive ball to break the cord at a lower speed. (Imagine whirling a bowling ball!)

Because the force causing the centripetal acceleration in this case is the force **T** exerted by the cord on the ball, Equation 6.1 yields for $\Sigma F_r = ma_r$

$$T = m\frac{v^2}{r}$$

Solving for v, we have

$$v = \sqrt{\frac{Tr}{m}}$$

This shows that v increases with T and decreases with larger m, as we expect to see—for a given v, a large mass requires a large tension and a small mass needs only a small tension. The maximum speed the ball can have corresponds to the maximum tension. Hence, we find

$$v_{\text{max}} = \sqrt{\frac{T_{\text{max}}r}{m}} = \sqrt{\frac{(50.0\text{ N})(1.50\text{ m})}{0.500\text{ kg}}}$$

$$= 12.2\text{ m/s}$$

Exercise Calculate the tension in the cord if the speed of the ball is 5.00 m/s.

Answer 8.33 N.

EXAMPLE 6.3 The Conical Pendulum

A small object of mass m is suspended from a string of length L. The object revolves with constant speed v in a horizontal circle of radius r, as shown in Figure 6.4. (Because the string sweeps out the surface of a cone, the system is known as a *conical pendulum*.) Find an expression for v.

Solution Let us choose θ to represent the angle between string and vertical. In the free-body diagram shown in Figure 6.4, the force **T** exerted by the string is resolved into a vertical component $T\cos\theta$ and a horizontal component $T\sin\theta$ acting toward the center of revolution. Because the object does not accelerate in the vertical direction, $\Sigma F_y = ma_y = 0$, and the upward vertical component of **T** must balance the downward force of gravity. Therefore,

$$(1) \qquad T\cos\theta = mg$$

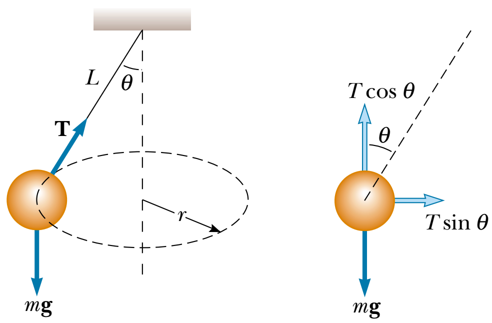


Figure 6.4 The conical pendulum and its free-body diagram.

Because the force providing the centripetal acceleration in this example is the component $T\sin\theta$, we can use Newton's second law and Equation 6.1 to obtain

$$(2) \qquad \sum F_r = T\sin\theta = ma_r = \frac{mv^2}{r}$$

Dividing (2) by (1) and remembering that $\sin\theta/\cos\theta = \tan\theta$, we eliminate T and find that

$$\tan\theta = \frac{v^2}{rg}$$

$$v = \sqrt{rg\tan\theta}$$

From the geometry in Figure 6.4, we note that $r = L\sin\theta$; therefore,

$$v = \boxed{\sqrt{Lg\sin\theta\tan\theta}}$$

Note that the speed is independent of the mass of the object.

EXAMPLE 6.4 What Is the Maximum Speed of the Car?

A 1 500-kg car moving on a flat, horizontal road negotiates a curve, as illustrated in Figure 6.5. If the radius of the curve is 35.0 m and the coefficient of static friction between the tires and dry pavement is 0.500, find the maximum speed the car can have and still make the turn successfully.

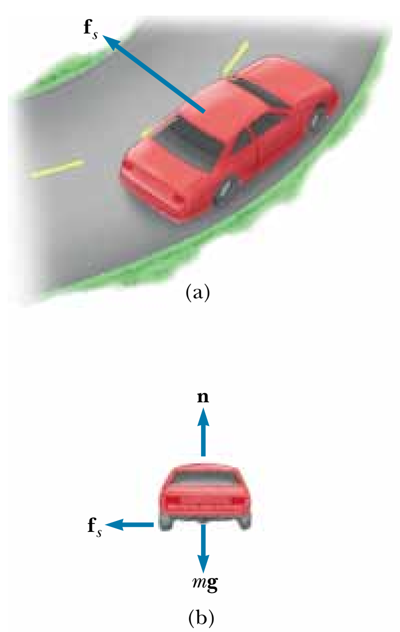


Figure 6.5 (a) The force of static friction directed toward the center of the curve keeps the car moving in a circular path. (b) The free-body diagram for the car.

Solution From experience, we should expect a maximum speed less than 50 m/s. (A convenient mental conversion is that 1 m/s is roughly 2 mi/h.) In this case, the force that enables the car to remain in its circular path is the force of static friction. (Because no slipping occurs at the point of contact between road and tires, the acting force is a force of static friction directed toward the center of the curve. If this force of static friction were zero—for example, if the car were on an icy road—the car would continue in a straight line and slide off the road.) Hence, from Equation 6.1 we have

$$(1) \qquad f_s = m\frac{v^2}{r}$$

The maximum speed the car can have around the curve is the speed at which it is on the verge of skidding outward. At this point, the friction force has its maximum value $f_{s,\max} = \mu_s n$. Because the car is on a horizontal road, the magnitude of the normal force equals the weight ($n = mg$) and thus $f_{s,\max} = \mu_s mg$. Substituting this value for f_s into (1), we find that the maximum speed is

$$v_{\max} = \sqrt{\frac{f_{s,\max}r}{m}} = \sqrt{\frac{\mu_s mgr}{m}} = \sqrt{\mu_s gr}$$

$$= \sqrt{(0.500)(9.80\ \text{m/s}^2)(35.0\ \text{m})} = \boxed{13.1\ \text{m/s}}$$

Note that the maximum speed does not depend on the mass of the car. That is why curved highways do not need multiple speed limit signs to cover the various masses of vehicles using the road.

Exercise On a wet day, the car begins to skid on the curve when its speed reaches 8.00 m/s. What is the coefficient of static friction in this case?

Answer 0.187.

EXAMPLE 6.5 The Banked Exit Ramp

A civil engineer wishes to design a curved exit ramp for a highway in such a way that a car will not have to rely on friction to round the curve without skidding. In other words, a car moving at the designated speed can negotiate the curve even when the road is covered with ice. Such a ramp is usually *banked;* this means the roadway is tilted toward the inside of the curve. Suppose the designated speed for the ramp is to be 13.4 m/s (30.0 mi/h) and the radius of the curve is 50.0 m. At what angle should the curve be banked?

Solution On a level (unbanked) road, the force that causes the centripetal acceleration is the force of static friction between car and road, as we saw in the previous example. However, if the road is banked at an angle θ, as shown in Figure 6.6, the normal force **n** has a horizontal component $n \sin \theta$ pointing toward the center of the curve. Because the ramp is to be designed so that the force of static friction is zero, only the component $n \sin \theta$ causes the centripetal acceleration. Hence, Newton's second law written for the radial direction gives

$$(1) \qquad \sum F_r = n \sin \theta = \frac{mv^2}{r}$$

The car is in equilibrium in the vertical direction. Thus, from $\Sigma F_y = 0$, we have

$$(2) \qquad n \cos \theta = mg$$

Dividing (1) by (2) gives

$$\tan \theta = \frac{v^2}{rg}$$

$$\theta = \tan^{-1}\left[\frac{(13.4 \text{ m/s})^2}{(50.0 \text{ m})(9.80 \text{ m/s}^2)}\right] = 20.1^\circ$$

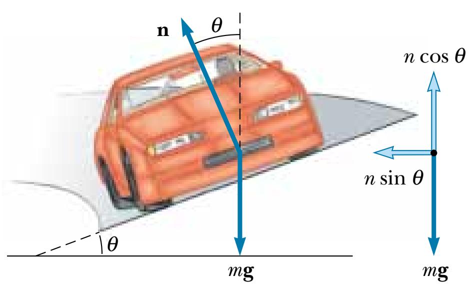


Figure 6.6 Car rounding a curve on a road banked at an angle θ to the horizontal. When friction is neglected, the force that causes the centripetal acceleration and keeps the car moving in its circular path is the horizontal component of the normal force. Note that **n** is the *sum* of the forces exerted by the road on the wheels.

If a car rounds the curve at a speed less than 13.4 m/s, friction is needed to keep it from sliding down the bank (to the left in Fig. 6.6). A driver who attempts to negotiate the curve at a speed greater than 13.4 m/s has to depend on friction to keep from sliding up the bank (to the right in Fig. 6.6). The banking angle is independent of the mass of the vehicle negotiating the curve.

Exercise Write Newton's second law applied to the radial direction when a frictional force $\mathbf{f}_s$ is directed down the bank, toward the center of the curve.

Answer $n \sin \theta + f_s \cos \theta = \frac{mv^2}{r}$

EXAMPLE 6.6 Satellite Motion

This example treats a satellite moving in a circular orbit around the Earth. To understand this situation, you must know that the gravitational force between spherical objects and small objects that can be modeled as particles having masses m_1 and m_2 and separated by a distance r is attractive and has a magnitude

$$F_g = G\frac{m_1 m_2}{r^2}$$

where $G = 6.673 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$. This is Newton's law of gravitation, which we study in Chapter 14.

Consider a satellite of mass m moving in a circular orbit around the Earth at a constant speed v and at an altitude h above the Earth's surface, as illustrated in Figure 6.7. Determine the speed of the satellite in terms of G, h, R_E (the radius of the Earth), and M_E (the mass of the Earth).

Solution The only external force acting on the satellite is the force of gravity, which acts toward the center of the Earth

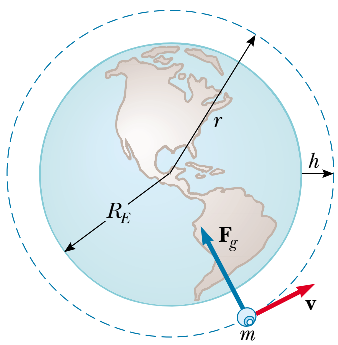


Figure 6.7 A satellite of mass m moving around the Earth at a constant speed v in a circular orbit of radius $r = R_E + h$. The force $\mathbf{F}_g$ acting on the satellite that causes the centripetal acceleration is the gravitational force exerted by the Earth on the satellite.

and keeps the satellite in its circular orbit. Therefore,

$$F_r = F_g = G\frac{M_E m}{r^2}$$

From Newton's second law and Equation 6.1 we obtain

$$G\frac{M_E m}{r^2} = m\frac{v^2}{r}$$

Solving for v and remembering that the distance r from the center of the Earth to the satellite is $r = R_E + h$, we obtain

$$(1) \qquad v = \sqrt{\frac{GM_E}{r}} = \boxed{\sqrt{\frac{GM_E}{R_E + h}}}$$

If the satellite were orbiting a different planet, its velocity would increase with the mass of the planet and decrease as the satellite's distance from the center of the planet increased.

Exercise A satellite is in a circular orbit around the Earth at an altitude of 1 000 km. The radius of the Earth is equal to 6.37×10^6 m, and its mass is 5.98×10^{24} kg. Find the speed of the satellite, and then find the *period*, which is the time it needs to make one complete revolution.

Answer 7.36×10^3 m/s; 6.29×10^3 s = 105 min.

EXAMPLE 6.7 Let's Go Loop-the-Loop!

A pilot of mass m in a jet aircraft executes a loop-the-loop, as shown in Figure 6.8a. In this maneuver, the aircraft moves in a vertical circle of radius 2.70 km at a constant speed of 225 m/s. Determine the force exerted by the seat on the pilot (a) at the bottom of the loop and (b) at the top of the loop. Express your answers in terms of the weight of the pilot mg.

Solution We expect the answer for (a) to be greater than that for (b) because at the bottom of the loop the normal and gravitational forces act in opposite directions, whereas at the top of the loop these two forces act in the same direction. It is the vector sum of these two forces that gives the force of constant magnitude that keeps the pilot moving in a circular path. To yield net force vectors with the same magnitude, the normal force at the bottom (where the normal and gravitational forces are in opposite directions) must be greater than that at the top (where the normal and gravitational forces are in the same direction). (a) The free-body diagram for the pilot at the bottom of the loop is shown in Figure 6.8b. The only forces acting on him are the downward force of gravity $\mathbf{F}_g = m\mathbf{g}$ and the upward force $\mathbf{n}_{\text{bot}}$ exerted by the seat. Because the net upward force that provides the centripetal acceleration has a magnitude $n_{\text{bot}} - mg$, Newton's second law for the radial direction combined with Equation 6.1 gives

$$\sum F_r = n_{\text{bot}} - mg = m\frac{v^2}{r}$$

$$n_{\text{bot}} = mg + m\frac{v^2}{r} = mg\left(1 + \frac{v^2}{rg}\right)$$

Substituting the values given for the speed and radius gives

$$n_{\text{bot}} = mg\left[1 + \frac{(225\ \text{m/s})^2}{(2.70 \times 10^3\ \text{m})(9.80\ \text{m/s}^2)}\right] = \boxed{2.91\,mg}$$

Hence, the magnitude of the force $\mathbf{n}_{\text{bot}}$ exerted by the seat on the pilot is *greater* than the weight of the pilot by a factor of 2.91. This means that the pilot experiences an apparent weight that is greater than his true weight by a factor of 2.91.

(b) The free-body diagram for the pilot at the top of the loop is shown in Figure 6.8c. As we noted earlier, both the gravitational force exerted by the Earth and the force $\mathbf{n}_{\text{top}}$ exerted by the seat on the pilot act downward, and so the net downward force that provides the centripetal acceleration has

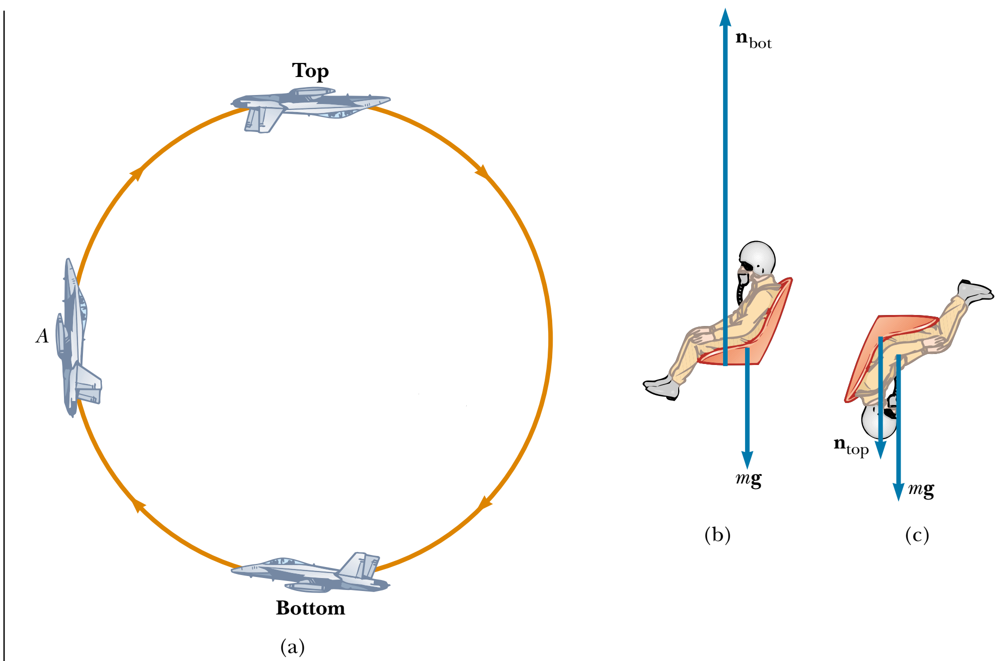


Figure 6.8 (a) An aircraft executes a loop-the-loop maneuver as it moves in a vertical circle at constant speed. (b) Free-body diagram for the pilot at the bottom of the loop. In this position the pilot experiences an apparent weight greater than his true weight. (c) Free-body diagram for the pilot at the top of the loop.

a magnitude $n_{top} + mg$. Applying Newton's second law yields

$$\sum F_r = n_{top} + mg = m\frac{v^2}{r}$$

$$n_{top} = m\frac{v^2}{r} - mg = mg\left(\frac{v^2}{rg} - 1\right)$$

$$n_{top} = mg\left[\frac{(225\text{ m/s})^2}{(2.70 \times 10^3\text{ m})(9.80\text{ m/s}^2)} - 1\right] = \boxed{0.913mg}$$

In this case, the magnitude of the force exerted by the seat on the pilot is *less* than his true weight by a factor of 0.913, and the pilot feels lighter.

Exercise Determine the magnitude of the radially directed force exerted on the pilot by the seat when the aircraft is at point A in Figure 6.8a, midway up the loop.

Answer $n_A = 1.913mg$ directed to the right.

Quick Quiz 6.3

A bead slides freely along a curved wire at constant speed, as shown in the overhead view of Figure 6.9. At each of the points Ⓐ, Ⓑ, and Ⓒ, draw the vector representing the force that the wire exerts on the bead in order to cause it to follow the path of the wire at that point.

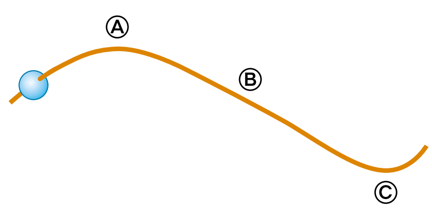


Figure 6.9

QuickLab

Hold a shoe by the end of its lace and spin it in a vertical circle. Can you feel the difference in the tension in the lace when the shoe is at top of the circle compared with when the shoe is at the bottom?

6.2 NONUNIFORM CIRCULAR MOTION

In Chapter 4 we found that if a particle moves with varying speed in a circular path, there is, in addition to the centripetal (radial) component of acceleration, a tangential component having magnitude dv/dt. Therefore, the force acting on the

Some examples of forces acting during circular motion. *(Left)* As these speed skaters round a curve, the force exerted by the ice on their skates provides the centripetal acceleration. *(Right)* Passengers on a "corkscrew" roller coaster. What are the origins of the forces in this example?

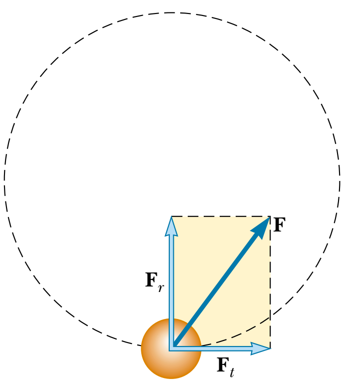


Figure 6.10 When the force acting on a particle moving in a circular path has a tangential component F_t, the particle's speed changes. The total force exerted on the particle in this case is the vector sum of the radial force and the tangential force. That is, $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_t$.

particle must also have a tangential and a radial component. Because the total acceleration is $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t$, the total force exerted on the particle is $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_t$, as shown in Figure 6.10. The vector $\mathbf{F}_r$ is directed toward the center of the circle and is responsible for the centripetal acceleration. The vector $\mathbf{F}_t$ tangent to the circle is responsible for the tangential acceleration, which represents a change in the speed of the particle with time. The following example demonstrates this type of motion.

EXAMPLE 6.8 Keep Your Eye on the Ball

A small sphere of mass m is attached to the end of a cord of length R and whirls in a *vertical* circle about a fixed point O, as illustrated in Figure 6.11a. Determine the tension in the cord at any instant when the speed of the sphere is v and the cord makes an angle θ with the vertical.

Solution Unlike the situation in Example 6.7, the speed is *not* uniform in this example because, at most points along the path, a tangential component of acceleration arises from the gravitational force exerted on the sphere. From the free-body diagram in Figure 6.11b, we see that the only forces acting on

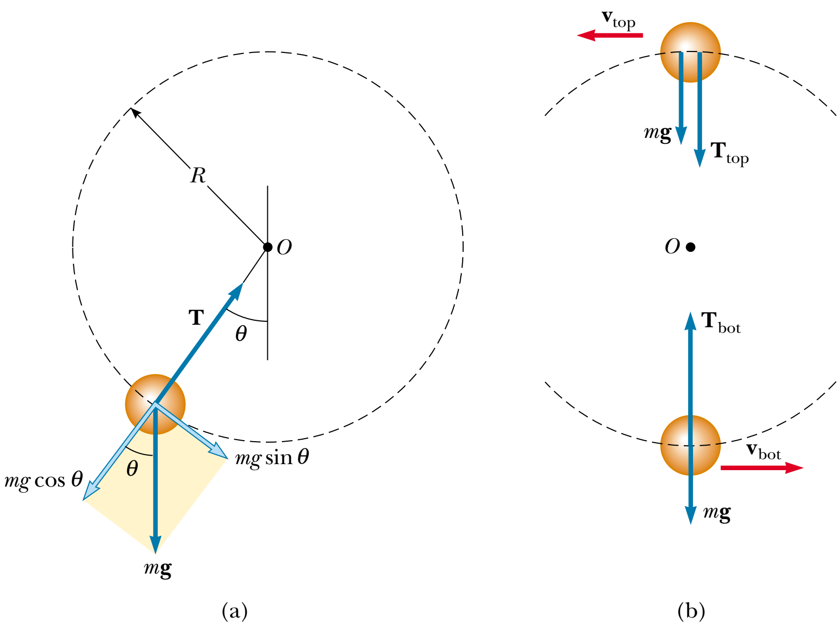


Figure 6.11 (a) Forces acting on a sphere of mass m connected to a cord of length R and rotating in a vertical circle centered at O. (b) Forces acting on the sphere at the top and bottom of the circle. The tension is a maximum at the bottom and a minimum at the top.

the sphere are the gravitational force $\mathbf{F}_g = m\mathbf{g}$ exerted by the Earth and the force $\mathbf{T}$ exerted by the cord. Now we resolve $\mathbf{F}_g$ into a tangential component $mg\sin\theta$ and a radial component $mg\cos\theta$. Applying Newton's second law to the forces acting on the sphere in the tangential direction yields

$$\sum F_t = mg\sin\theta = ma_t$$

$$a_t = g\sin\theta$$

This tangential component of the acceleration causes v to change in time because $a_t = dv/dt$.

Applying Newton's second law to the forces acting on the sphere in the radial direction and noting that both $\mathbf{T}$ and $\mathbf{a}_r$ are directed toward O, we obtain

$$\sum F_r = T - mg\cos\theta = \frac{mv^2}{R}$$

$$T = m\left(\frac{v^2}{R} + g\cos\theta\right)$$

Special Cases At the top of the path, where $\theta = 180°$, we have $\cos 180° = -1$, and the tension equation becomes

$$T_{top} = m\left(\frac{v_{top}^2}{R} - g\right)$$

This is the minimum value of T. Note that at this point $a_t = 0$ and therefore the acceleration is purely radial and directed downward.

At the bottom of the path, where $\theta = 0$, we see that, because $\cos 0 = 1$,

$$T_{bot} = m\left(\frac{v_{bot}^2}{R} + g\right)$$

This is the maximum value of T. At this point, a_t is again 0 and the acceleration is now purely radial and directed upward.

Exercise At what position of the sphere would the cord most likely break if the average speed were to increase?

Answer At the bottom, where T has its maximum value.

Optional Section

6.3 MOTION IN ACCELERATED FRAMES

When Newton's laws of motion were introduced in Chapter 5, we emphasized that they are valid only when observations are made in an inertial frame of reference. In this section, we analyze how an observer in a noninertial frame of reference (one that is accelerating) applies Newton's second law.

To understand the motion of a system that is noninertial because an object is moving along a curved path, consider a car traveling along a highway at a high speed and approaching a curved exit ramp, as shown in Figure 6.12a. As the car takes the sharp left turn onto the ramp, a person sitting in the passenger seat slides to the right and hits the door. At that point, the force exerted on her by the door keeps her from being ejected from the car. What causes her to move toward the door? A popular, but improper, explanation is that some mysterious force acting from left to right pushes her outward. (This is often called the "centrifugal" force, but we shall not use this term because it often creates confusion.) The passenger invents this **fictitious force** to explain what is going on in her accelerated frame of reference, as shown in Figure 6.12b. (The driver also experiences this effect but holds on to the steering wheel to keep from sliding to the right.)

The phenomenon is correctly explained as follows. Before the car enters the ramp, the passenger is moving in a straight-line path. As the car enters the ramp and travels a curved path, the passenger tends to move along the original straight-line path. This is in accordance with Newton's first law: The natural tendency of a body is to continue moving in a straight line. However, if a sufficiently large force (toward the center of curvature) acts on the passenger, as in Figure 6.12c, she will move in a curved path along with the car. The origin of this force is the force of friction between her and the car seat. If this frictional force is not large enough, she will slide to the right as the car turns to the left under her. Eventually, she encounters the door, which provides a force large enough to enable her to follow the same curved path as the car. She slides toward the door not because of some mysterious outward force but because **the force of friction is not sufficiently great to allow her to travel along the circular path followed by the car.**

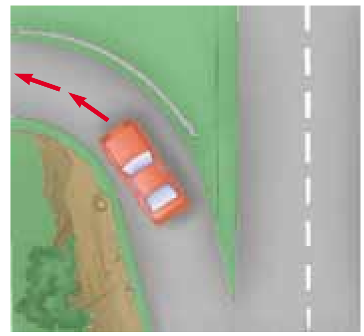
4.8

In general, if a particle moves with an acceleration **a** relative to an observer in an inertial frame, that observer may use Newton's second law and correctly claim that $\Sigma\mathbf{F} = m\mathbf{a}$. If another observer in an accelerated frame tries to apply Newton's second law to the motion of the particle, the person must introduce fictitious forces to make Newton's second law work. These forces "invented" by the observer in the accelerating frame appear to be real. However, we emphasize that **these fictitious forces do not exist when the motion is observed in an inertial frame.** Fictitious forces are used only in an accelerating frame and do not represent "real" forces acting on the particle. (By real forces, we mean the interaction of the particle with its environment.) If the fictitious forces are properly defined in the accelerating frame, the description of motion in this frame is equivalent to the description given by an inertial observer who considers only real forces. Usually, we analyze motions using inertial reference frames, but there are cases in which it is more convenient to use an accelerating frame.

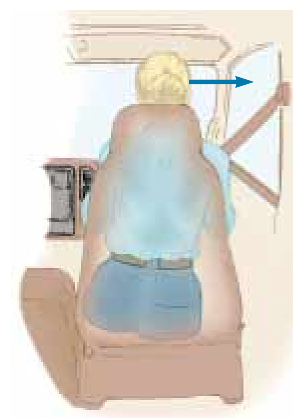
QuickLab

Use a string, a small weight, and a protractor to measure your acceleration as you start sprinting from a standing position.

Fictitious forces



(a)



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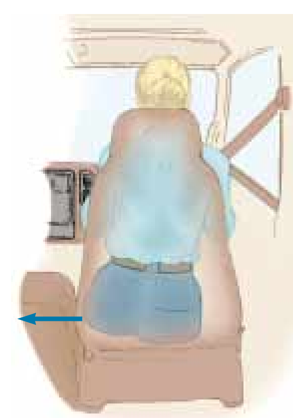


(c)

Figure 6.12 (a) A car approaching a curved exit ramp. What causes a front-seat passenger to move toward the right-hand door? (b) From the frame of reference of the passenger, a (fictitious) force pushes her toward the right door. (c) Relative to the reference frame of the Earth, the car seat applies a leftward force to the passenger, causing her to change direction along with the rest of the car.

EXAMPLE 6.9 Fictitious Forces in Linear Motion

A small sphere of mass m is hung by a cord from the ceiling of a boxcar that is accelerating to the right, as shown in Figure 6.13. According to the inertial observer at rest (Fig. 6.13a), the forces on the sphere are the force $\mathbf{T}$ exerted by the cord and the force of gravity. The inertial observer concludes that the acceleration of the sphere is the same as that of the boxcar and that this acceleration is provided by the horizontal component of $\mathbf{T}$. Also, the vertical component of $\mathbf{T}$ balances the force of gravity. Therefore, she writes Newton's second law as $\Sigma \mathbf{F} = \mathbf{T} + m\mathbf{g} = m\mathbf{a}$, which in component form becomes

$$\text{Inertial observer} \begin{cases} (1) \quad \sum F_x = T\sin\theta = ma \\ (2) \quad \sum F_y = T\cos\theta - mg = 0 \end{cases}$$

Thus, by solving (1) and (2) simultaneously for a, the inertial observer can determine the magnitude of the car's acceleration through the relationship

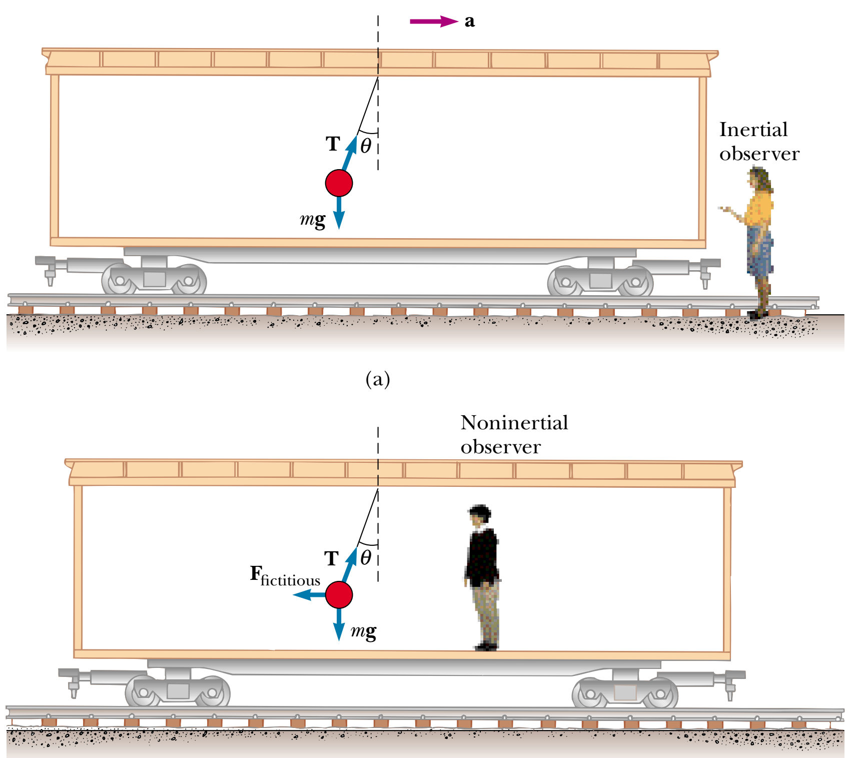
$$a = g\tan\theta$$

Because the deflection of the cord from the vertical serves as a measure of acceleration, *a simple pendulum can be used as an accelerometer.*

According to the noninertial observer riding in the car (Fig. 6.13b), the cord still makes an angle θ with the vertical; however, to her the sphere is at rest and so its acceleration is zero. Therefore, she introduces a fictitious force to balance the horizontal component of $\mathbf{T}$ and claims that the net force on the sphere is *zero!* In this noninertial frame of reference, Newton's second law in component form yields

$$\text{Noninertial observer} \begin{cases} \sum F_x' = T\sin\theta - F_{\text{fictitious}} = 0 \\ \sum F_y' = T\cos\theta - mg = 0 \end{cases}$$

If we recognize that $F_{\text{fictitious}} = ma_{\text{inertial}} = ma$, then these expressions are equivalent to (1) and (2); therefore, the noninertial observer obtains the same mathematical results as the inertial observer does. However, the physical interpretation of the deflection of the cord differs in the two frames of reference.



(a)

(b)

Figure 6.13 A small sphere suspended from the ceiling of a boxcar accelerating to the right is deflected as shown. (a) An inertial observer at rest outside the car claims that the acceleration of the sphere is provided by the horizontal component of $\mathbf{T}$. (b) A noninertial observer riding in the car says that the net force on the sphere is zero and that the deflection of the cord from the vertical is due to a fictitious force $\mathbf{F}_{\text{fictitious}}$ that balances the horizontal component of $\mathbf{T}$.

EXAMPLE 6.10 Fictitious Force in a Rotating System

Suppose a block of mass m lying on a horizontal, frictionless turntable is connected to a string attached to the center of the turntable, as shown in Figure 6.14. According to an inertial observer, if the block rotates uniformly, it undergoes an acceleration of magnitude v^2/r, where v is its linear speed. The inertial observer concludes that this centripetal acceleration is provided by the force **T** exerted by the string and writes Newton's second law as $T = mv^2/r$.

According to a noninertial observer attached to the turntable, the block is at rest and its acceleration is zero. Therefore, she must introduce a fictitious outward force of magnitude mv^2/r to balance the inward force exerted by the string. According to her, the net force on the block is zero, and she writes Newton's second law as $T - mv^2/r = 0$.

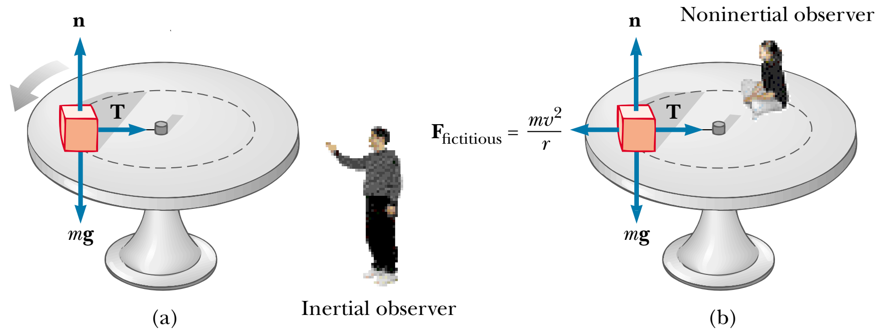


Figure 6.14 A block of mass m connected to a string tied to the center of a rotating turntable. (a) The inertial observer claims that the force causing the circular motion is provided by the force **T** exerted by the string on the block. (b) The noninertial observer claims that the block is not accelerating, and therefore she introduces a fictitious force of magnitude mv^2/r that acts outward and balances the force **T**.

Optional Section

6.4 MOTION IN THE PRESENCE OF RESISTIVE FORCES

4.9

In the preceding chapter we described the force of kinetic friction exerted on an object moving on some surface. We completely ignored any interaction between the object and the medium through which it moves. Now let us consider the effect of that medium, which can be either a liquid or a gas. The medium exerts a **resistive force R** on the object moving through it. Some examples are the air resistance associated with moving vehicles (sometimes called *air drag*) and the viscous forces that act on objects moving through a liquid. The magnitude of **R** depends on such factors as the speed of the object, and the direction of **R** is always opposite the direction of motion of the object relative to the medium. The magnitude of **R** nearly always increases with increasing speed.

The magnitude of the resistive force can depend on speed in a complex way, and here we consider only two situations. In the first situation, we assume the resistive force is proportional to the speed of the moving object; this assumption is valid for objects falling slowly through a liquid and for very small objects, such as dust particles, moving through air. In the second situation, we assume a resistive force that is proportional to the square of the speed of the moving object; large objects, such as a skydiver moving through air in free fall, experience such a force.

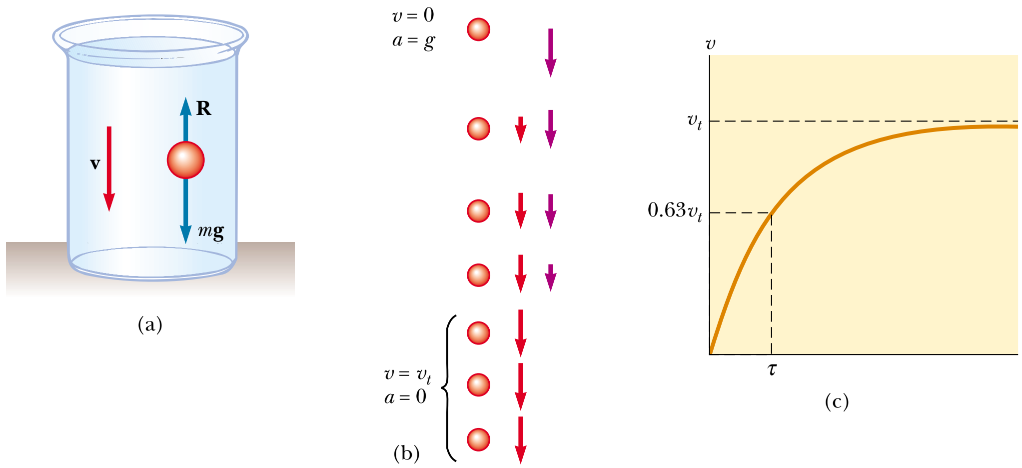


Figure 6.15 (a) A small sphere falling through a liquid. (b) Motion diagram of the sphere as it falls. (c) Speed–time graph for the sphere. The sphere reaches a maximum, or terminal, speed v_t, and the time constant τ is the time it takes to reach $0.63v_t$.

Resistive Force Proportional to Object Speed

If we assume that the resistive force acting on an object moving through a liquid or gas is proportional to the object's speed, then the magnitude of the resistive force can be expressed as

$$R = bv \tag{6.2}$$

where v is the speed of the object and b is a constant whose value depends on the properties of the medium and on the shape and dimensions of the object. If the object is a sphere of radius r, then b is proportional to r.

Consider a small sphere of mass m released from rest in a liquid, as in Figure 6.15a. Assuming that the only forces acting on the sphere are the resistive force bv and the force of gravity F_g, let us describe its motion.[1] Applying Newton's second law to the vertical motion, choosing the downward direction to be positive, and noting that $\Sigma F_y = mg - bv$, we obtain

$$mg - bv = ma = m\frac{dv}{dt} \tag{6.3}$$

where the acceleration dv/dt is downward. Solving this expression for the acceleration gives

$$\frac{dv}{dt} = g - \frac{b}{m}v \tag{6.4}$$

This equation is called a *differential equation,* and the methods of solving it may not be familiar to you as yet. However, note that initially, when $v = 0$, the resistive force $-bv$ is also zero and the acceleration dv/dt is simply g. As t increases, the resistive force increases and the acceleration decreases. Eventually, the acceleration becomes zero when the magnitude of the resistive force equals the sphere's weight. At this point, the sphere reaches its **terminal speed** v_t, and from then on

Terminal speed

[1] There is also a *buoyant force* acting on the submerged object. This force is constant, and its magnitude is equal to the weight of the displaced liquid. This force changes the apparent weight of the sphere by a constant factor, so we will ignore the force here. We discuss buoyant forces in Chapter 15.

it continues to move at this speed with zero acceleration, as shown in Figure 6.15b. We can obtain the terminal speed from Equation 6.3 by setting $a = dv/dt = 0$. This gives

$$mg - bv_t = 0 \qquad \text{or} \qquad v_t = mg/b$$

The expression for v that satisfies Equation 6.4 with $v = 0$ at $t = 0$ is

$$v = \frac{mg}{b}(1 - e^{-bt/m}) = v_t(1 - e^{-t/\tau}) \qquad \textbf{(6.5)}$$

This function is plotted in Figure 6.15c. The **time constant** $\tau = m/b$ (Greek letter tau) is the time it takes the sphere to reach 63.2% ($= 1 - 1/e$) of its terminal speed. This can be seen by noting that when $t = \tau$, Equation 6.5 yields $v = 0.632v_t$.

We can check that Equation 6.5 is a solution to Equation 6.4 by direct differentiation:

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{mg}{b} - \frac{mg}{b}e^{-bt/m}\right) = -\frac{mg}{b}\frac{d}{dt}e^{-bt/m} = ge^{-bt/m}$$

(See Appendix Table B.4 for the derivative of e raised to some power.) Substituting into Equation 6.4 both this expression for dv/dt and the expression for v given by Equation 6.5 shows that our solution satisfies the differential equation.

Aerodynamic car. A streamlined body reduces air drag and increases fuel efficiency.

EXAMPLE 6.11 Sphere Falling in Oil

A small sphere of mass 2.00 g is released from rest in a large vessel filled with oil, where it experiences a resistive force proportional to its speed. The sphere reaches a terminal speed of 5.00 cm/s. Determine the time constant τ and the time it takes the sphere to reach 90% of its terminal speed.

Solution Because the terminal speed is given by $v_t = mg/b$, the coefficient b is

$$b = \frac{mg}{v_t} = \frac{(2.00\text{ g})(980\text{ cm/s}^2)}{5.00\text{ cm/s}} = 392\text{ g/s}$$

Therefore, the time constant τ is

$$\tau = \frac{m}{b} = \frac{2.00\text{ g}}{392\text{ g/s}} = \boxed{5.10 \times 10^{-3}\text{ s}}$$

The speed of the sphere as a function of time is given by Equation 6.5. To find the time t it takes the sphere to reach a speed of $0.900v_t$, we set $v = 0.900v_t$ in Equation 6.5 and solve for t:

$$0.900v_t = v_t(1 - e^{-t/\tau})$$

$$1 - e^{-t/\tau} = 0.900$$

$$e^{-t/\tau} = 0.100$$

$$-\frac{t}{\tau} = \ln(0.100) = -2.30$$

$$t = 2.30\tau = 2.30(5.10 \times 10^{-3}\text{ s}) = 11.7 \times 10^{-3}\text{ s}$$

$$= \boxed{11.7\text{ ms}}$$

Thus, the sphere reaches 90% of its terminal (maximum) speed in a very short time.

Exercise What is the sphere's speed through the oil at $t = 11.7$ ms? Compare this value with the speed the sphere would have if it were falling in a vacuum and so were influenced only by gravity.

Answer 4.50 cm/s in oil versus 11.5 cm/s in free fall.

Air Drag at High Speeds

For objects moving at high speeds through air, such as airplanes, sky divers, cars, and baseballs, the resistive force is approximately proportional to the square of the speed. In these situations, the magnitude of the resistive force can be expressed as

$$R = \tfrac{1}{2}D\rho Av^2 \qquad \textbf{(6.6)}$$

where ρ is the density of air, A is the cross-sectional area of the falling object measured in a plane perpendicular to its motion, and D is a dimensionless empirical quantity called the *drag coefficient.* The drag coefficient has a value of about 0.5 for spherical objects but can have a value as great as 2 for irregularly shaped objects.

Let us analyze the motion of an object in free fall subject to an upward air resistive force of magnitude $R = \frac{1}{2}D\rho A v^2$. Suppose an object of mass m is released from rest. As Figure 6.16 shows, the object experiences two external forces: the downward force of gravity $\mathbf{F}_g = m\mathbf{g}$ and the upward resistive force $\mathbf{R}$. (There is also an upward buoyant force that we neglect.) Hence, the magnitude of the net force is

$$\sum F = mg - \tfrac{1}{2}D\rho A v^2 \qquad \textbf{(6.7)}$$

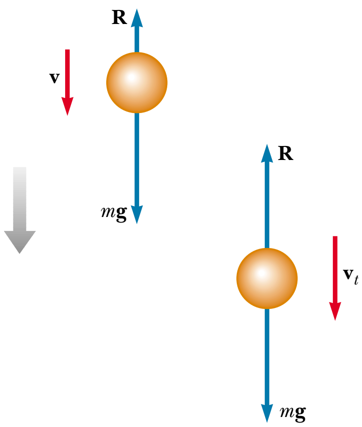


Figure 6.16 An object falling through air experiences a resistive force $\mathbf{R}$ and a gravitational force $\mathbf{F}_g = m\mathbf{g}$. The object reaches terminal speed (on the right) when the net force acting on it is zero, that is, when $\mathbf{R} = -\mathbf{F}_g$ or $R = mg$. Before this occurs, the acceleration varies with speed according to Equation 6.8.

where we have taken downward to be the positive vertical direction. Substituting $\Sigma F = ma$ into Equation 6.7, we find that the object has a downward acceleration of magnitude

$$a = g - \left(\frac{D\rho A}{2m}\right)v^2 \qquad \textbf{(6.8)}$$

We can calculate the terminal speed v_t by using the fact that when the force of gravity is balanced by the resistive force, the net force on the object is zero and therefore its acceleration is zero. Setting $a = 0$ in Equation 6.8 gives

$$g - \left(\frac{D\rho A}{2m}\right)v_t^2 = 0$$

$$v_t = \sqrt{\frac{2mg}{D\rho A}} \qquad \textbf{(6.9)}$$

Using this expression, we can determine how the terminal speed depends on the dimensions of the object. Suppose the object is a sphere of radius r. In this case, $A \propto r^2$ (from $A = \pi r^2$) and $m \propto r^3$ (because the mass is proportional to the volume of the sphere, which is $V = \frac{4}{3}\pi r^3$). Therefore, $v_t \propto \sqrt{r}$.

Table 6.1 lists the terminal speeds for several objects falling through air.

The high cost of fuel has prompted many truck owners to install wind deflectors on their cabs to reduce drag.

TABLE 6.1 Terminal Speed for Various Objects Falling Through Air

Object	Mass (kg)	Cross-Sectional Area (m^2)	v_t (m/s)
Sky diver	75	0.70	60
Baseball (radius 3.7 cm)	0.145	4.2×10^{-3}	43
Golf ball (radius 2.1 cm)	0.046	1.4×10^{-3}	44
Hailstone (radius 0.50 cm)	4.8×10^{-4}	7.9×10^{-5}	14
Raindrop (radius 0.20 cm)	3.4×10^{-5}	1.3×10^{-5}	9.0

CONCEPTUAL EXAMPLE 6.12

Consider a sky surfer who jumps from a plane with her feet attached firmly to her surfboard, does some tricks, and then opens her parachute. Describe the forces acting on her during these maneuvers.

Solution When the surfer first steps out of the plane, she has no vertical velocity. The downward force of gravity causes her to accelerate toward the ground. As her downward speed increases, so does the upward resistive force exerted by the air on her body and the board. This upward force reduces their acceleration, and so their speed increases more slowly. Eventually, they are going so fast that the upward resistive force matches the downward force of gravity. Now the net force is zero and they no longer accelerate, but reach their terminal speed. At some point after reaching terminal speed, she opens her parachute, resulting in a drastic increase in the upward resistive force. The net force (and thus the acceleration) is now upward, in the direction opposite the direction of the velocity. This causes the downward velocity to decrease rapidly; this means the resistive force on the chute also decreases. Eventually the upward resistive force and the downward force of gravity balance each other and a much smaller terminal speed is reached, permitting a safe landing.

(Contrary to popular belief, the velocity vector of a sky diver never points upward. You may have seen a videotape in which a sky diver appeared to "rocket" upward once the chute opened. In fact, what happened is that the diver slowed down while the person holding the camera continued falling at high speed.)

A sky surfer takes advantage of the upward force of the air on her board. (

EXAMPLE 6.13 Falling Coffee Filters

The dependence of resistive force on speed is an empirical relationship. In other words, it is based on observation rather than on a theoretical model. A series of stacked filters is dropped, and the terminal speeds are measured. Table 6.2 presents data for these coffee filters as they fall through the air. The time constant τ is small, so that a dropped filter quickly reaches terminal speed. Each filter has a mass of 1.64 g. When the filters are nested together, they stack in such a way that the front-facing surface area does not increase. Determine the relationship between the resistive force exerted by the air and the speed of the falling filters.

Solution At terminal speed, the upward resistive force balances the downward force of gravity. So, a single filter falling at its terminal speed experiences a resistive force of

$$R = mg = \left(\frac{1.64\text{ g}}{1000\text{ g/kg}}\right)(9.80\text{ m/s}^2) = 0.016\,1\text{ N}$$

Two filters nested together experience 0.032 2 N of resistive force, and so forth. A graph of the resistive force on the filters as a function of terminal speed is shown in Figure 6.17a. A straight line would not be a good fit, indicating that the resistive force is not proportional to the speed. The curved line is for a second-order polynomial, indicating a proportionality of the resistive force to the square of the speed. This proportionality is more clearly seen in Figure 6.17b, in which the resistive force is plotted as a function of the square of the terminal speed.

TABLE 6.2
Terminal Speed for Stacked Coffee Filters

Number of Filters	v_t (m/s)[a]
1	1.01
2	1.40
3	1.63
4	2.00
5	2.25
6	2.40
7	2.57
8	2.80
9	3.05
10	3.22

[a] All values of v_t are approximate.

Pleated coffee filters can be nested together so that the force of air resistance can be studied.
(

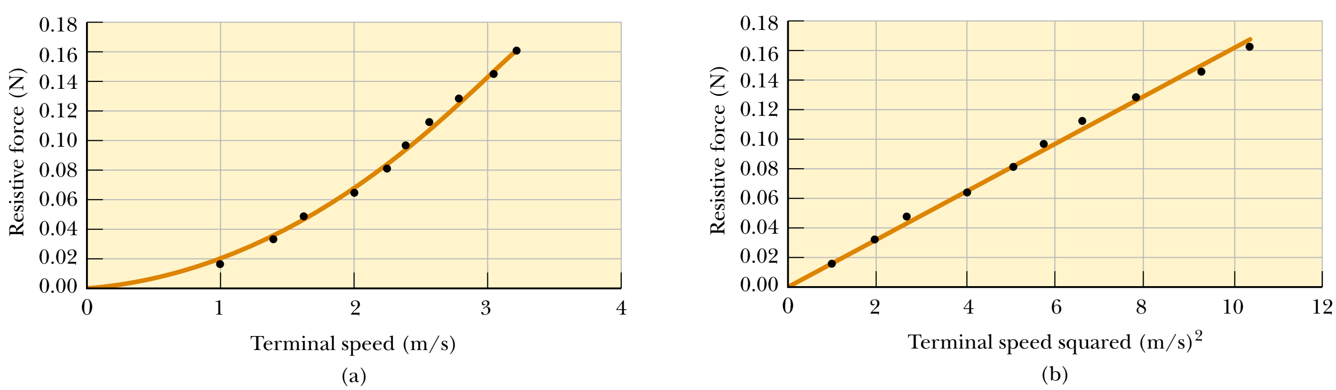


Figure 6.17 (a) Relationship between the resistive force acting on falling coffee filters and their terminal speed. The curved line is a second-order polynomial fit. (b) Graph relating the resistive force to the square of the terminal speed. The fit of the straight line to the data points indicates that the resistive force is proportional to the terminal speed squared. Can you find the proportionality constant?

EXAMPLE 6.14 Resistive Force Exerted on a Baseball

A pitcher hurls a 0.145-kg baseball past a batter at 40.2 m/s (= 90 mi/h). Find the resistive force acting on the ball at this speed.

Solution We do not expect the air to exert a huge force on the ball, and so the resistive force we calculate from Equation 6.6 should not be more than a few newtons. First, we must determine the drag coefficient *D*. We do this by imagining that we drop the baseball and allow it to reach terminal speed. We solve Equation 6.9 for *D* and substitute the appropriate values for m, v_t, and A from Table 6.1. Taking the density of air as 1.29 kg/m^3, we obtain

$$D = \frac{2\,mg}{v_t^2\,\rho A} = \frac{2(0.145\text{ kg})(9.80\text{ m/s}^2)}{(43\text{ m/s})^2\,(1.29\text{ kg/m}^3)(4.2\times10^{-3}\text{ m}^2)}$$
$$= 0.284$$

This number has no dimensions. We have kept an extra digit beyond the two that are significant and will drop it at the end of our calculation.

We can now use this value for *D* in Equation 6.6 to find the magnitude of the resistive force:

$$R = \tfrac{1}{2}D\rho A v^2$$
$$= \tfrac{1}{2}(0.284)(1.29\text{ kg/m}^3)(4.2\times10^{-3}\text{ m}^2)(40.2\text{ m/s})^2$$
$$= \boxed{1.2\text{ N}}$$

Optional Section

6.5 NUMERICAL MODELING IN PARTICLE DYNAMICS[2]

As we have seen in this and the preceding chapter, the study of the dynamics of a particle focuses on describing the position, velocity, and acceleration as functions of time. Cause-and-effect relationships exist among these quantities: Velocity causes position to change, and acceleration causes velocity to change. Because acceleration is the direct result of applied forces, any analysis of the dynamics of a particle usually begins with an evaluation of the net force being exerted on the particle.

Up till now, we have used what is called the *analytical method* to investigate the position, velocity, and acceleration of a moving particle. Let us review this method briefly before learning about a second way of approaching problems in dynamics. (Because we confine our discussion to one-dimensional motion in this section, boldface notation will not be used for vector quantities.)

If a particle of mass m moves under the influence of a net force ΣF, Newton's second law tells us that the acceleration of the particle is $a = \Sigma F/m$. In general, we apply the analytical method to a dynamics problem using the following procedure:

1. Sum all the forces acting on the particle to get the net force ΣF.
2. Use this net force to determine the acceleration from the relationship $a = \Sigma F/m$.
3. Use this acceleration to determine the velocity from the relationship $dv/dt = a$.
4. Use this velocity to determine the position from the relationship $dx/dt = v$.

The following straightforward example illustrates this method.

EXAMPLE 6.15 An Object Falling in a Vacuum—Analytical Method

Consider a particle falling in a vacuum under the influence of the force of gravity, as shown in Figure 6.18. Use the analytical method to find the acceleration, velocity, and position of the particle.

Solution The only force acting on the particle is the downward force of gravity of magnitude F_g, which is also the net force. Applying Newton's second law, we set the net force acting on the particle equal to the mass of the particle times

[2] The authors are most grateful to Colonel James Head of the U.S. Air Force Academy for preparing this section. See the *Student Tools CD-ROM* for some assistance with numerical modeling.

its acceleration (taking upward to be the positive y direction):

$$F_g = ma_y = -mg$$

Thus, $a_y = -g$, which means the acceleration is constant. Because $dv_y/dt = a_y$, we see that $dv_y/dt = -g$, which may be integrated to yield

$$v_y(t) = v_{yi} - gt$$

Then, because $v_y = dy/dt$, the position of the particle is obtained from another integration, which yields the well-known result

$$y(t) = y_i + v_{yi}t - \tfrac{1}{2}gt^2$$

In these expressions, y_i and v_{yi} represent the position and speed of the particle at $t_i = 0$.

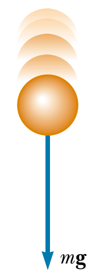


Figure 6.18 An object falling in vacuum under the influence of gravity.

The analytical method is straightforward for many physical situations. In the "real world," however, complications often arise that make analytical solutions difficult and perhaps beyond the mathematical abilities of most students taking introductory physics. For example, the net force acting on a particle may depend on the particle's position, as in cases where the gravitational acceleration varies with height. Or the force may vary with velocity, as in cases of resistive forces caused by motion through a liquid or gas.

Another complication arises because the expressions relating acceleration, velocity, position, and time are differential equations rather than algebraic ones. Differential equations are usually solved using integral calculus and other special techniques that introductory students may not have mastered.

When such situations arise, scientists often use a procedure called *numerical modeling* to study motion. The simplest numerical model is called the Euler method, after the Swiss mathematician Leonhard Euler (1707–1783).

The Euler Method

In the **Euler method** for solving differential equations, derivatives are approximated as ratios of finite differences. Considering a small increment of time Δt, we can approximate the relationship between a particle's speed and the magnitude of its acceleration as

$$a(t) \approx \frac{\Delta v}{\Delta t} = \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

Then the speed $v(t + \Delta t)$ of the particle at the end of the time interval Δt is approximately equal to the speed $v(t)$ at the beginning of the time interval plus the magnitude of the acceleration during the interval multiplied by Δt:

$$v(t + \Delta t) \approx v(t) + a(t)\Delta t \qquad \textbf{(6.10)}$$

Because the acceleration is a function of time, this estimate of $v(t + \Delta t)$ is accurate only if the time interval Δt is short enough that the change in acceleration during it is very small (as is discussed later). Of course, Equation 6.10 is exact if the acceleration is constant.

The position $x(t + \Delta t)$ of the particle at the end of the interval Δt can be found in the same manner:

$$v(t) \approx \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

$$x(t + \Delta t) \approx x(t) + v(t)\Delta t \quad \textbf{(6.11)}$$

You may be tempted to add the term $\frac{1}{2}\, a(\Delta t)^2$ to this result to make it look like the familiar kinematics equation, but this term is not included in the Euler method because Δt is assumed to be so small that Δt^2 is nearly zero.

If the acceleration at any instant t is known, the particle's velocity and position at a time $t + \Delta t$ can be calculated from Equations 6.10 and 6.11. The calculation then proceeds in a series of finite steps to determine the velocity and position at any later time. The acceleration is determined from the net force acting on the particle, and this force may depend on position, velocity, or time:

$$a(x, v, t) = \frac{\sum F(x, v, t)}{m} \quad \textbf{(6.12)}$$

It is convenient to set up the numerical solution to this kind of problem by numbering the steps and entering the calculations in a table, a procedure that is illustrated in Table 6.3.

The equations in the table can be entered into a spreadsheet and the calculations performed row by row to determine the velocity, position, and acceleration as functions of time. The calculations can also be carried out by using a program written in either BASIC, C++, or FORTRAN or by using commercially available mathematics packages for personal computers. Many small increments can be taken, and accurate results can usually be obtained with the help of a computer. Graphs of velocity versus time or position versus time can be displayed to help you visualize the motion.

One advantage of the Euler method is that the dynamics is not obscured—the fundamental relationships between acceleration and force, velocity and acceleration, and position and velocity are clearly evident. Indeed, these relationships form the heart of the calculations. There is no need to use advanced mathematics, and the basic physics governs the dynamics.

The Euler method is completely reliable for infinitesimally small time increments, but for practical reasons a finite increment size must be chosen. For the finite difference approximation of Equation 6.10 to be valid, the time increment must be small enough that the acceleration can be approximated as being constant during the increment. We can determine an appropriate size for the time in-

See the spreadsheet file "Baseball with Drag" on the Student Web site (address below) for an example of how this technique can be applied to find the initial speed of the baseball described in Example 6.14. We cannot use our regular approach because our kinematics equations assume constant acceleration. Euler's method provides a way to circumvent this difficulty.

A detailed solution to Problem 41 involving iterative integration appears in the *Student Solutions Manual and Study Guide* and is posted on the Web at **http://www.saunderscollege.com/physics**

TABLE 6.3 The Euler Method for Solving Dynamics Problems

Step	Time	Position	Velocity	Acceleration
0	t_0	x_0	v_0	$a_0 = F(x_0, v_0, t_0)/m$
1	$t_1 = t_0 + \Delta t$	$x_1 = x_0 + v_0\,\Delta t$	$v_1 = v_0 + a_0\,\Delta t$	$a_1 = F(x_1, v_1, t_1)/m$
2	$t_2 = t_1 + \Delta t$	$x_2 = x_1 + v_1\,\Delta t$	$v_2 = v_1 + a_1\,\Delta t$	$a_2 = F(x_2, v_2, t_2)/m$
3	$t_3 = t_2 + \Delta t$	$x_3 = x_2 + v_2\,\Delta t$	$v_3 = v_2 + a_2\,\Delta t$	$a_3 = F(x_3, v_3, t_3)/m$
	⋮	⋮	⋮	⋮
n	t_n	x_n	v_n	a_n

crement by examining the particular problem being investigated. The criterion for the size of the time increment may need to be changed during the course of the motion. In practice, however, we usually choose a time increment appropriate to the initial conditions and use the same value throughout the calculations.

The size of the time increment influences the accuracy of the result, but unfortunately it is not easy to determine the accuracy of an Euler-method solution without a knowledge of the correct analytical solution. One method of determining the accuracy of the numerical solution is to repeat the calculations with a smaller time increment and compare results. If the two calculations agree to a certain number of significant figures, you can assume that the results are correct to that precision.

SUMMARY

Newton's second law applied to a particle moving in uniform circular motion states that the net force causing the particle to undergo a centripetal acceleration is

$$\sum F_r = ma_r = \frac{mv^2}{r} \qquad \textbf{(6.1)}$$

You should be able to use this formula in situations where the force providing the centripetal acceleration could be the force of gravity, a force of friction, a force of string tension, or a normal force.

A particle moving in nonuniform circular motion has both a centripetal component of acceleration and a nonzero tangential component of acceleration. In the case of a particle rotating in a vertical circle, the force of gravity provides the tangential component of acceleration and part or all of the centripetal component of acceleration. Be sure you understand the directions and magnitudes of the velocity and acceleration vectors for nonuniform circular motion.

An observer in a noninertial (accelerating) frame of reference must introduce **fictitious forces** when applying Newton's second law in that frame. If these fictitious forces are properly defined, the description of motion in the noninertial frame is equivalent to that made by an observer in an inertial frame. However, the observers in the two frames do not agree on the causes of the motion. You should be able to distinguish between inertial and noninertial frames and identify the fictitious forces acting in a noninertial frame.

A body moving through a liquid or gas experiences a **resistive force** that is speed-dependent. This resistive force, which opposes the motion, generally increases with speed. The magnitude of the resistive force depends on the shape of the body and on the properties of the medium through which the body is moving. In the limiting case for a falling body, when the magnitude of the resistive force equals the body's weight, the body reaches its **terminal speed.** You should be able to apply Newton's laws to analyze the motion of objects moving under the influence of resistive forces. You may need to apply **Euler's method** if the force depends on velocity, as it does for air drag.

QUESTIONS

1. Because the Earth rotates about its axis and revolves around the Sun, it is a noninertial frame of reference. Assuming the Earth is a uniform sphere, why would the apparent weight of an object be greater at the poles than at the equator?
2. Explain why the Earth bulges at the equator.

3. Why is it that an astronaut in a space capsule orbiting the Earth experiences a feeling of weightlessness?
4. Why does mud fly off a rapidly turning automobile tire?
5. Imagine that you attach a heavy object to one end of a spring and then whirl the spring and object in a horizontal circle (by holding the free end of the spring). Does the spring stretch? If so, why? Discuss this in terms of the force causing the circular motion.
6. It has been suggested that rotating cylinders about 10 mi in length and 5 mi in diameter be placed in space and used as colonies. The purpose of the rotation is to simulate gravity for the inhabitants. Explain this concept for producing an effective gravity.
7. Why does a pilot tend to black out when pulling out of a steep dive?
8. Describe a situation in which a car driver can have a centripetal acceleration but no tangential acceleration.
9. Describe the path of a moving object if its acceleration is constant in magnitude at all times and (a) perpendicular to the velocity; (b) parallel to the velocity.
10. Analyze the motion of a rock falling through water in terms of its speed and acceleration as it falls. Assume that the resistive force acting on the rock increases as the speed increases.
11. Consider a small raindrop and a large raindrop falling through the atmosphere. Compare their terminal speeds. What are their accelerations when they reach terminal speed?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Section 6.1 Newton's Second Law Applied to Uniform Circular Motion

1. A toy car moving at constant speed completes one lap around a circular track (a distance of 200 m) in 25.0 s. (a) What is its average speed? (b) If the mass of the car is 1.50 kg, what is the magnitude of the force that keeps it in a circle?
2. A 55.0-kg ice skater is moving at 4.00 m/s when she grabs the loose end of a rope, the opposite end of which is tied to a pole. She then moves in a circle of radius 0.800 m around the pole. (a) Determine the force exerted by the rope on her arms. (b) Compare this force with her weight.
3. A light string can support a stationary hanging load of 25.0 kg before breaking. A 3.00-kg mass attached to the string rotates on a horizontal, frictionless table in a circle of radius 0.800 m. What range of speeds can the mass have before the string breaks?
4. In the Bohr model of the hydrogen atom, the speed of the electron is approximately 2.20×10^6 m/s. Find (a) the force acting on the electron as it revolves in a circular orbit of radius 0.530×10^{-10} m and (b) the centripetal acceleration of the electron.
5. In a cyclotron (one type of particle accelerator), a deuteron (of atomic mass 2.00 u) reaches a final speed of 10.0% of the speed of light while moving in a circular path of radius 0.480 m. The deuteron is maintained in the circular path by a magnetic force. What magnitude of force is required?
6. A satellite of mass 300 kg is in a circular orbit around the Earth at an altitude equal to the Earth's mean radius (see Example 6.6). Find (a) the satellite's orbital speed, (b) the period of its revolution, and (c) the gravitational force acting on it.
7. Whenever two Apollo astronauts were on the surface of the Moon, a third astronaut orbited the Moon. Assume the orbit to be circular and 100 km above the surface of the Moon. If the mass of the Moon is 7.40×10^{22} kg and its radius is 1.70×10^6 m, determine (a) the orbiting astronaut's acceleration, (b) his orbital speed, and (c) the period of the orbit.
8. The speed of the tip of the minute hand on a town clock is 1.75×10^{-3} m/s. (a) What is the speed of the tip of the second hand of the same length? (b) What is the centripetal acceleration of the tip of the second hand?
9. A coin placed 30.0 cm from the center of a rotating, horizontal turntable slips when its speed is 50.0 cm/s. (a) What provides the force in the radial direction when the coin is stationary relative to the turntable? (b) What is the coefficient of static friction between coin and turntable?
10. The cornering performance of an automobile is evaluated on a skid pad, where the maximum speed that a car can maintain around a circular path on a dry, flat surface is measured. The centripetal acceleration, also called the lateral acceleration, is then calculated as a multiple of the free-fall acceleration *g*. The main factors affecting the performance are the tire characteristics and the suspension system of the car. A Dodge Viper GTS can negotiate a skid pad of radius 61.0 m at 86.5 km/h. Calculate its maximum lateral acceleration.
11. A crate of eggs is located in the middle of the flatbed of a pickup truck as the truck negotiates an unbanked

curve in the road. The curve may be regarded as an arc of a circle of radius 35.0 m. If the coefficient of static friction between crate and truck is 0.600, how fast can the truck be moving without the crate sliding?

12. A car initially traveling eastward turns north by traveling in a circular path at uniform speed as in Figure P6.12. The length of the arc *ABC* is 235 m, and the car completes the turn in 36.0 s. (a) What is the acceleration when the car is at *B* located at an angle of 35.0°? Express your answer in terms of the unit vectors **i** and **j**. Determine (b) the car's average speed and (c) its average acceleration during the 36.0-s interval.

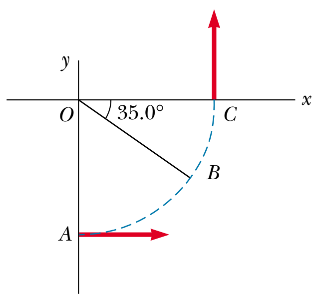


Figure P6.12

13. Consider a conical pendulum with an 80.0-kg bob on a 10.0-m wire making an angle of $\theta = 5.00°$ with the vertical (Fig. P6.13). Determine (a) the horizontal and vertical components of the force exerted by the wire on the pendulum and (b) the radial acceleration of the bob.

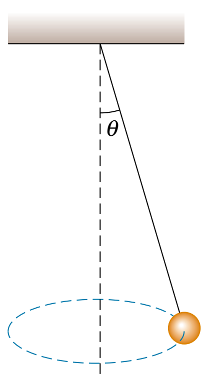


Figure P6.13

Section 6.2 Nonuniform Circular Motion

14. A car traveling on a straight road at 9.00 m/s goes over a hump in the road. The hump may be regarded as an arc of a circle of radius 11.0 m. (a) What is the apparent weight of a 600-N woman in the car as she rides over the hump? (b) What must be the speed of the car over the hump if she is to experience weightlessness? (That is, if her apparent weight is zero.)

WEB **15.** Tarzan ($m = 85.0$ kg) tries to cross a river by swinging from a vine. The vine is 10.0 m long, and his speed at the bottom of the swing (as he just clears the water) is 8.00 m/s. Tarzan doesn't know that the vine has a breaking strength of 1 000 N. Does he make it safely across the river?

16. A hawk flies in a horizontal arc of radius 12.0 m at a constant speed of 4.00 m/s. (a) Find its centripetal acceleration. (b) It continues to fly along the same horizontal arc but steadily increases its speed at the rate of 1.20 m/s^2. Find the acceleration (magnitude and direction) under these conditions.

17. A 40.0-kg child sits in a swing supported by two chains, each 3.00 m long. If the tension in each chain at the lowest point is 350 N, find (a) the child's speed at the lowest point and (b) the force exerted by the seat on the child at the lowest point. (Neglect the mass of the seat.)

18. A child of mass *m* sits in a swing supported by two chains, each of length *R*. If the tension in each chain at the lowest point is *T*, find (a) the child's speed at the lowest point and (b) the force exerted by the seat on the child at the lowest point. (Neglect the mass of the seat.)

WEB **19.** A pail of water is rotated in a vertical circle of radius 1.00 m. What must be the minimum speed of the pail at the top of the circle if no water is to spill out?

20. A 0.400-kg object is swung in a vertical circular path on a string 0.500 m long. If its speed is 4.00 m/s at the top of the circle, what is the tension in the string there?

21. A roller-coaster car has a mass of 500 kg when fully loaded with passengers (Fig. P6.21). (a) If the car has a speed of 20.0 m/s at point *A*, what is the force exerted by the track on the car at this point? (b) What is the maximum speed the car can have at *B* and still remain on the track?

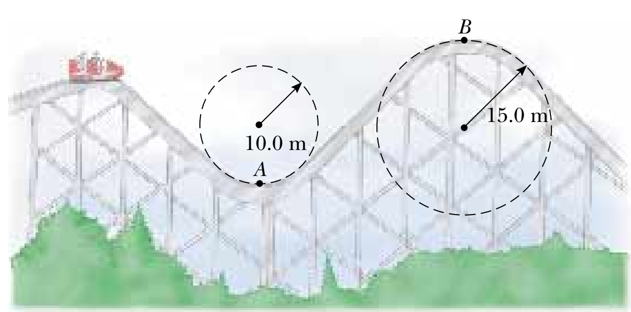


Figure P6.21

22. A roller coaster at the Six Flags Great America amusement park in Gurnee, Illinois, incorporates some of the latest design technology and some basic physics. Each vertical loop, instead of being circular, is shaped like a teardrop (Fig. P6.22). The cars ride on the inside of the loop at the top, and the speeds are high enough to ensure that the cars remain on the track. The biggest loop is 40.0 m high, with a maximum speed of 31.0 m/s (nearly 70 mi/h) at the bottom. Suppose the speed at the top is 13.0 m/s and the corresponding centripetal acceleration is $2g$. (a) What is the radius of the arc of the teardrop at the top? (b) If the total mass of the cars plus people is M, what force does the rail exert on this total mass at the top? (c) Suppose the roller coaster had a loop of radius 20.0 m. If the cars have the same speed, 13.0 m/s at the top, what is the centripetal acceleration at the top? Comment on the normal force at the top in this situation.

Figure P6.22 *(Frank Cezus/FPG International)*

(Optional)
Section 6.3 Motion in Accelerated Frames

23. A merry-go-round makes one complete revolution in 12.0 s. If a 45.0-kg child sits on the horizontal floor of the merry-go-round 3.00 m from the center, find (a) the child's acceleration and (b) the horizontal force of friction that acts on the child. (c) What minimum coefficient of static friction is necessary to keep the child from slipping?

24. A 5.00-kg mass attached to a spring scale rests on a frictionless, horizontal surface as in Figure P6.24. The spring scale, attached to the front end of a boxcar, reads 18.0 N when the car is in motion. (a) If the spring scale reads zero when the car is at rest, determine the acceleration of the car. (b) What will the spring scale read if the car moves with constant velocity? (c) Describe the forces acting on the mass as observed by someone in the car and by someone at rest outside the car.

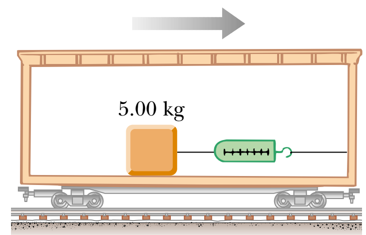


Figure P6.24

25. A 0.500-kg object is suspended from the ceiling of an accelerating boxcar as was seen in Figure 6.13. If $a = 3.00\ \text{m/s}^2$, find (a) the angle that the string makes with the vertical and (b) the tension in the string.

26. The Earth rotates about its axis with a period of 24.0 h. Imagine that the rotational speed can be increased. If an object at the equator is to have zero apparent weight, (a) what must the new period be? (b) By what factor would the speed of the object be increased when the planet is rotating at the higher speed? (*Hint:* See Problem 53 and note that the apparent weight of the object becomes zero when the normal force exerted on it is zero. Also, the distance traveled during one period is $2\pi R$, where R is the Earth's radius.)

27. A person stands on a scale in an elevator. As the elevator starts, the scale has a constant reading of 591 N. As the elevator later stops, the scale reading is 391 N. Assume the magnitude of the acceleration is the same during starting and stopping, and determine (a) the weight of the person, (b) the person's mass, and (c) the acceleration of the elevator.

28. A child on vacation wakes up. She is lying on her back. The tension in the muscles on both sides of her neck is 55.0 N as she raises her head to look past her toes and out the motel window. Finally, it is not raining! Ten minutes later she is screaming and sliding feet first down a water slide at a constant speed of 5.70 m/s, riding high on the outside wall of a horizontal curve of radius 2.40 m (Fig. P6.28). She raises her head to look forward past her toes; find the tension in the muscles on both sides of her neck.

Figure P6.28

29. A plumb bob does not hang exactly along a line directed to the center of the Earth, because of the Earth's rotation. How much does the plumb bob deviate from a radial line at 35.0° north latitude? Assume that the Earth is spherical.

(Optional)
Section 6.4 Motion in the Presence of Resistive Forces

30. A sky diver of mass 80.0 kg jumps from a slow-moving aircraft and reaches a terminal speed of 50.0 m/s. (a) What is the acceleration of the sky diver when her speed is 30.0 m/s? What is the drag force exerted on the diver when her speed is (b) 50.0 m/s? (c) 30.0 m/s?

31. A small piece of Styrofoam packing material is dropped from a height of 2.00 m above the ground. Until it reaches terminal speed, the magnitude of its acceleration is given by $a = g - bv$. After falling 0.500 m, the Styrofoam effectively reaches its terminal speed, and then takes 5.00 s more to reach the ground. (a) What is the value of the constant b? (b) What is the acceleration at $t = 0$? (c) What is the acceleration when the speed is 0.150 m/s?

32. (a) Estimate the terminal speed of a wooden sphere (density 0.830 g/cm^3) falling through the air if its radius is 8.00 cm. (b) From what height would a freely falling object reach this speed in the absence of air resistance?

33. Calculate the force required to pull a copper ball of radius 2.00 cm upward through a fluid at the constant speed 9.00 cm/s. Take the drag force to be proportional to the speed, with proportionality constant 0.950 kg/s. Ignore the buoyant force.

34. A fire helicopter carries a 620-kg bucket at the end of a cable 20.0 m long as in Figure P6.34. As the helicopter flies to a fire at a constant speed of 40.0 m/s, the cable makes an angle of 40.0° with respect to the vertical. The bucket presents a cross-sectional area of 3.80 m^2 in a plane perpendicular to the air moving past it. Determine the drag coefficient assuming that the resistive force is proportional to the square of the bucket's speed.

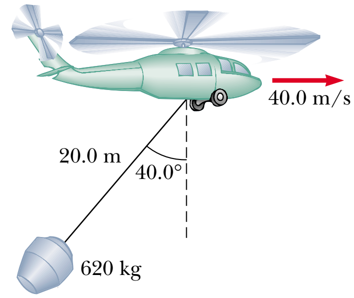


Figure P6.34

35. A small, spherical bead of mass 3.00 g is released from rest at $t = 0$ in a bottle of liquid shampoo. The terminal speed is observed to be $v_t = 2.00$ cm/s. Find (a) the value of the constant b in Equation 6.4, (b) the time τ the bead takes to reach $0.632v_t$, and (c) the value of the resistive force when the bead reaches terminal speed.

36. The mass of a sports car is 1 200 kg. The shape of the car is such that the aerodynamic drag coefficient is 0.250 and the frontal area is 2.20 m^2. Neglecting all other sources of friction, calculate the initial acceleration of the car if, after traveling at 100 km/h, it is shifted into neutral and is allowed to coast.

WEB **37.** A motorboat cuts its engine when its speed is 10.0 m/s and coasts to rest. The equation governing the motion of the motorboat during this period is $v = v_i e^{-ct}$, where v is the speed at time t, v_i is the initial speed, and c is a constant. At $t = 20.0$ s, the speed is 5.00 m/s. (a) Find the constant c. (b) What is the speed at $t = 40.0$ s? (c) Differentiate the expression for $v(t)$ and thus show that the acceleration of the boat is proportional to the speed at any time.

38. Assume that the resistive force acting on a speed skater is $f = -kmv^2$, where k is a constant and m is the skater's mass. The skater crosses the finish line of a straight-line race with speed v_f and then slows down by coasting on his skates. Show that the skater's speed at any time t after crossing the finish line is $v(t) = v_f/(1 + ktv_f)$.

39. You can feel a force of air drag on your hand if you stretch your arm out of the open window of a speeding car. (*Note:* Do not get hurt.) What is the order of magnitude of this force? In your solution, state the quantities you measure or estimate and their values.

(Optional)
6.5 Numerical Modeling in Particle Dynamics

40. A 3.00-g leaf is dropped from a height of 2.00 m above the ground. Assume the net downward force exerted on the leaf is $F = mg - bv$, where the drag factor is $b = 0.030\ 0$ kg/s. (a) Calculate the terminal speed of the leaf. (b) Use Euler's method of numerical analysis to find the speed and position of the leaf as functions of time, from the instant it is released until 99% of terminal speed is reached. (*Hint:* Try $\Delta t = 0.005$ s.)

WEB 41. A hailstone of mass 4.80×10^{-4} kg falls through the air and experiences a net force given by

$$F = -mg + Cv^2$$

where $C = 2.50 \times 10^{-5}$ kg/m. (a) Calculate the terminal speed of the hailstone. (b) Use Euler's method of numerical analysis to find the speed and position of the hailstone at 0.2-s intervals, taking the initial speed to be zero. Continue the calculation until the hailstone reaches 99% of terminal speed.

42. A 0.142-kg baseball has a terminal speed of 42.5 m/s (95 mi/h). (a) If a baseball experiences a drag force of magnitude $R = Cv^2$, what is the value of the constant C? (b) What is the magnitude of the drag force when the speed of the baseball is 36.0 m/s? (c) Use a computer to determine the motion of a baseball thrown vertically upward at an initial speed of 36.0 m/s. What maximum height does the ball reach? How long is it in the air? What is its speed just before it hits the ground?

43. A 50.0-kg parachutist jumps from an airplane and falls with a drag force proportional to the square of the speed $R = Cv^2$. Take $C = 0.200$ kg/m with the parachute closed and $C = 20.0$ kg/m with the chute open. (a) Determine the terminal speed of the parachutist in both configurations, before and after the chute is opened. (b) Set up a numerical analysis of the motion and compute the speed and position as functions of time, assuming the jumper begins the descent at 1 000 m above the ground and is in free fall for 10.0 s before opening the parachute. (*Hint:* When the parachute opens, a sudden large acceleration takes place; a smaller time step may be necessary in this region.)

44. Consider a 10.0-kg projectile launched with an initial speed of 100 m/s, at an angle of 35.0° elevation. The resistive force is $\mathbf{R} = -b\mathbf{v}$, where $b = 10.0$ kg/s. (a) Use a numerical method to determine the horizontal and vertical positions of the projectile as functions of time. (b) What is the range of this projectile? (c) Determine the elevation angle that gives the maximum range for the projectile. (*Hint:* Adjust the elevation angle by trial and error to find the greatest range.)

45. A professional golfer hits a golf ball of mass 46.0 g with her 5-iron, and the ball first strikes the ground 155 m (170 yards) away. The ball experiences a drag force of magnitude $R = Cv^2$ and has a terminal speed of 44.0 m/s. (a) Calculate the drag constant C for the golf ball. (b) Use a numerical method to analyze the trajectory of this shot. If the initial velocity of the ball makes an angle of 31.0° (the loft angle) with the horizontal, what initial speed must the ball have to reach the 155-m distance? (c) If the same golfer hits the ball with her 9-iron (47.0° loft) and it first strikes the ground 119 m away, what is the initial speed of the ball? Discuss the differences in trajectories between the two shots.

ADDITIONAL PROBLEMS

46. An 1 800-kg car passes over a bump in a road that follows the arc of a circle of radius 42.0 m as in Figure P6.46. (a) What force does the road exert on the car as the car passes the highest point of the bump if the car travels at 16.0 m/s? (b) What is the maximum speed the car can have as it passes this highest point before losing contact with the road?

47. A car of mass m passes over a bump in a road that follows the arc of a circle of radius R as in Figure P6.46. (a) What force does the road exert on the car as the car passes the highest point of the bump if the car travels at a speed v? (b) What is the maximum speed the car can have as it passes this highest point before losing contact with the road?

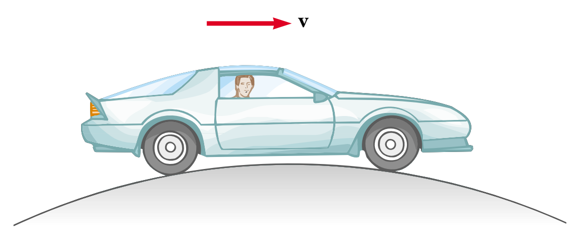


Figure P6.46 Problems 46 and 47.

48. In one model of a hydrogen atom, the electron in orbit around the proton experiences an attractive force of about 8.20×10^{-8} N. If the radius of the orbit is 5.30×10^{-11} m, how many revolutions does the electron make each second? (This number of revolutions per unit time is called the *frequency* of the motion.) See the inside front cover for additional data.

49. A student builds and calibrates an accelerometer, which she uses to determine the speed of her car around a certain unbanked highway curve. The accelerometer is a plumb bob with a protractor that she attaches to the roof of her car. A friend riding in the car with her observes that the plumb bob hangs at an angle of 15.0° from the vertical when the car has a speed of 23.0 m/s. (a) What is the centripetal acceleration of the car rounding the curve? (b) What is the radius of the curve? (c) What is the speed of the car if the plumb bob deflection is 9.00° while the car is rounding the same curve?

50. Suppose the boxcar shown in Figure 6.13 is moving with constant acceleration a up a hill that makes an angle ϕ with the horizontal. If the hanging pendulum makes a constant angle θ with the perpendicular to the ceiling, what is a?

51. An air puck of mass 0.250 kg is tied to a string and allowed to revolve in a circle of radius 1.00 m on a fric-

tionless horizontal table. The other end of the string passes through a hole in the center of the table, and a mass of 1.00 kg is tied to it (Fig. P6.51). The suspended mass remains in equilibrium while the puck on the tabletop revolves. What are (a) the tension in the string, (b) the force exerted by the string on the puck, and (c) the speed of the puck?

52. An air puck of mass m_1 is tied to a string and allowed to revolve in a circle of radius R on a frictionless horizontal table. The other end of the string passes through a hole in the center of the table, and a mass m_2 is tied to it (Fig. P6.51). The suspended mass remains in equilibrium while the puck on the tabletop revolves. What are (a) the tension in the string? (b) the central force exerted on the puck? (c) the speed of the puck?

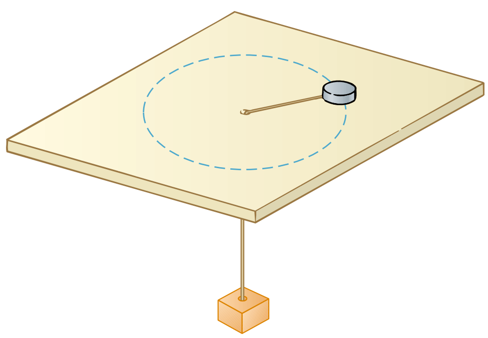


Figure P6.51 Problems 51 and 52.

WEB **53.** Because the Earth rotates about its axis, a point on the equator experiences a centripetal acceleration of 0.033 7 m/s^2, while a point at one of the poles experiences no centripetal acceleration. (a) Show that at the equator the gravitational force acting on an object (the true weight) must exceed the object's apparent weight. (b) What is the apparent weight at the equator and at the poles of a person having a mass of 75.0 kg? (Assume the Earth is a uniform sphere and take $g = 9.800$ m/s^2.)

54. A string under a tension of 50.0 N is used to whirl a rock in a horizontal circle of radius 2.50 m at a speed of 20.4 m/s. The string is pulled in and the speed of the rock increases. When the string is 1.00 m long and the speed of the rock is 51.0 m/s, the string breaks. What is the breaking strength (in newtons) of the string?

55. A child's toy consists of a small wedge that has an acute angle θ (Fig. P6.55). The sloping side of the wedge is frictionless, and a mass m on it remains at constant height if the wedge is spun at a certain constant speed. The wedge is spun by rotating a vertical rod that is firmly attached to the wedge at the bottom end. Show that, when the mass sits a distance L up along the sloping side, the speed of the mass must be

$$v = (gL \sin \theta)^{1/2}$$

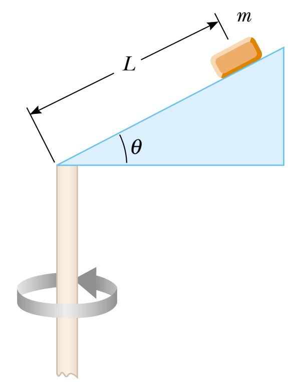


Figure P6.55

56. The pilot of an airplane executes a constant-speed loop-the-loop maneuver. His path is a vertical circle. The speed of the airplane is 300 mi/h, and the radius of the circle is 1 200 ft. (a) What is the pilot's apparent weight at the lowest point if his true weight is 160 lb? (b) What is his apparent weight at the highest point? (c) Describe how the pilot could experience apparent weightlessness if both the radius and the speed can be varied. (*Note:* His apparent weight is equal to the force that the seat exerts on his body.)

57. For a satellite to move in a stable circular orbit at a constant speed, its centripetal acceleration must be inversely proportional to the square of the radius r of the orbit. (a) Show that the tangential speed of a satellite is proportional to $r^{-1/2}$. (b) Show that the time required to complete one orbit is proportional to $r^{3/2}$.

58. A penny of mass 3.10 g rests on a small 20.0-g block supported by a spinning disk (Fig. P6.58). If the coeffi-

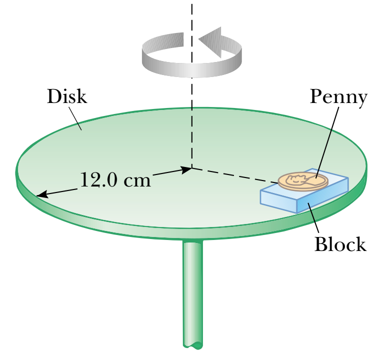


Figure P6.58

cients of friction between block and disk are 0.750 (static) and 0.640 (kinetic) while those for the penny and block are 0.450 (kinetic) and 0.520 (static), what is the maximum rate of rotation (in revolutions per minute) that the disk can have before either the block or the penny starts to slip?

59. Figure P6.59 shows a Ferris wheel that rotates four times each minute and has a diameter of 18.0 m. (a) What is the centripetal acceleration of a rider? What force does the seat exert on a 40.0-kg rider (b) at the lowest point of the ride and (c) at the highest point of the ride? (d) What force (magnitude and direction) does the seat exert on a rider when the rider is halfway between top and bottom?

Figure P6.59 *(Color Box/FPG)*

60. A space station, in the form of a large wheel 120 m in diameter, rotates to provide an "artificial gravity" of 3.00 m/s^2 for persons situated at the outer rim. Find the rotational frequency of the wheel (in revolutions per minute) that will produce this effect.

61. An amusement park ride consists of a rotating circular platform 8.00 m in diameter from which 10.0-kg seats are suspended at the end of 2.50-m massless chains (Fig. P6.61). When the system rotates, the chains make an angle $\theta = 28.0°$ with the vertical. (a) What is the speed of each seat? (b) Draw a free-body diagram of a 40.0-kg child riding in a seat and find the tension in the chain.

62. A piece of putty is initially located at point A on the rim of a grinding wheel rotating about a horizontal axis. The putty is dislodged from point A when the diameter through A is horizontal. The putty then rises vertically and returns to A the instant the wheel completes one revolution. (a) Find the speed of a point on the rim of the wheel in terms of the acceleration due to gravity and the radius R of the wheel. (b) If the mass of the putty is m, what is the magnitude of the force that held it to the wheel?

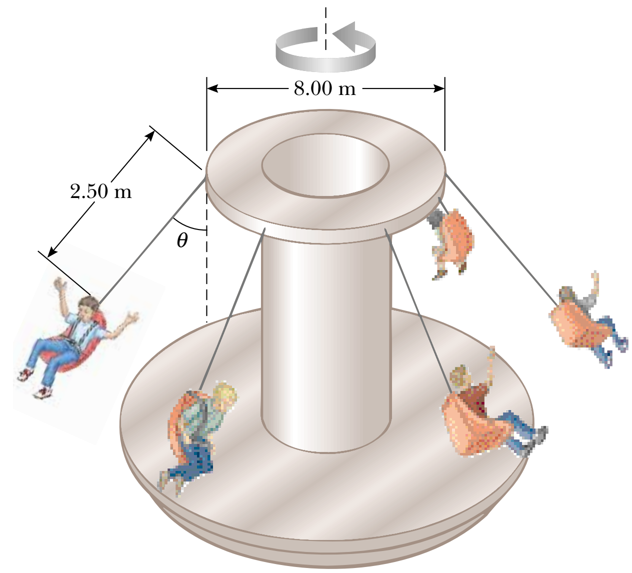


Figure P6.61

63. An amusement park ride consists of a large vertical cylinder that spins about its axis fast enough that any person inside is held up against the wall when the floor drops away (Fig. P6.63). The coefficient of static friction between person and wall is μ_s, and the radius of the cylinder is R. (a) Show that the maximum period of revolution necessary to keep the person from falling is $T = (4\pi^2 R\mu_s/g)^{1/2}$. (b) Obtain a numerical value for T

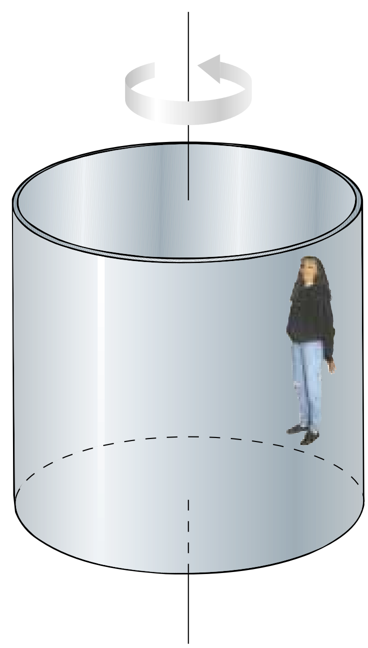


Figure P6.63

if $R = 4.00$ m and $\mu_s = 0.400$. How many revolutions per minute does the cylinder make?

64. *An example of the Coriolis effect.* Suppose air resistance is negligible for a golf ball. A golfer tees off from a location precisely at $\phi_i = 35.0°$ north latitude. He hits the ball due south, with range 285 m. The ball's initial velocity is at 48.0° above the horizontal. (a) For what length of time is the ball in flight? The cup is due south of the golfer's location, and he would have a hole-in-one if the Earth were not rotating. As shown in Figure P6.64, the Earth's rotation makes the tee move in a circle of radius $R_E \cos \phi_i = (6.37 \times 10^6 \text{ m}) \cos 35.0°$, completing one revolution each day. (b) Find the eastward speed of the tee, relative to the stars. The hole is also moving eastward, but it is 285 m farther south and thus at a slightly lower latitude ϕ_f. Because the hole moves eastward in a slightly larger circle, its speed must be greater than that of the tee. (c) By how much does the hole's speed exceed that of the tee? During the time the ball is in flight, it moves both upward and downward, as well as southward with the projectile motion you studied in Chapter 4, but it also moves eastward with the speed you found in part (b). The hole moves to the east at a faster speed, however, pulling ahead of the ball with the relative speed you found in part (c). (d) How far to the west of the hole does the ball land?

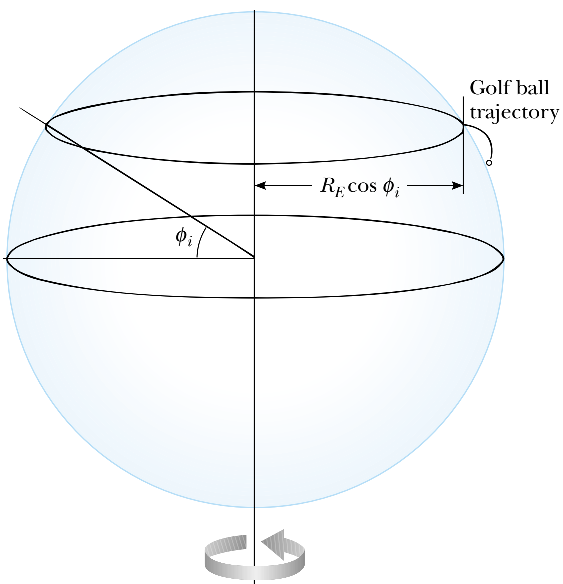


Figure P6.64

65. A curve in a road forms part of a horizontal circle. As a car goes around it at constant speed 14.0 m/s, the total force exerted on the driver has magnitude 130 N. What are the magnitude and direction of the total force exerted on the driver if the speed is 18.0 m/s instead?

66. A car rounds a banked curve as shown in Figure 6.6. The radius of curvature of the road is R, the banking angle is θ, and the coefficient of static friction is μ_s. (a) Determine the range of speeds the car can have without slipping up or down the banked surface. (b) Find the minimum value for μ_s such that the minimum speed is zero. (c) What is the range of speeds possible if $R = 100$ m, $\theta = 10.0°$, and $\mu_s = 0.100$ (slippery conditions)?

67. A single bead can slide with negligible friction on a wire that is bent into a circle of radius 15.0 cm, as in Figure P6.67. The circle is always in a vertical plane and rotates steadily about its vertical diameter with a period of 0.450 s. The position of the bead is described by the angle θ that the radial line from the center of the loop to the bead makes with the vertical. (a) At what angle up from the lowest point can the bead stay motionless relative to the turning circle? (b) Repeat the problem if the period of the circle's rotation is 0.850 s.

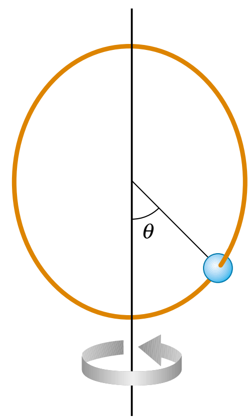


Figure P6.67

68. The expression $F = arv + br^2v^2$ gives the magnitude of the resistive force (in newtons) exerted on a sphere of radius r (in meters) by a stream of air moving at speed v (in meters per second), where a and b are constants with appropriate SI units. Their numerical values are $a = 3.10 \times 10^{-4}$ and $b = 0.870$. Using this formula, find the terminal speed for water droplets falling under their own weight in air, taking the following values for the drop radii: (a) 10.0 μm, (b) 100 μm, (c) 1.00 mm. Note that for (a) and (c) you can obtain accurate answers without solving a quadratic equation, by considering which of the two contributions to the air resistance is dominant and ignoring the lesser contribution.

69. A model airplane of mass 0.750 kg flies in a horizontal circle at the end of a 60.0-m control wire, with a speed of 35.0 m/s. Compute the tension in the wire if it makes a constant angle of 20.0° with the horizontal. The forces exerted on the airplane are the pull of the control wire, its own weight, and aerodynamic lift, which acts at 20.0° inward from the vertical as shown in Figure P6.69.

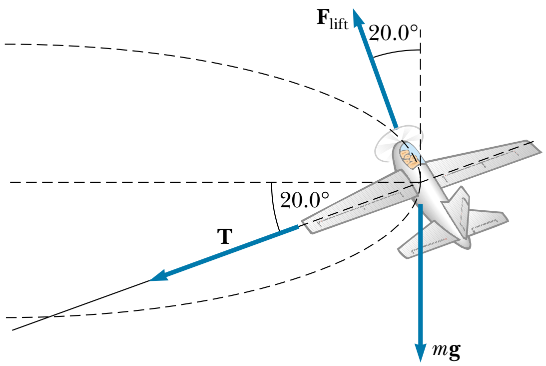


Figure P6.69

70. A 9.00-kg object starting from rest falls through a viscous medium and experiences a resistive force $\mathbf{R} = -b\mathbf{v}$, where $\mathbf{v}$ is the velocity of the object. If the object's speed reaches one-half its terminal speed in 5.54 s, (a) determine the terminal speed. (b) At what time is the speed of the object three-fourths the terminal speed? (c) How far has the object traveled in the first 5.54 s of motion?

71. Members of a skydiving club were given the following data to use in planning their jumps. In the table, d is the distance fallen from rest by a sky diver in a "free-fall stable spread position" versus the time of fall t. (a) Convert the distances in feet into meters. (b) Graph d (in meters) versus t. (c) Determine the value of the terminal speed v_t by finding the slope of the straight portion of the curve. Use a least-squares fit to determine this slope.

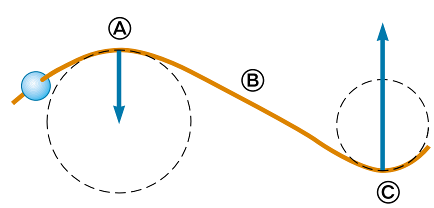
t (s)	d (ft)
1	16
2	62
3	138
4	242
5	366
6	504
7	652
8	808
9	971
10	1 138
11	1 309
12	1 483
13	1 657
14	1 831
15	2 005
16	2 179
17	2 353
18	2 527
19	2 701
20	2 875

Answers to Quick Quizzes

6.1 No. The tangential acceleration changes just the speed part of the velocity vector. For the car to move in a circle, the *direction* of its velocity vector must change, and the only way this can happen is for there to be a centripetal acceleration.

6.2 (a) The ball travels in a circular path that has a larger radius than the original circular path, and so there must be some external force causing the change in the velocity vector's direction. The external force must not be as strong as the original tension in the string because if it were, the ball would follow the original path. (b) The ball again travels in an arc, implying some kind of external force. As in part (a), the external force is directed toward the center of the new arc and not toward the center of the original circular path. (c) The ball undergoes an abrupt change in velocity—from tangent to the circle to perpendicular to it—and so must have experienced a large force that had one component opposite the ball's velocity (tangent to the circle) and another component radially outward. (d) The ball travels in a straight line tangent to the original path. If there is an external force, it cannot have a component perpendicular to this line because if it did, the path would curve. In fact, if the string breaks and there is no other force acting on the ball, Newton's first law says the ball will travel along such a tangent line at constant speed.

6.3 At Ⓐ the path is along the circumference of the larger circle. Therefore, the wire must be exerting a force on the bead directed toward the center of the circle. Because the speed is constant, there is no tangential force component. At Ⓑ the path is not curved, and so the wire exerts no force on the bead. At Ⓒ the path is again curved, and so the wire is again exerting a force on the bead. This time the force is directed toward the center of the smaller circle. Because the radius of this circle is smaller, the magnitude of the force exerted on the bead is larger here than at Ⓐ.



PUZZLER

Chum salmon "climbing a ladder" in the McNeil River in Alaska. Why are fish ladders like this often built around dams? Do the ladders reduce the amount of work that the fish must do to get past the dam? *(Daniel J. Cox/Tony Stone Images)*

chapter

7

Work and Kinetic Energy

Chapter Outline

The concept of energy is one of the most important topics in science and engineering. In everyday life, we think of energy in terms of fuel for transportation and heating, electricity for lights and appliances, and foods for consumption. However, these ideas do not really define energy. They merely tell us that fuels are needed to do a job and that those fuels provide us with something we call *energy*.

In this chapter, we first introduce the concept of work. Work is done by a force acting on an object when the point of application of that force moves through some distance and the force has a component along the line of motion. Next, we define kinetic energy, which is energy an object possesses because of its motion. In general, we can think of *energy* as the capacity that an object has for performing work. We shall see that the concepts of work and kinetic energy can be applied to the dynamics of a mechanical system without resorting to Newton's laws. In a complex situation, in fact, the "energy approach" can often allow a much simpler analysis than the direct application of Newton's second law. However, it is important to note that the work–energy concepts are based on Newton's laws and therefore allow us to make predictions that are always in agreement with these laws.

This alternative method of describing motion is especially useful when the force acting on a particle varies with the position of the particle. In this case, the acceleration is not constant, and we cannot apply the kinematic equations developed in Chapter 2. Often, a particle in nature is subject to a force that varies with the position of the particle. Such forces include the gravitational force and the force exerted on an object attached to a spring. Although we could analyze situations like these by applying numerical methods such as those discussed in Section 6.5, utilizing the ideas of work and energy is often much simpler. We describe techniques for treating complicated systems with the help of an extremely important theorem called the *work–kinetic energy theorem,* which is the central topic of this chapter.

7.1 WORK DONE BY A CONSTANT FORCE

5.1 Almost all the terms we have used thus far—velocity, acceleration, force, and so on—convey nearly the same meaning in physics as they do in everyday life. Now, however, we encounter a term whose meaning in physics is distinctly different from its everyday meaning. That new term is *work*.

To understand what *work* means to the physicist, consider the situation illustrated in Figure 7.1. A force is applied to a chalkboard eraser, and the eraser slides along the tray. If we are interested in how effective the force is in moving the

(a) (b) (c)

Figure 7.1 An eraser being pushed along a chalkboard tray. *(Charles D. Winters)*

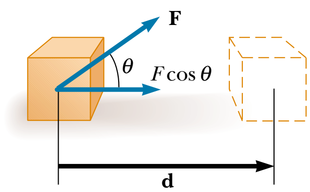


Figure 7.2 If an object undergoes a displacement **d** under the action of a constant force **F**, the work done by the force is $(F \cos \theta) d$.

eraser, we need to consider not only the magnitude of the force but also its direction. If we assume that the magnitude of the applied force is the same in all three photographs, it is clear that the push applied in Figure 7.1b does more to move the eraser than the push in Figure 7.1a. On the other hand, Figure 7.1c shows a situation in which the applied force does not move the eraser at all, regardless of how hard it is pushed. (Unless, of course, we apply a force so great that we break something.) So, in analyzing forces to determine the work they do, we must consider the vector nature of forces. We also need to know how far the eraser moves along the tray if we want to determine the work required to cause that motion. Moving the eraser 3 m requires more work than moving it 2 cm.

Let us examine the situation in Figure 7.2, where an object undergoes a displacement **d** along a straight line while acted on by a constant force **F** that makes an angle θ with **d**.

Work done by a constant force

The **work** W done on an object by an agent exerting a constant force on the object is the product of the component of the force in the direction of the displacement and the magnitude of the displacement:

$$W = Fd \cos \theta \quad \textbf{(7.1)}$$

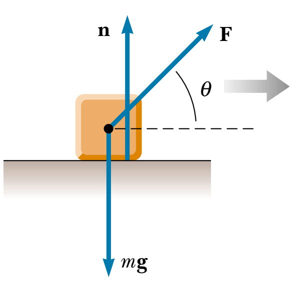


Figure 7.3 When an object is displaced on a frictionless, horizontal, surface, the normal force **n** and the force of gravity $m\mathbf{g}$ do no work on the object. In the situation shown here, **F** is the only force doing work on the object.

As an example of the distinction between this definition of work and our everyday understanding of the word, consider holding a heavy chair at arm's length for 3 min. At the end of this time interval, your tired arms may lead you to think that you have done a considerable amount of work on the chair. According to our definition, however, you have done no work on it whatsoever.[1] You exert a force to support the chair, but you do not move it. A force does no work on an object if the object does not move. This can be seen by noting that if $d = 0$, Equation 7.1 gives $W = 0$—the situation depicted in Figure 7.1c.

Also note from Equation 7.1 that the work done by a force on a moving object is zero when the force applied is perpendicular to the object's displacement. That is, if $\theta = 90°$, then $W = 0$ because $\cos 90° = 0$. For example, in Figure 7.3, the work done by the normal force on the object and the work done by the force of gravity on the object are both zero because both forces are perpendicular to the displacement and have zero components in the direction of **d**.

The sign of the work also depends on the direction of **F** relative to **d**. The work done by the applied force is positive when the vector associated with the component $F \cos \theta$ is in the same direction as the displacement. For example, when an object is lifted, the work done by the applied force is positive because the direction of that force is upward, that is, in the same direction as the displacement. When the vector associated with the component $F \cos \theta$ is in the direction opposite the displacement, W is negative. For example, as an object is lifted, the work done by the gravitational force on the object is negative. The factor $\cos \theta$ in the definition of W (Eq. 7.1) automatically takes care of the sign. It is important to note that **work is an energy transfer;** if energy is transferred *to* the system (object), W is positive; if energy is transferred *from* the system, W is negative.

5.3

[1] Actually, you do work while holding the chair at arm's length because your muscles are continuously contracting and relaxing; this means that they are exerting internal forces on your arm. Thus, work is being done by your body—but internally on itself rather than on the chair.

If an applied force **F** acts along the direction of the displacement, then $\theta = 0$ and $\cos 0 = 1$. In this case, Equation 7.1 gives

$$W = Fd$$

Work is a scalar quantity, and its units are force multiplied by length. Therefore, the SI unit of work is the **newton·meter** (N·m). This combination of units is used so frequently that it has been given a name of its own: the **joule** (J).

Quick Quiz 7.1

Can the component of a force that gives an object a centripetal acceleration do any work on the object? (One such force is that exerted by the Sun on the Earth that holds the Earth in a circular orbit around the Sun.)

In general, a particle may be moving with either a constant or a varying velocity under the influence of several forces. In these cases, because work is a scalar quantity, the total work done as the particle undergoes some displacement is the algebraic sum of the amounts of work done by all the forces.

EXAMPLE 7.1 Mr. Clean

A man cleaning a floor pulls a vacuum cleaner with a force of magnitude $F = 50.0$ N at an angle of 30.0° with the horizontal (Fig. 7.4a). Calculate the work done by the force on the vacuum cleaner as the vacuum cleaner is displaced 3.00 m to the right.

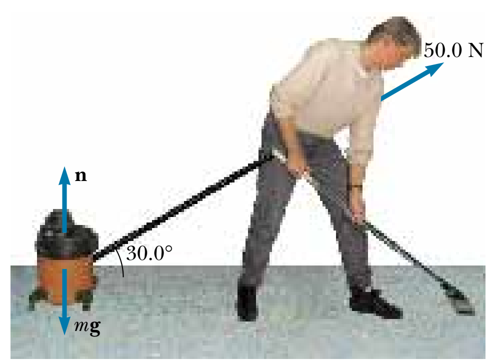
Solution Because they aid us in clarifying which forces are acting on the object being considered, drawings like Figure 7.4b are helpful when we are gathering information and organizing a solution. For our analysis, we use the definition of work (Eq. 7.1):

$$\begin{aligned} W &= (F\cos\theta)\,d \\ &= (50.0\ \text{N})(\cos 30.0°)(3.00\ \text{m}) = 130\ \text{N·m} \\ &= \boxed{130\ \text{J}} \end{aligned}$$

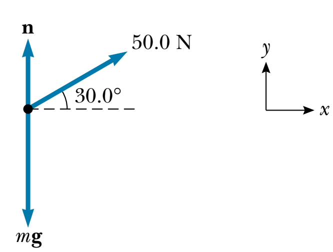
One thing we should learn from this problem is that the normal force **n**, the force of gravity $\mathbf{F}_g = m\mathbf{g}$, and the upward component of the applied force (50.0 N) (sin 30.0°) do no work on the vacuum cleaner because these forces are perpendicular to its displacement.

Exercise Find the work done by the man on the vacuum cleaner if he pulls it 3.0 m with a horizontal force of 32 N.

Answer 96 J.

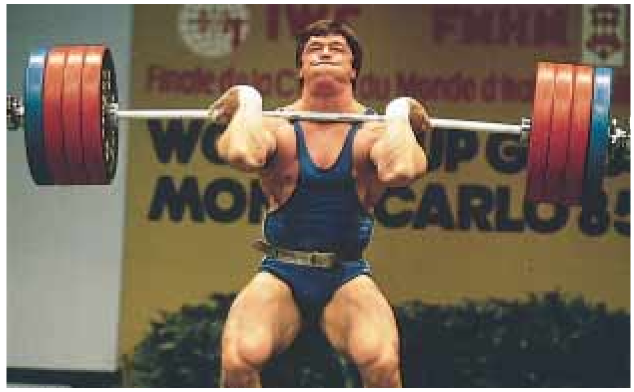


(a)



(b)

Figure 7.4 (a) A vacuum cleaner being pulled at an angle of 30.0° with the horizontal. (b) Free-body diagram of the forces acting on the vacuum cleaner.



The weightlifter does no work on the weights as he holds them on his shoulders. (If he could rest the bar on his shoulders and lock his knees, he would be able to support the weights for quite some time.) Did he do any work when he raised the weights to this height?

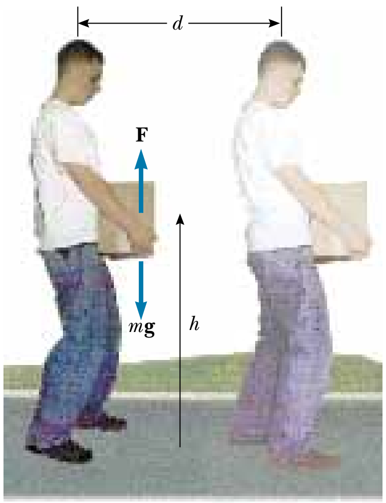


Figure 7.5 A person lifts a box of mass m a vertical distance h and then walks horizontally a distance d.

Quick Quiz 7.2

A person lifts a heavy box of mass m a vertical distance h and then walks horizontally a distance d while holding the box, as shown in Figure 7.5. Determine (a) the work he does on the box and (b) the work done on the box by the force of gravity.

7.2 THE SCALAR PRODUCT OF TWO VECTORS

2.6 Because of the way the force and displacement vectors are combined in Equation 7.1, it is helpful to use a convenient mathematical tool called the **scalar product.** This tool allows us to indicate how **F** and **d** interact in a way that depends on how close to parallel they happen to be. We write this scalar product $\mathbf{F}\cdot\mathbf{d}$. (Because of the dot symbol, the scalar product is often called the **dot product.**) Thus, we can express Equation 7.1 as a scalar product:

Work expressed as a dot product

$$W = \mathbf{F}\cdot\mathbf{d} = Fd\cos\theta \tag{7.2}$$

In other words, $\mathbf{F}\cdot\mathbf{d}$ (read "F dot d") is a shorthand notation for $Fd\cos\theta$.

Scalar product of any two vectors **A** and **B**

In general, the scalar product of any two vectors **A** and **B** is a scalar quantity equal to the product of the magnitudes of the two vectors and the cosine of the angle θ between them:

$$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta \tag{7.3}$$

This relationship is shown in Figure 7.6. Note that **A** and **B** need not have the same units.

In Figure 7.6, $B \cos \theta$ is the projection of **B** onto **A**. Therefore, Equation 7.3 says that $\mathbf{A} \cdot \mathbf{B}$ is the product of the magnitude of **A** and the projection of **B** onto **A**.[2]

From the right-hand side of Equation 7.3 we also see that the scalar product is **commutative.**[3] That is,

The order of the dot product can be reversed

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

Finally, the scalar product obeys the **distributive law of multiplication,** so that

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

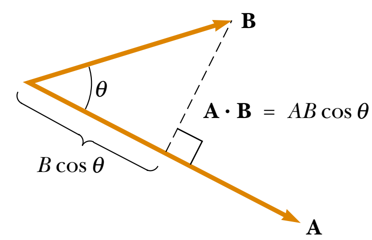


Figure 7.6 The scalar product $\mathbf{A} \cdot \mathbf{B}$ equals the magnitude of **A** multiplied by $B \cos \theta$, which is the projection of **B** onto **A**.

The dot product is simple to evaluate from Equation 7.3 when **A** is either perpendicular or parallel to **B**. If **A** is perpendicular to **B** ($\theta = 90°$), then $\mathbf{A} \cdot \mathbf{B} = 0$. (The equality $\mathbf{A} \cdot \mathbf{B} = 0$ also holds in the more trivial case when either **A** or **B** is zero.) If vector **A** is parallel to vector **B** and the two point in the same direction ($\theta = 0$), then $\mathbf{A} \cdot \mathbf{B} = AB$. If vector **A** is parallel to vector **B** but the two point in opposite directions ($\theta = 180°$), then $\mathbf{A} \cdot \mathbf{B} = -AB$. The scalar product is negative when $90° < \theta < 180°$.

The unit vectors **i**, **j**, and **k**, which were defined in Chapter 3, lie in the positive x, y, and z directions, respectively, of a right-handed coordinate system. Therefore, it follows from the definition of $\mathbf{A} \cdot \mathbf{B}$ that the scalar products of these unit vectors are

Dot products of unit vectors

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \quad \textbf{(7.4)}$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0 \quad \textbf{(7.5)}$$

Equations 3.18 and 3.19 state that two vectors **A** and **B** can be expressed in component vector form as

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$$

$$\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

Using the information given in Equations 7.4 and 7.5 shows that the scalar product of **A** and **B** reduces to

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad \textbf{(7.6)}$$

(Details of the derivation are left for you in Problem 7.10.) In the special case in which $\mathbf{A} = \mathbf{B}$, we see that

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 = A^2$$

Quick Quiz 7.3

If the dot product of two vectors is positive, must the vectors have positive rectangular components?

[2] This is equivalent to stating that $\mathbf{A} \cdot \mathbf{B}$ equals the product of the magnitude of **B** and the projection of **A** onto **B**.

[3] This may seem obvious, but in Chapter 11 you will see another way of combining vectors that proves useful in physics and is not commutative.

EXAMPLE 7.2 The Scalar Product

The vectors **A** and **B** are given by $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. (a) Determine the scalar product $\mathbf{A}\cdot\mathbf{B}$.

Solution

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (2\mathbf{i} + 3\mathbf{j})\cdot(-\mathbf{i} + 2\mathbf{j}) \\ &= -2\mathbf{i}\cdot\mathbf{i} + 2\mathbf{i}\cdot 2\mathbf{j} - 3\mathbf{j}\cdot\mathbf{i} + 3\mathbf{j}\cdot 2\mathbf{j} \\ &= -2(1) + 4(0) - 3(0) + 6(1) \\ &= -2 + 6 = \boxed{4}\end{aligned}$$

where we have used the facts that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$. The same result is obtained when we use Equation 7.6 directly, where $A_x = 2$, $A_y = 3$, $B_x = -1$, and $B_y = 2$.

(b) Find the angle θ between **A** and **B**.

Solution The magnitudes of **A** and **B** are

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$$

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$$

Using Equation 7.3 and the result from part (a) we find that

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{4}{\sqrt{13}\sqrt{5}} = \frac{4}{\sqrt{65}}$$

$$\theta = \cos^{-1}\frac{4}{8.06} = \boxed{60.2^\circ}$$

EXAMPLE 7.3 Work Done by a Constant Force

A particle moving in the xy plane undergoes a displacement $\mathbf{d} = (2.0\mathbf{i} + 3.0\mathbf{j})$ m as a constant force $\mathbf{F} = (5.0\mathbf{i} + 2.0\mathbf{j})$ N acts on the particle. (a) Calculate the magnitude of the displacement and that of the force.

Solution

$$d = \sqrt{x^2 + y^2} = \sqrt{(2.0)^2 + (3.0)^2} = \boxed{3.6\text{ m}}$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2} = \boxed{5.4\text{ N}}$$

(b) Calculate the work done by **F**.

Solution Substituting the expressions for **F** and **d** into Equations 7.4 and 7.5, we obtain

$$\begin{aligned}W = \mathbf{F}\cdot\mathbf{d} &= (5.0\mathbf{i} + 2.0\mathbf{j})\cdot(2.0\mathbf{i} + 3.0\mathbf{j})\text{ N}\cdot\text{m} \\ &= 5.0\mathbf{i}\cdot 2.0\mathbf{i} + 5.0\mathbf{i}\cdot 3.0\mathbf{j} + 2.0\mathbf{j}\cdot 2.0\mathbf{i} + 2.0\mathbf{j}\cdot 3.0\mathbf{j} \\ &= 10 + 0 + 0 + 6 = 16\text{ N}\cdot\text{m} = \boxed{16\text{ J}}\end{aligned}$$

Exercise Calculate the angle between **F** and **d**.

Answer 35°.

7.3 WORK DONE BY A VARYING FORCE

5.2

Consider a particle being displaced along the x axis under the action of a varying force. The particle is displaced in the direction of increasing x from $x = x_i$ to $x = x_f$. In such a situation, we cannot use $W = (F\cos\theta)d$ to calculate the work done by the force because this relationship applies only when **F** is constant in magnitude and direction. However, if we imagine that the particle undergoes a very small displacement Δx, shown in Figure 7.7a, then the x component of the force F_x is approximately constant over this interval; for this small displacement, we can express the work done by the force as

$$\Delta W = F_x\,\Delta x$$

This is just the area of the shaded rectangle in Figure 7.7a. If we imagine that the F_x versus x curve is divided into a large number of such intervals, then the total work done for the displacement from x_i to x_f is approximately equal to the sum of a large number of such terms:

$$W \approx \sum_{x_i}^{x_f} F_x\,\Delta x$$

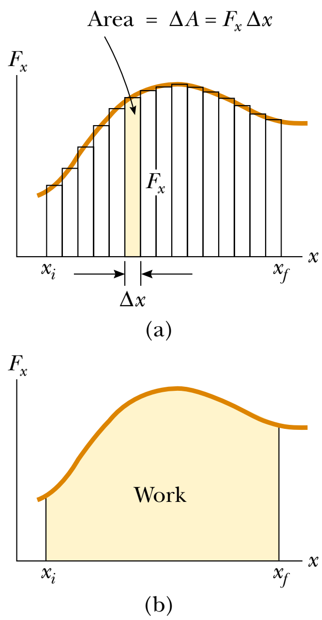


Figure 7.7 (a) The work done by the force component F_x for the small displacement Δx is $F_x \Delta x$, which equals the area of the shaded rectangle. The total work done for the displacement from x_i to x_f is approximately equal to the sum of the areas of all the rectangles. (b) The work done by the component F_x of the varying force as the particle moves from x_i to x_f is exactly equal to the area under this curve.

If the displacements are allowed to approach zero, then the number of terms in the sum increases without limit but the value of the sum approaches a definite value equal to the area bounded by the F_x curve and the x axis:

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx$$

This definite integral is numerically equal to the area under the F_x-versus-x curve between x_i and x_f. Therefore, we can express the work done by F_x as the particle moves from x_i to x_f as

$$W = \int_{x_i}^{x_f} F_x \, dx \qquad \textbf{(7.7)}$$

Work done by a varying force

This equation reduces to Equation 7.1 when the component $F_x = F \cos \theta$ is constant.

If more than one force acts on a particle, the total work done is just the work done by the resultant force. If we express the resultant force in the x direction as ΣF_x, then the total work, or *net work*, done as the particle moves from x_i to x_f is

$$\sum W = W_{\text{net}} = \int_{x_i}^{x_f} \left(\sum F_x \right) dx \qquad \textbf{(7.8)}$$

EXAMPLE 7.4 Calculating Total Work Done from a Graph

A force acting on a particle varies with x, as shown in Figure 7.8. Calculate the work done by the force as the particle moves from $x = 0$ to $x = 6.0$ m.

Solution The work done by the force is equal to the area under the curve from $x_A = 0$ to $x_C = 6.0$ m. This area is equal to the area of the rectangular section from Ⓐ to Ⓑ plus

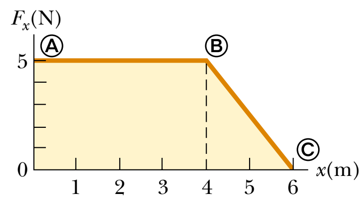


Figure 7.8 The force acting on a particle is constant for the first 4.0 m of motion and then decreases linearly with x from x_B = 4.0 m to x_C = 6.0 m. The net work done by this force is the area under the curve.

the area of the triangular section from Ⓑ to Ⓒ. The area of the rectangle is (4.0)(5.0) N·m = 20 J, and the area of the triangle is $\frac{1}{2}$(2.0)(5.0) N·m = 5.0 J. Therefore, the total work done is 25 J.

EXAMPLE 7.5 Work Done by the Sun on a Probe

The interplanetary probe shown in Figure 7.9a is attracted to the Sun by a force of magnitude

$$F = -1.3 \times 10^{22}/x^2$$

where x is the distance measured outward from the Sun to the probe. Graphically and analytically determine how much work is done by the Sun on the probe as the probe–Sun separation changes from 1.5×10^{11} m to 2.3×10^{11} m.

Graphical Solution The minus sign in the formula for the force indicates that the probe is attracted to the Sun. Because the probe is moving away from the Sun, we expect to calculate a negative value for the work done on it.

A spreadsheet or other numerical means can be used to generate a graph like that in Figure 7.9b. Each small square of the grid corresponds to an area $(0.05 \text{ N})(0.1 \times 10^{11} \text{ m}) = 5 \times 10^8$ N·m. The work done is equal to the shaded area in Figure 7.9b. Because there are approximately 60 squares shaded, the total area (which is negative because it is below the x axis) is about -3×10^{10} N·m. This is the work done by the Sun on the probe.

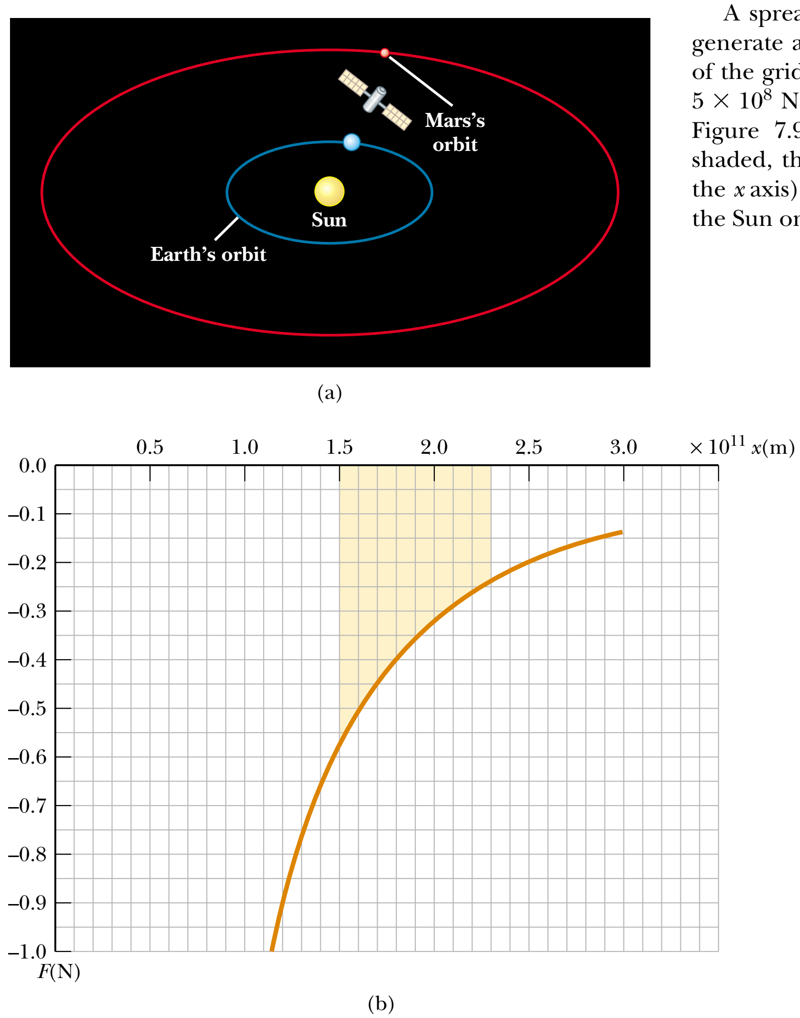


Figure 7.9 (a) An interplanetary probe moves from a position near the Earth's orbit radially outward from the Sun, ending up near the orbit of Mars. (b) Attractive force versus distance for the interplanetary probe.

Analytical Solution We can use Equation 7.7 to calculate a more precise value for the work done on the probe by the Sun. To solve this integral, we use the first formula of Table B.5 in Appendix B with $n = -2$:

$$W = \int_{1.5\times10^{11}}^{2.3\times10^{11}} \left(\frac{-1.3\times10^{22}}{x^2}\right) dx$$

$$= (-1.3\times10^{22}) \int_{1.5\times10^{11}}^{2.3\times10^{11}} x^{-2}\, dx$$

$$= (-1.3\times10^{22})(-x^{-1}) \Big|_{1.5\times10^{11}}^{2.3\times10^{11}}$$

$$= (-1.3\times10^{22})\left(\frac{-1}{2.3\times10^{11}} - \frac{-1}{1.5\times10^{11}}\right)$$

$$= -3.0\times10^{10}\ \text{J}$$

Exercise Does it matter whether the path of the probe is not directed along a radial line away from the Sun?

Answer No; the value of W depends only on the initial and final positions, not on the path taken between these points.

Work Done by a Spring

A common physical system for for which the force varies with position is shown in Figure 7.10. A block on a horizontal, frictionless surface is connected to a spring. If
5.3
the spring is either stretched or compressed a small distance from its unstretched (equilibrium) configuration, it exerts on the block a force of magnitude

$$F_s = -kx \tag{7.9}$$

Spring force

where x is the displacement of the block from its unstretched ($x = 0$) position and k is a positive constant called the **force constant** of the spring. In other words, the force required to stretch or compress a spring is proportional to the amount of stretch or compression x. This force law for springs, known as **Hooke's law,** is valid only in the limiting case of small displacements. The value of k is a measure of the *stiffness* of the spring. Stiff springs have large k values, and soft springs have small k values.

Quick Quiz 7.4

What are the units for k, the force constant in Hooke's law?

The negative sign in Equation 7.9 signifies that the force exerted by the spring is always directed *opposite* the displacement. When $x > 0$ as in Figure 7.10a, the spring force is directed to the left, in the negative x direction. When $x < 0$ as in Figure 7.10c, the spring force is directed to the right, in the positive x direction. When $x = 0$ as in Figure 7.10b, the spring is unstretched and $F_s = 0$. Because the spring force always acts toward the equilibrium position ($x = 0$), it sometimes is called a *restoring force.* If the spring is compressed until the block is at the point $-x_{\text{max}}$ and is then released, the block moves from $-x_{\text{max}}$ through zero to $+x_{\text{max}}$. If the spring is instead stretched until the block is at the point x_{max} and is then released, the block moves from $+x_{\text{max}}$ through zero to $-x_{\text{max}}$. It then reverses direction, returns to $+x_{\text{max}}$, and continues oscillating back and forth.

Suppose the block has been pushed to the left a distance x_{max} from equilibrium and is then released. Let us calculate the work W_s done by the spring force as the block moves from $x_i = -x_{\text{max}}$ to $x_f = 0$. Applying Equation 7.7 and assuming the block may be treated as a particle, we obtain

$$W_s = \int_{x_i}^{x_f} F_s\, dx = \int_{-x_{\text{max}}}^{0} (-kx)\, dx = \tfrac{1}{2}kx_{\text{max}}^2 \tag{7.10}$$

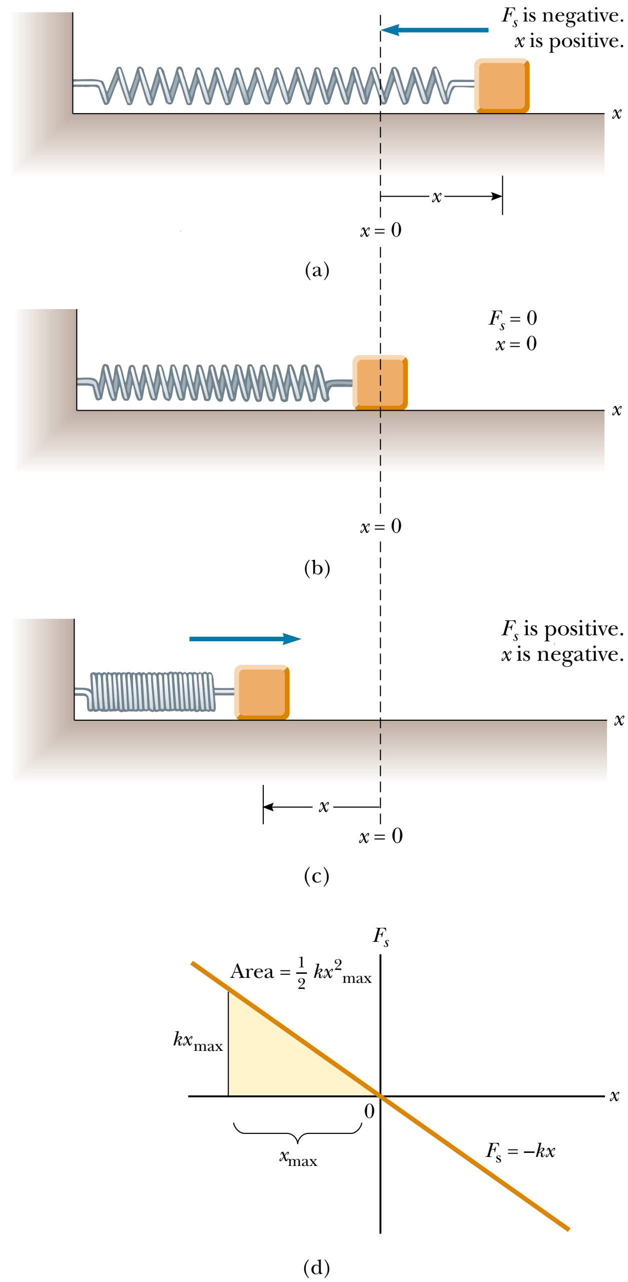


Figure 7.10 The force exerted by a spring on a block varies with the block's displacement x from the equilibrium position $x = 0$. (a) When x is positive (stretched spring), the spring force is directed to the left. (b) When x is zero (natural length of the spring), the spring force is zero. (c) When x is negative (compressed spring), the spring force is directed to the right. (d) Graph of F_s versus x for the block–spring system. The work done by the spring force as the block moves from $-x_{max}$ to 0 is the area of the shaded triangle, $\frac{1}{2}kx^2_{max}$.

where we have used the indefinite integral $\int x^n dx = x^{n+1}/(n+1)$ with $n = 1$. The work done by the spring force is positive because the force is in the same direction as the displacement (both are to the right). When we consider the work done by the spring force as the block moves from $x_i = 0$ to $x_f = x_{max}$, we find that

$W_s = -\frac{1}{2}kx_{max}^2$ because for this part of the motion the displacement is to the right and the spring force is to the left. Therefore, the *net* work done by the spring force as the block moves from $x_i = -x_{max}$ to $x_f = x_{max}$ is *zero.*

Figure 7.10d is a plot of F_s versus x. The work calculated in Equation 7.10 is the area of the shaded triangle, corresponding to the displacement from $-x_{max}$ to 0. Because the triangle has base x_{max} and height kx_{max}, its area is $\frac{1}{2}kx_{max}^2$, the work done by the spring as given by Equation 7.10.

If the block undergoes an arbitrary displacement from $x = x_i$ to $x = x_f$, the work done by the spring force is

$$W_s = \int_{x_i}^{x_f} (-kx)\,dx = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \qquad \textbf{(7.11)}$$

Work done by a spring

For example, if the spring has a force constant of 80 N/m and is compressed 3.0 cm from equilibrium, the work done by the spring force as the block moves from $x_i = -3.0$ cm to its unstretched position $x_f = 0$ is 3.6×10^{-2} J. From Equation 7.11 we also see that the work done by the spring force is zero for any motion that ends where it began ($x_i = x_f$). We shall make use of this important result in Chapter 8, in which we describe the motion of this system in greater detail.

Equations 7.10 and 7.11 describe the work done by the spring on the block. Now let us consider the work done on the spring by an *external agent* that stretches the spring very slowly from $x_i = 0$ to $x_f = x_{max}$, as in Figure 7.11. We can calculate this work by noting that at any value of the displacement, the *applied force* $\mathbf{F}_{app}$ is equal to and opposite the spring force $\mathbf{F}_s$, so that $F_{app} = -(-kx) = kx$. Therefore, the work done by this applied force (the external agent) is

$$W_{F_{app}} = \int_0^{x_{max}} F_{app}\,dx = \int_0^{x_{max}} kx\,dx = \tfrac{1}{2}kx_{max}^2$$

This work is equal to the negative of the work done by the spring force for this displacement.

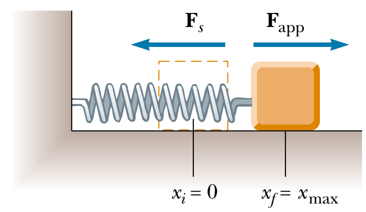


Figure 7.11 A block being pulled from $x_i = 0$ to $x_f = x_{max}$ on a frictionless surface by a force $\mathbf{F}_{app}$. If the process is carried out very slowly, the applied force is equal to and opposite the spring force at all times.

Example 7.6 Measuring *k* for a Spring

A common technique used to measure the force constant of a spring is described in Figure 7.12. The spring is hung vertically, and an object of mass m is attached to its lower end. Under the action of the "load" mg, the spring stretches a distance d from its equilibrium position. Because the spring force is upward (opposite the displacement), it must balance the downward force of gravity $m\mathbf{g}$ when the system is at rest. In this case, we can apply Hooke's law to give $|\mathbf{F}_s| = kd = mg$, or

$$k = \frac{mg}{d}$$

For example, if a spring is stretched 2.0 cm by a suspended object having a mass of 0.55 kg, then the force constant is

$$k = \frac{mg}{d} = \frac{(0.55\text{ kg})(9.80\text{ m/s}^2)}{2.0 \times 10^{-2}\text{ m}} = \boxed{2.7 \times 10^2\text{ N/m}}$$

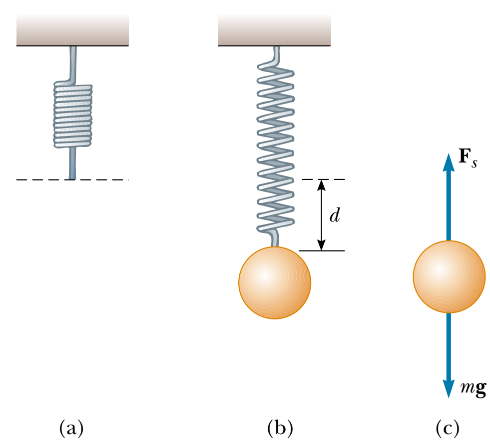


Figure 7.12 Determining the force constant k of a spring. The elongation d is caused by the attached object, which has a weight mg. Because the spring force balances the force of gravity, it follows that $k = mg/d$.

7.4 KINETIC ENERGY AND THE WORK–KINETIC ENERGY THEOREM

5.7

It can be difficult to use Newton's second law to solve motion problems involving complex forces. An alternative approach is to relate the speed of a moving particle to its displacement under the influence of some net force. If the work done by the net force on a particle can be calculated for a given displacement, then the change in the particle's speed can be easily evaluated.

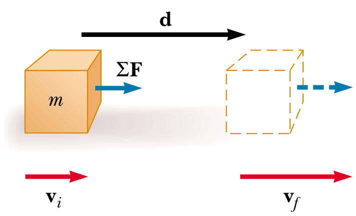


Figure 7.13 A particle undergoing a displacement **d** and a change in velocity under the action of a constant net force $\Sigma\mathbf{F}$.

Figure 7.13 shows a particle of mass m moving to the right under the action of a constant net force $\Sigma\mathbf{F}$. Because the force is constant, we know from Newton's second law that the particle moves with a constant acceleration **a**. If the particle is displaced a distance d, the net work done by the total force $\Sigma\mathbf{F}$ is

$$\sum W = \left(\sum F\right) d = (ma)\,d \qquad \textbf{(7.12)}$$

In Chapter 2 we found that the following relationships are valid when a particle undergoes constant acceleration:

$$d = \tfrac{1}{2}(v_i + v_f)\,t \qquad a = \frac{v_f - v_i}{t}$$

where v_i is the speed at $t = 0$ and v_f is the speed at time t. Substituting these expressions into Equation 7.12 gives

$$\sum W = m\left(\frac{v_f - v_i}{t}\right)\tfrac{1}{2}(v_i + v_f)\,t$$

$$\sum W = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 \qquad \textbf{(7.13)}$$

The quantity $\frac{1}{2}mv^2$ represents the energy associated with the motion of the particle. This quantity is so important that it has been given a special name—**kinetic energy.** The net work done on a particle by a constant net force $\Sigma\mathbf{F}$ acting on it equals the change in kinetic energy of the particle.

In general, the kinetic energy K of a particle of mass m moving with a speed v is defined as

Kinetic energy is energy associated with the motion of a body

$$K \equiv \tfrac{1}{2}mv^2 \qquad \textbf{(7.14)}$$

TABLE 7.1 Kinetic Energies for Various Objects

Object	Mass (kg)	Speed (m/s)	Kinetic Energy (J)
Earth orbiting the Sun	5.98×10^{24}	2.98×10^{4}	2.65×10^{33}
Moon orbiting the Earth	7.35×10^{22}	1.02×10^{3}	3.82×10^{28}
Rocket moving at escape speed[a]	500	1.12×10^{4}	3.14×10^{10}
Automobile at 55 mi/h	2 000	25	6.3×10^{5}
Running athlete	70	10	3.5×10^{3}
Stone dropped from 10 m	1.0	14	9.8×10^{1}
Golf ball at terminal speed	0.046	44	4.5×10^{1}
Raindrop at terminal speed	3.5×10^{-5}	9.0	1.4×10^{-3}
Oxygen molecule in air	5.3×10^{-26}	500	6.6×10^{-21}

[a] *Escape speed* is the minimum speed an object must attain near the Earth's surface if it is to escape the Earth's gravitational force.

5.4 Kinetic energy is a scalar quantity and has the same units as work. For example, a 2.0-kg object moving with a speed of 4.0 m/s has a kinetic energy of 16 J. Table 7.1 lists the kinetic energies for various objects.

It is often convenient to write Equation 7.13 in the form

$$\sum W = K_f - K_i = \Delta K \tag{7.15}$$

Work–kinetic energy theorem

That is, $K_i + \sum W = K_f$.

Equation 7.15 is an important result known as the **work–kinetic energy theorem.** It is important to note that when we use this theorem, we must include *all* of the forces that do work on the particle in the calculation of the net work done. From this theorem, we see that the speed of a particle increases if the net work done on it is positive because the final kinetic energy is greater than the initial kinetic energy. The particle's speed decreases if the net work done is negative because the final kinetic energy is less than the initial kinetic energy.

The work–kinetic energy theorem as expressed by Equation 7.15 allows us to think of kinetic energy as the work a particle can do in coming to rest, or the amount of energy stored in the particle. For example, suppose a hammer (our particle) is on the verge of striking a nail, as shown in Figure 7.14. The moving hammer has kinetic energy and so can do work on the nail. The work done on the nail is equal to Fd, where F is the average force exerted on the nail by the hammer and d is the distance the nail is driven into the wall.[4]

Figure 7.14 The moving hammer has kinetic energy and thus can do work on the nail, driving it into the wall.

We derived the work–kinetic energy theorem under the assumption of a constant net force, but it also is valid when the force varies. To see this, suppose the net force acting on a particle in the x direction is ΣF_x. We can apply Newton's second law, $\Sigma F_x = ma_x$, and use Equation 7.8 to express the net work done as

$$\sum W = \int_{x_i}^{x_f} \left(\sum F_x\right) dx = \int_{x_i}^{x_f} ma_x \, dx$$

If the resultant force varies with x, the acceleration and speed also depend on x. Because we normally consider acceleration as a function of t, we now use the following chain rule to express a in a slightly different way:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Substituting this expression for a into the above equation for ΣW gives

$$\sum W = \int_{x_i}^{x_f} mv \frac{dv}{dx}\, dx = \int_{v_i}^{v_f} mv \, dv$$

$$\sum W = \tfrac{1}{2}mv_f^{2} - \tfrac{1}{2}mv_i^{2} \tag{7.16}$$

The net work done on a particle equals the change in its kinetic energy

The limits of the integration were changed from x values to v values because the variable was changed from x to v. Thus, we conclude that the net work done on a particle by the net force acting on it is equal to the change in the kinetic energy of the particle. This is true whether or not the net force is constant.

[4] Note that because the nail and the hammer are *systems* of particles rather than single particles, part of the hammer's kinetic energy goes into warming the hammer and the nail upon impact. Also, as the nail moves into the wall in response to the impact, the large frictional force between the nail and the wood results in the continuous transformation of the kinetic energy of the nail into further temperature increases in the nail and the wood, as well as in deformation of the wall. Energy associated with temperature changes is called *internal energy* and will be studied in detail in Chapter 20.

Situations Involving Kinetic Friction

One way to include frictional forces in analyzing the motion of an object sliding on a *horizontal* surface is to describe the kinetic energy lost because of friction. Suppose a book moving on a horizontal surface is given an initial horizontal velocity $\mathbf{v}_i$ and slides a distance d before reaching a final velocity $\mathbf{v}_f$ as shown in Figure 7.15. The external force that causes the book to undergo an acceleration in the negative x direction is the force of kinetic friction $\mathbf{f}_k$ acting to the left, opposite the motion. The initial kinetic energy of the book is $\frac{1}{2}mv_i^2$, and its final kinetic energy is $\frac{1}{2}mv_f^2$. Applying Newton's second law to the book can show this. Because the only force acting on the book in the x direction is the friction force, Newton's second law gives $-f_k = ma_x$. Multiplying both sides of this expression by d and using Equation 2.12 in the form $v_{xf}^2 - v_{xi}^2 = 2a_x d$ for motion under constant acceleration give $-f_k d = (ma_x)d = \frac{1}{2}mv_{xf}^2 - \frac{1}{2}mv_{xi}^2$ or

Loss in kinetic energy due to friction

$$\Delta K_{\text{friction}} = -f_k d \tag{7.17a}$$

This result specifies that the amount by which the force of kinetic friction changes the kinetic energy of the book is equal to $-f_k d$. Part of this lost kinetic energy goes into warming up the book, and the rest goes into warming up the surface over which the book slides. In effect, the quantity $-f_k d$ is equal to the work done by kinetic friction on the book *plus* the work done by kinetic friction on the surface. (We shall study the relationship between temperature and energy in Part III of this text.) When friction—as well as other forces—acts on an object, the work–kinetic energy theorem reads

$$K_i + \sum W_{\text{other}} - f_k d = K_f \tag{7.17b}$$

Here, $\sum W_{\text{other}}$ represents the sum of the amounts of work done on the object by forces other than kinetic friction.

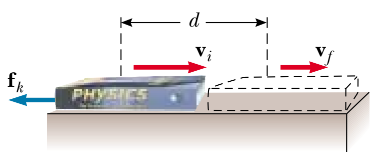


Figure 7.15 A book sliding to the right on a horizontal surface slows down in the presence of a force of kinetic friction acting to the left. The initial velocity of the book is $\mathbf{v}_i$, and its final velocity is $\mathbf{v}_f$. The normal force and the force of gravity are not included in the diagram because they are perpendicular to the direction of motion and therefore do not influence the book's velocity.

Quick Quiz 7.5

Can frictional forces ever *increase* an object's kinetic energy?

EXAMPLE 7.7 A Block Pulled on a Frictionless Surface

A 6.0-kg block initially at rest is pulled to the right along a horizontal, frictionless surface by a constant horizontal force of 12 N. Find the speed of the block after it has moved 3.0 m.

Solution We have made a drawing of this situation in Figure 7.16a. We could apply the equations of kinematics to determine the answer, but let us use the energy approach for

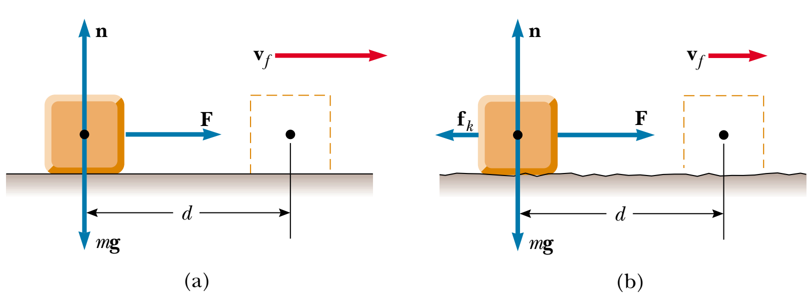


Figure 7.16 A block pulled to the right by a constant horizontal force. (a) Frictionless surface. (b) Rough surface.

practice. The normal force balances the force of gravity on the block, and neither of these vertically acting forces does work on the block because the displacement is horizontal. Because there is no friction, the net external force acting on the block is the 12-N force. The work done by this force is

$$W = Fd = (12\ \text{N})(3.0\ \text{m}) = 36\ \text{N}\cdot\text{m} = 36\ \text{J}$$

Using the work–kinetic energy theorem and noting that the initial kinetic energy is zero, we obtain

$$W = K_f - K_i = \tfrac{1}{2}mv_f^{\,2} - 0$$

$$v_f^{\,2} = \frac{2W}{m} = \frac{2(36\ \text{J})}{6.0\ \text{kg}} = 12\ \text{m}^2/\text{s}^2$$

$$v_f = \boxed{3.5\ \text{m/s}}$$

Exercise Find the acceleration of the block and determine its final speed, using the kinematics equation $v_{xf}^{\,2} = v_{xi}^{\,2} + 2a_x d$.

Answer $a_x = 2.0\ \text{m/s}^2$; $v_f = 3.5\ \text{m/s}$.

EXAMPLE 7.8 A Block Pulled on a Rough Surface

Find the final speed of the block described in Example 7.7 if the surface is not frictionless but instead has a coefficient of kinetic friction of 0.15.

Solution The applied force does work just as in Example 7.7:

$$W = Fd = (12\ \text{N})(3.0\ \text{m}) = 36\ \text{J}$$

In this case we must use Equation 7.17a to calculate the kinetic energy lost to friction $\Delta K_{\text{friction}}$. The magnitude of the frictional force is

$$f_k = \mu_k n = \mu_k mg = (0.15)(6.0\ \text{kg})(9.80\ \text{m/s}^2) = 8.82\ \text{N}$$

The change in kinetic energy due to friction is

$$\Delta K_{\text{friction}} = -f_k d = -(8.82\ \text{N})(3.0\ \text{m}) = -26.5\ \text{J}$$

The final speed of the block follows from Equation 7.17b:

$$\tfrac{1}{2}\,mv_i^{\,2} + \sum W_{\text{other}} - f_k d = \tfrac{1}{2}\,mv_f^{\,2}$$

$$0 + 36\ \text{J} - 26.5\ \text{J} = \tfrac{1}{2}\,(6.0\ \text{kg})\ v_f^{\,2}$$

$$v_f^{\,2} = 2(9.5\ \text{J})/(6.0\ \text{kg}) = 3.18\ \text{m}^2/\text{s}^2$$

$$v_f = \boxed{1.8\ \text{m/s}}$$

After sliding the 3-m distance on the rough surface, the block is moving at a speed of 1.8 m/s; in contrast, after covering the same distance on a frictionless surface (see Example 7.7), its speed was 3.5 m/s.

Exercise Find the acceleration of the block from Newton's second law and determine its final speed, using equations of kinematics.

Answer $a_x = 0.53\ \text{m/s}^2$; $v_f = 1.8\ \text{m/s}$.

CONCEPTUAL EXAMPLE 7.9 Does the Ramp Lessen the Work Required?

A man wishes to load a refrigerator onto a truck using a ramp, as shown in Figure 7.17. He claims that less work would be required to load the truck if the length L of the ramp were increased. Is his statement valid?

Solution No. Although less force is required with a longer ramp, that force must act over a greater distance if the same amount of work is to be done. Suppose the refrigerator is wheeled on a dolly up the ramp at constant speed. The

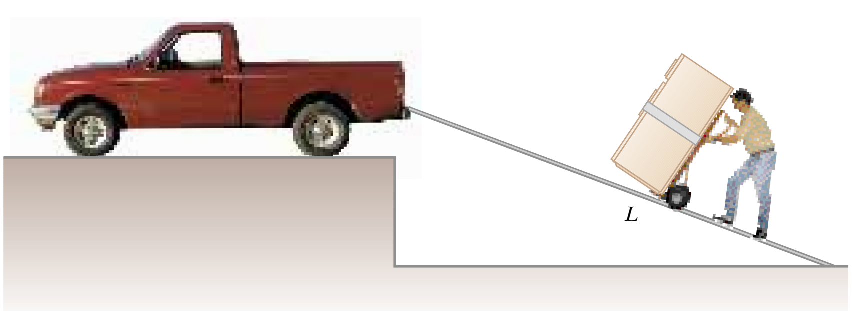


Figure 7.17 A refrigerator attached to a frictionless wheeled dolly is moved up a ramp at constant speed.

normal force exerted by the ramp on the refrigerator is directed 90° to the motion and so does no work on the refrigerator. Because $\Delta K = 0$, the work–kinetic energy theorem gives

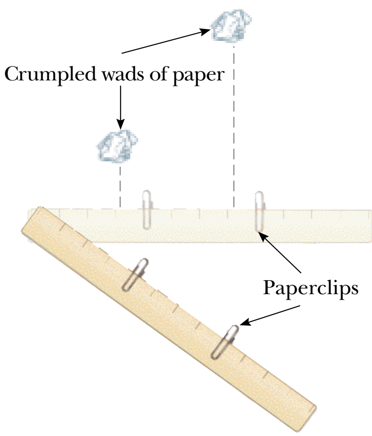
$$\sum W = W_{\text{by man}} + W_{\text{by gravity}} = 0$$

The work done by the force of gravity equals the weight of the refrigerator mg times the vertical height h through which it is displaced times cos 180°, or $W_{\text{by gravity}} = -mgh$. (The minus sign arises because the downward force of gravity is opposite the displacement.) Thus, the man must do work mgh on the refrigerator, regardless of the length of the ramp.

Consider the chum salmon attempting to swim upstream in the photograph at the beginning of this chapter. The "steps" of a fish ladder built around a dam do not change the total amount of work that must be done by the salmon as they leap through some vertical distance. However, the ladder allows the fish to perform that work in a series of smaller jumps, and the net effect is to raise the vertical position of the fish by the height of the dam.

QuickLab

Attach two paperclips to a ruler so that one of the clips is twice the distance from the end as the other. Place the ruler on a table with two small wads of paper against the clips, which act as stops. Sharply swing the ruler through a small angle, stopping it abruptly with your finger. The outer paper wad will have twice the speed of the inner paper wad as the two slide on the table away from the ruler. Compare how far the two wads slide. How does this relate to the results of Conceptual Example 7.10?



These cyclists are working hard and expending energy as they pedal uphill in Marin County, CA.

CONCEPTUAL EXAMPLE 7.10 Useful Physics for Safer Driving

A certain car traveling at an initial speed v slides a distance d to a halt after its brakes lock. Assuming that the car's initial speed is instead $2v$ at the moment the brakes lock, estimate the distance it slides.

Solution Let us assume that the force of kinetic friction between the car and the road surface is constant and the same for both speeds. The net force multiplied by the displacement of the car is equal to the initial kinetic energy of the car (because $K_f = 0$). If the speed is doubled, as it is in this example, the kinetic energy is quadrupled. For a given constant applied force (in this case, the frictional force), the distance traveled is four times as great when the initial speed is doubled, and so the estimated distance that the car slides is $4d$.

EXAMPLE 7.11 A Block–Spring System

A block of mass 1.6 kg is attached to a horizontal spring that has a force constant of 1.0×10^3 N/m, as shown in Figure 7.10. The spring is compressed 2.0 cm and is then released from rest. (a) Calculate the speed of the block as it passes through the equilibrium position $x = 0$ if the surface is frictionless.

Solution In this situation, the block starts with $v_i = 0$ at $x_i = -2.0$ cm, and we want to find v_f at $x_f = 0$. We use Equation 7.10 to find the work done by the spring with $x_{max} = x_i = -2.0 \text{ cm} = -2.0 \times 10^{-2}$ m:

$$W_s = \tfrac{1}{2}kx_{max}^2 = \tfrac{1}{2}(1.0 \times 10^3 \text{ N/m})(-2.0 \times 10^{-2} \text{ m})^2 = 0.20 \text{ J}$$

Using the work–kinetic energy theorem with $v_i = 0$, we obtain the change in kinetic energy of the block due to the work done on it by the spring:

$$W_s = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

$$0.20 \text{ J} = \tfrac{1}{2}(1.6 \text{ kg})v_f^2 - 0$$

$$v_f^2 = \frac{0.40 \text{ J}}{1.6 \text{ kg}} = 0.25 \text{ m}^2/\text{s}^2$$

$$v_f = 0.50 \text{ m/s}$$

(b) Calculate the speed of the block as it passes through the equilibrium position if a constant frictional force of 4.0 N retards its motion from the moment it is released.

Solution Certainly, the answer has to be less than what we found in part (a) because the frictional force retards the motion. We use Equation 7.17 to calculate the kinetic energy lost because of friction and add this negative value to the kinetic energy found in the absence of friction. The kinetic energy lost due to friction is

$$\Delta K = -f_k d = -(4.0 \text{ N})(2.0 \times 10^{-2} \text{ m}) = -0.080 \text{ J}$$

In part (a), the final kinetic energy without this loss was found to be 0.20 J. Therefore, the final kinetic energy in the presence of friction is

$$K_f = 0.20 \text{ J} - 0.080 \text{ J} = 0.12 \text{ J} = \tfrac{1}{2}mv_f^2$$

$$\tfrac{1}{2}(1.6 \text{ kg})v_f^2 = 0.12 \text{ J}$$

$$v_f^2 = \frac{0.24 \text{ J}}{1.6 \text{ kg}} = 0.15 \text{ m}^2/\text{s}^2$$

$$v_f = 0.39 \text{ m/s}$$

As expected, this value is somewhat less than the 0.50 m/s we found in part (a). If the frictional force were greater, then the value we obtained as our answer would have been even smaller.

7.5 POWER

5.8

Imagine two identical models of an automobile: one with a base-priced four-cylinder engine; and the other with the highest-priced optional engine, a mighty eight-cylinder powerplant. Despite the differences in engines, the two cars have the same mass. Both cars climb a roadway up a hill, but the car with the optional engine takes much less time to reach the top. Both cars have done the same amount of work against gravity, but in different time periods. From a practical viewpoint, it is interesting to know not only the work done by the vehicles but also the *rate* at which it is done. In taking the ratio of the amount of work done to the time taken to do it, we have a way of quantifying this concept. The time rate of doing work is called **power.**

If an external force is applied to an object (which we assume acts as a particle), and if the work done by this force in the time interval Δt is W, then the **average power** expended during this interval is defined as

Average power

$$\overline{\mathcal{P}} \equiv \frac{W}{\Delta t}$$

The work done on the object contributes to the increase in the energy of the object. Therefore, a more general definition of power is the *time rate of energy transfer.* In a manner similar to how we approached the definition of velocity and acceleration, we can define the **instantaneous power** $\mathcal{P}$ as the limiting value of the average power as Δt approaches zero:

$$\mathcal{P} \equiv \lim_{\Delta t \to 0} \frac{W}{\Delta t} = \frac{dW}{dt}$$

where we have represented the increment of work done by dW. We find from Equation 7.2, letting the displacement be expressed as $d\mathbf{s}$, that $dW = \mathbf{F} \cdot d\mathbf{s}$. Therefore, the instantaneous power can be written

Instantaneous power

$$\mathcal{P} = \frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{7.18}$$

where we use the fact that $\mathbf{v} = d\mathbf{s}/dt$.

The SI unit of power is joules per second (J/s), also called the *watt* (W) (after James Watt, the inventor of the steam engine):

The watt

$$1 \text{ W} = 1 \text{ J/s} = 1 \text{ kg}\cdot\text{m}^2/\text{s}^3$$

The symbol W (not italic) for watt should not be confused with the symbol W (italic) for work.

A unit of power in the British engineering system is the horsepower (hp):

$$1 \text{ hp} = 746 \text{ W}$$

A unit of energy (or work) can now be defined in terms of the unit of power. One **kilowatt hour** (kWh) is the energy converted or consumed in 1 h at the constant rate of 1 kW = 1 000 J/s. The numerical value of 1 kWh is

The kilowatt hour is a unit of energy

$$1 \text{ kWh} = (10^3 \text{ W})(3\,600 \text{ s}) = 3.60 \times 10^6 \text{ J}$$

It is important to realize that a kilowatt hour is a unit of energy, not power. When you pay your electric bill, you pay the power company for the total electrical energy you used during the billing period. This energy is the power used multiplied by the time during which it was used. For example, a 300-W lightbulb run for 12 h would convert (0.300 kW)(12 h) = 3.6 kWh of electrical energy.

Quick Quiz 7.6

Suppose that an old truck and a sports car do the same amount of work as they climb a hill but that the truck takes much longer to accomplish this work. How would graphs of $\mathcal{P}$ versus t compare for the two vehicles?

EXAMPLE 7.12 Power Delivered by an Elevator Motor

An elevator car has a mass of 1 000 kg and is carrying passengers having a combined mass of 800 kg. A constant frictional force of 4 000 N retards its motion upward, as shown in Figure 7.18a. (a) What must be the minimum power delivered by the motor to lift the elevator car at a constant speed of 3.00 m/s?

Solution The motor must supply the force of magnitude T that pulls the elevator car upward. Reading that the speed is constant provides the hint that $a = 0$, and therefore we know from Newton's second law that $\Sigma F_y = 0$. We have drawn a free-body diagram in Figure 7.18b and have arbitrarily specified that the upward direction is positive. From Newton's second law we obtain

$$\sum F_y = T - f - Mg = 0$$

where M is the *total* mass of the system (car plus passengers), equal to 1 800 kg. Therefore,

$$\begin{aligned} T &= f + Mg \\ &= 4.00 \times 10^3 \text{ N} + (1.80 \times 10^3 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 2.16 \times 10^4 \text{ N} \end{aligned}$$

Using Equation 7.18 and the fact that **T** is in the same direction as **v**, we find that

$$\mathcal{P} = \mathbf{T}\cdot\mathbf{v} = Tv$$
$$= (2.16 \times 10^4 \text{ N})(3.00 \text{ m/s}) = \boxed{6.48 \times 10^4 \text{ W}}$$

(b) What power must the motor deliver at the instant its speed is v if it is designed to provide an upward acceleration of 1.00 m/s^2?

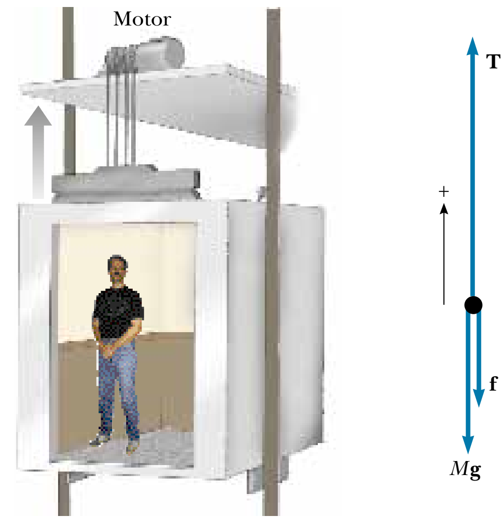
Solution Now we expect to obtain a value greater than we did in part (a), where the speed was constant, because the motor must now perform the additional task of accelerating the car. The only change in the setup of the problem is that now $a > 0$. Applying Newton's second law to the car gives

$$\sum F_y = T - f - Mg = Ma$$
$$T = M(a + g) + f$$
$$= (1.80 \times 10^3 \text{ kg})(1.00 + 9.80)\text{m/s}^2 + 4.00 \times 10^3 \text{ N}$$
$$= 2.34 \times 10^4 \text{ N}$$

Therefore, using Equation 7.18, we obtain for the required power

$$\mathcal{P} = Tv = \boxed{(2.34 \times 10^4 v) \text{ W}}$$

where v is the instantaneous speed of the car in meters per second. The power is less than that obtained in part (a) as long as the speed is less than $\mathcal{P}/T = 2.77$ m/s, but it is greater when the elevator's speed exceeds this value.



(a) (b)

Figure 7.18 (a) The motor exerts an upward force **T** on the elevator car. The magnitude of this force is the tension T in the cable connecting the car and motor. The downward forces acting on the car are a frictional force **f** and the force of gravity $\mathbf{F}_g = M\mathbf{g}$. (b) The free-body diagram for the elevator car.

CONCEPTUAL EXAMPLE 7.13

In part (a) of the preceding example, the motor delivers power to lift the car, and yet the car moves at constant speed. A student analyzing this situation notes that the kinetic energy of the car does not change because its speed does not change. This student then reasons that, according to the work–kinetic energy theorem, $W = \Delta K = 0$. Knowing that $\mathcal{P} = W/t$, the student concludes that the power delivered by the motor also must be zero. How would you explain this apparent paradox?

Solution The work–kinetic energy theorem tells us that the *net* force acting on the system multiplied by the displacement is equal to the change in the kinetic energy of the system. In our elevator case, the net force is indeed zero (that is, $T - Mg - f = 0$), and so $W = (\Sigma F_y)d = 0$. However, the power from the motor is calculated not from the *net* force but rather from the force exerted by the motor acting in the direction of motion, which in this case is T and not zero.

Optional Section

7.6 ENERGY AND THE AUTOMOBILE

Automobiles powered by gasoline engines are very inefficient machines. Even under ideal conditions, less than 15% of the chemical energy in the fuel is used to power the vehicle. The situation is much worse under stop-and-go driving conditions in a city. In this section, we use the concepts of energy, power, and friction to analyze automobile fuel consumption.

Many mechanisms contribute to energy loss in an automobile. About 67% of the energy available from the fuel is lost in the engine. This energy ends up in the atmosphere, partly via the exhaust system and partly via the cooling system. (As we shall see in Chapter 22, the great energy loss from the exhaust and cooling systems is required by a fundamental law of thermodynamics.) Approximately 10% of the available energy is lost to friction in the transmission, drive shaft, wheel and axle bearings, and differential. Friction in other moving parts dissipates approximately 6% of the energy, and 4% of the energy is used to operate fuel and oil pumps and such accessories as power steering and air conditioning. This leaves a mere 13% of the available energy to propel the automobile! This energy is used mainly to balance the energy loss due to flexing of the tires and the friction caused by the air, which is more commonly referred to as *air resistance.*

Let us examine the power required to provide a force in the forward direction that balances the combination of the two frictional forces. The coefficient of rolling friction μ between the tires and the road is about 0.016. For a 1 450-kg car, the weight is 14 200 N and the force of rolling friction has a magnitude of $\mu n = \mu mg = 227$ N. As the speed of the car increases, a small reduction in the normal force occurs as a result of a decrease in atmospheric pressure as air flows over the top of the car. (This phenomenon is discussed in Chapter 15.) This reduction in the normal force causes a slight reduction in the force of rolling friction f_r with increasing speed, as the data in Table 7.2 indicate.

Now let us consider the effect of the resistive force that results from the movement of air past the car. For large objects, the resistive force f_a associated with air friction is proportional to the square of the speed (in meters per second; see Section 6.4) and is given by Equation 6.6:

$$f_a = \tfrac{1}{2}D\rho A v^2$$

where D is the drag coefficient, ρ is the density of air, and A is the cross-sectional area of the moving object. We can use this expression to calculate the f_a values in Table 7.2, using $D = 0.50$, $\rho = 1.293\ \text{kg/m}^3$, and $A \approx 2\ \text{m}^2$.

The magnitude of the total frictional force f_t is the sum of the rolling frictional force and the air resistive force:

$$f_t = f_r + f_a$$

At low speeds, road friction is the predominant resistive force, but at high speeds air drag predominates, as shown in Table 7.2. Road friction can be decreased by a reduction in tire flexing (for example, by an increase in the air pres-

TABLE 7.2 Frictional Forces and Power Requirements for a Typical Car[a]

v (m/s)	n (N)	f_r (N)	f_a (N)	f_t (N)	$\mathcal{P} = f_t v$ (kW)
0	14 200	227	0	227	0
8.9	14 100	226	51	277	2.5
17.8	13 900	222	204	426	7.6
26.8	13 600	218	465	683	18.3
35.9	13 200	211	830	1 041	37.3
44.8	12 600	202	1 293	1 495	67.0

[a] In this table, n is the normal force, f_r is road friction, f_a is air friction, f_t is total friction, and $\mathcal{P}$ is the power delivered to the wheels.

sure slightly above recommended values) and by the use of radial tires. Air drag can be reduced through the use of a smaller cross-sectional area and by streamlining the car. Although driving a car with the windows open increases air drag and thus results in a 3% decrease in mileage, driving with the windows closed and the air conditioner running results in a 12% decrease in mileage.

The total power needed to maintain a constant speed v is $f_t v$, and it is this power that must be delivered to the wheels. For example, from Table 7.2 we see that at $v = 26.8$ m/s (60 mi/h) the required power is

$$\mathcal{P} = f_t v = (683 \text{ N})\left(26.8 \frac{\text{m}}{\text{s}}\right) = 18.3 \text{ kW}$$

This power can be broken down into two parts: (1) the power $f_r v$ needed to compensate for road friction, and (2) the power $f_a v$ needed to compensate for air drag. At $v = 26.8$ m/s, we obtain the values

$$\mathcal{P}_r = f_r v = (218 \text{ N})\left(26.8 \frac{\text{m}}{\text{s}}\right) = 5.84 \text{ kW}$$

$$\mathcal{P}_a = f_a v = (465 \text{ N})\left(26.8 \frac{\text{m}}{\text{s}}\right) = 12.5 \text{ kW}$$

Note that $\mathcal{P} = \mathcal{P}_r + \mathcal{P}_a$.

On the other hand, at $v = 44.8$ m/s (100 mi/h), $\mathcal{P}_r = 9.05$ kW, $\mathcal{P}_a = 57.9$ kW, and $\mathcal{P} = 67.0$ kW. This shows the importance of air drag at high speeds.

EXAMPLE 7.14 Gas Consumed by a Compact Car

A compact car has a mass of 800 kg, and its efficiency is rated at 18%. (That is, 18% of the available fuel energy is delivered to the wheels.) Find the amount of gasoline used to accelerate the car from rest to 27 m/s (60 mi/h). Use the fact that the energy equivalent of 1 gal of gasoline is 1.3×10^8 J.

Solution The energy required to accelerate the car from rest to a speed v is its final kinetic energy $\frac{1}{2}mv^2$:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(800 \text{ kg})(27 \text{ m/s})^2 = 2.9 \times 10^5 \text{ J}$$

If the engine were 100% efficient, each gallon of gasoline would supply 1.3×10^8 J of energy. Because the engine is only 18% efficient, each gallon delivers only $(0.18)(1.3 \times 10^8 \text{ J}) = 2.3 \times 10^7$ J. Hence, the number of gallons used to accelerate the car is

$$\text{Number of gallons} = \frac{2.9 \times 10^5 \text{ J}}{2.3 \times 10^7 \text{ J/gal}} = \boxed{0.013 \text{ gal}}$$

At cruising speed, this much gasoline is sufficient to propel the car nearly 0.5 mi. This demonstrates the extreme energy requirements of stop-and-start driving.

EXAMPLE 7.15 Power Delivered to Wheels

Suppose the compact car in Example 7.14 gets 35 mi/gal at 60 mi/h. How much power is delivered to the wheels?

Solution By simply canceling units, we determine that the car consumes 60 mi/h ÷ 35 mi/gal = 1.7 gal/h. Using the fact that each gallon is equivalent to 1.3×10^8 J, we find that the total power used is

$$\mathcal{P} = \frac{(1.7 \text{ gal/h})(1.3 \times 10^8 \text{ J/gal})}{3.6 \times 10^3 \text{ s/h}}$$

$$= \frac{2.2 \times 10^8 \text{ J}}{3.6 \times 10^3 \text{ s}} = 62 \text{ kW}$$

Because 18% of the available power is used to propel the car, the power delivered to the wheels is $(0.18)(62 \text{ kW}) = \boxed{11 \text{ kW.}}$ This is 40% less than the 18.3-kW value obtained for the 1 450-kg car discussed in the text. Vehicle mass is clearly an important factor in power-loss mechanisms.

EXAMPLE 7.16 Car Accelerating Up a Hill

Consider a car of mass m that is accelerating up a hill, as shown in Figure 7.19. An automotive engineer has measured the magnitude of the total resistive force to be

$$f_t = (218 + 0.70v^2)\ \text{N}$$

where v is the speed in meters per second. Determine the power the engine must deliver to the wheels as a function of speed.

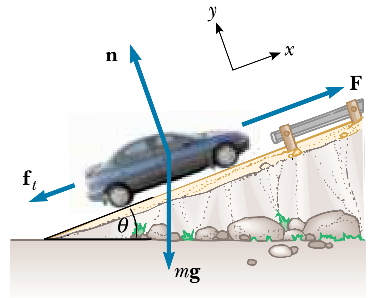


Figure 7.19

Solution The forces on the car are shown in Figure 7.19, in which **F** is the force of friction from the road that propels the car; the remaining forces have their usual meaning. Applying Newton's second law to the motion along the road surface, we find that

$$\sum F_x = F - f_t - mg\sin\theta = ma$$

$$F = ma + mg\sin\theta + f_t$$

$$= ma + mg\sin\theta + (218 + 0.70v^2)$$

Therefore, the power required to move the car forward is

$$\mathcal{P} = Fv = mva + mvg\sin\theta + 218v + 0.70v^3$$

The term mva represents the power that the engine must deliver to accelerate the car. If the car moves at constant speed, this term is zero and the total power requirement is reduced. The term $mvg\sin\theta$ is the power required to provide a force to balance a component of the force of gravity as the car moves up the incline. This term would be zero for motion on a horizontal surface. The term $218v$ is the power required to provide a force to balance road friction, and the term $0.70v^3$ is the power needed to do work on the air.

If we take $m = 1\,450$ kg, $v = 27$ m/s ($=60$ mi/h), $a = 1.0$ m/s^2, and $\theta = 10°$, then the various terms in $\mathcal{P}$ are calculated to be

$$mva = (1\,450\ \text{kg})(27\ \text{m/s})(1.0\ \text{m/s}^2)$$
$$= 39\ \text{kW} = 52\ \text{hp}$$

$$mvg\sin\theta = (1\,450\ \text{kg})(27\ \text{m/s})(9.80\ \text{m/s}^2)(\sin 10°)$$
$$= 67\ \text{kW} = 89\ \text{hp}$$

$$218v = 218(27\ \text{m/s}) = 5.9\ \text{kW} = 7.9\ \text{hp}$$

$$0.70v^3 = 0.70(27\ \text{m/s})^3 = 14\ \text{kW} = 19\ \text{hp}$$

Hence, the total power required is 126 kW, or 168 hp.

Note that the power requirements for traveling at constant speed on a horizontal surface are only 20 kW, or 27 hp (the sum of the last two terms). Furthermore, if the mass were halved (as in the case of a compact car), then the power required also is reduced by almost the same factor.

Optional Section

7.7 KINETIC ENERGY AT HIGH SPEEDS

The laws of Newtonian mechanics are valid only for describing the motion of particles moving at speeds that are small compared with the speed of light in a vacuum c ($=3.00 \times 10^8$ m/s). When speeds are comparable to c, the equations of Newtonian mechanics must be replaced by the more general equations predicted by the theory of relativity. One consequence of the theory of relativity is that the kinetic energy of a particle of mass m moving with a speed v is no longer given by $K = mv^2/2$. Instead, one must use the relativistic form of the kinetic energy:

Relativistic kinetic energy

$$K = mc^2\left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right) \qquad \textbf{(7.19)}$$

According to this expression, speeds greater than c are not allowed because, as v approaches c, K approaches ∞. This limitation is consistent with experimental observations on subatomic particles, which have shown that no particles travel at speeds greater than c. (In other words, c is the ultimate speed.) From this relativistic point of view, the work–kinetic energy theorem says that v can only approach c because it would take an infinite amount of work to attain the speed $v = c$.

All formulas in the theory of relativity must reduce to those in Newtonian mechanics at low particle speeds. It is instructive to show that this is the case for the kinetic energy relationship by analyzing Equation 7.19 when v is small compared with c. In this case, we expect K to reduce to the Newtonian expression. We can check this by using the binomial expansion (Appendix B.5) applied to the quantity $[1 - (v/c)^2]^{-1/2}$, with $v/c \ll 1$. If we let $x = (v/c)^2$, the expansion gives

$$\frac{1}{(1-x)^{1/2}} = 1 + \frac{x}{2} + \frac{3}{8}x^2 + \cdots$$

Making use of this expansion in Equation 7.19 gives

$$\begin{aligned} K &= mc^2\left(1 + \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots - 1\right) \\ &= \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots \\ &= \frac{1}{2}mv^2 \qquad \text{for} \qquad \frac{v}{c} \ll 1 \end{aligned}$$

Thus, we see that the relativistic kinetic energy expression does indeed reduce to the Newtonian expression for speeds that are small compared with c. We shall return to the subject of relativity in Chapter 39.

SUMMARY

The work done by a constant force **F** acting on a particle is defined as the product of the component of the force in the direction of the particle's displacement and the magnitude of the displacement. Given a force **F** that makes an angle θ with the displacement vector **d** of a particle acted on by the force, you should be able to determine the work done by **F** using the equation

$$W \equiv Fd\cos\theta \qquad \textbf{(7.1)}$$

The **scalar product** (dot product) of two vectors **A** and **B** is defined by the relationship

$$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta \qquad \textbf{(7.3)}$$

where the result is a scalar quantity and θ is the angle between the two vectors. The scalar product obeys the commutative and distributive laws.

If a varying force does work on a particle as the particle moves along the x axis from x_i to x_f, you must use the expression

$$W \equiv \int_{x_i}^{x_f} F_x\,dx \qquad \textbf{(7.7)}$$

where F_x is the component of force in the x direction. If several forces are acting on the particle, the net work done by all of the forces is the sum of the amounts of work done by all of the forces.

The **kinetic energy** of a particle of mass m moving with a speed v (where v is small compared with the speed of light) is

$$K \equiv \frac{1}{2}mv^2 \tag{7.14}$$

The **work–kinetic energy theorem** states that the net work done on a particle by external forces equals the change in kinetic energy of the particle:

$$\sum W = K_f - K_i = \frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_i^{\,2} \tag{7.16}$$

If a frictional force acts, then the work–kinetic energy theorem can be modified to give

$$K_i + \sum W_{\text{other}} - f_k d = K_f \tag{7.17b}$$

The **instantaneous power** $\mathcal{P}$ is defined as the time rate of energy transfer. If an agent applies a force $\mathbf{F}$ to an object moving with a velocity $\mathbf{v}$, the power delivered by that agent is

$$\mathcal{P} \equiv \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{7.18}$$

QUESTIONS

1. Consider a tug-of-war in which two teams pulling on a rope are evenly matched so that no motion takes place. Assume that the rope does not stretch. Is work done on the rope? On the pullers? On the ground? Is work done on anything?
2. For what values of θ is the scalar product (a) positive and (b) negative?
3. As the load on a spring hung vertically is increased, one would not expect the F_s-versus-x curve to always remain linear, as shown in Figure 7.10d. Explain qualitatively what you would expect for this curve as m is increased.
4. Can the kinetic energy of an object be negative? Explain.
5. (a) If the speed of a particle is doubled, what happens to its kinetic energy? (b) If the net work done on a particle is zero, what can be said about the speed?
6. In Example 7.16, does the required power increase or decrease as the force of friction is reduced?
7. An automobile sales representative claims that a "souped-up" 300-hp engine is a necessary option in a compact car (instead of a conventional 130-hp engine). Suppose you intend to drive the car within speed limits ($\leq$ 55 mi/h) and on flat terrain. How would you counter this sales pitch?
8. One bullet has twice the mass of another bullet. If both bullets are fired so that they have the same speed, which has the greater kinetic energy? What is the ratio of the kinetic energies of the two bullets?
9. When a punter kicks a football, is he doing any work on the ball while his toe is in contact with it? Is he doing any work on the ball after it loses contact with his toe? Are any forces doing work on the ball while it is in flight?
10. Discuss the work done by a pitcher throwing a baseball. What is the approximate distance through which the force acts as the ball is thrown?
11. Two sharpshooters fire 0.30-caliber rifles using identical shells. The barrel of rifle A is 2.00 cm longer than that of rifle B. Which rifle will have the higher muzzle speed? (*Hint:* The force of the expanding gases in the barrel accelerates the bullets.)
12. As a simple pendulum swings back and forth, the forces acting on the suspended mass are the force of gravity, the tension in the supporting cord, and air resistance. (a) Which of these forces, if any, does no work on the pendulum? (b) Which of these forces does negative work at all times during its motion? (c) Describe the work done by the force of gravity while the pendulum is swinging.
13. The kinetic energy of an object depends on the frame of reference in which its motion is measured. Give an example to illustrate this point.
14. An older model car accelerates from 0 to a speed v in 10 s. A newer, more powerful sports car accelerates from 0 to $2v$ in the same time period. What is the ratio of powers expended by the two cars? Consider the energy coming from the engines to appear only as kinetic energy of the cars.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 7.1 Work Done by a Constant Force

1. A tugboat exerts a constant force of 5 000 N on a ship moving at constant speed through a harbor. How much work does the tugboat do on the ship in a distance of 3.00 km?

2. A shopper in a supermarket pushes a cart with a force of 35.0 N directed at an angle of 25.0° downward from the horizontal. Find the work done by the shopper as she moves down an aisle 50.0 m in length.

3. A raindrop ($m = 3.35 \times 10^{-5}$ kg) falls vertically at constant speed under the influence of gravity and air resistance. After the drop has fallen 100 m, what is the work done (a) by gravity and (b) by air resistance?

4. A sledge loaded with bricks has a total mass of 18.0 kg and is pulled at constant speed by a rope. The rope is inclined at 20.0° above the horizontal, and the sledge moves a distance of 20.0 m on a horizontal surface. The coefficient of kinetic friction between the sledge and the surface is 0.500. (a) What is the tension of the rope? (b) How much work is done on the sledge by the rope? (c) What is the energy lost due to friction?

5. A block of mass 2.50 kg is pushed 2.20 m along a frictionless horizontal table by a constant 16.0-N force directed 25.0° below the horizontal. Determine the work done by (a) the applied force, (b) the normal force exerted by the table, and (c) the force of gravity. (d) Determine the total work done on the block.

6. A 15.0-kg block is dragged over a rough, horizontal surface by a 70.0-N force acting at 20.0° above the horizontal. The block is displaced 5.00 m, and the coefficient of kinetic friction is 0.300. Find the work done by (a) the 70-N force, (b) the normal force, and (c) the force of gravity. (d) What is the energy loss due to friction? (e) Find the total change in the block's kinetic energy.

WEB **7.** Batman, whose mass is 80.0 kg, is holding onto the free end of a 12.0-m rope, the other end of which is fixed to a tree limb above. He is able to get the rope in motion as only Batman knows how, eventually getting it to swing enough so that he can reach a ledge when the rope makes a 60.0° angle with the vertical. How much work was done against the force of gravity in this maneuver?

Section 7.2 The Scalar Product of Two Vectors

In Problems 8 to 14, calculate all numerical answers to three significant figures.

8. Vector **A** has a magnitude of 5.00 units, and vector **B** has a magnitude of 9.00 units. The two vectors make an angle of 50.0° with each other. Find **A · B.**

9. Vector **A** extends from the origin to a point having polar coordinates (7, 70°), and vector **B** extends from the origin to a point having polar coordinates (4, 130°). Find **A · B.**

10. Given two arbitrary vectors **A** and **B**, show that **A·B** = $A_xB_x + A_yB_y + A_zB_z$. (*Hint:* Write **A** and **B** in unit vector form and use Equations 7.4 and 7.5.)

WEB **11.** A force **F** = (6**i** − 2**j**) N acts on a particle that undergoes a displacement **d** = (3**i** + **j**)m. Find (a) the work done by the force on the particle and (b) the angle between **F** and **d**.

12. For **A** = 3**i** + **j** − **k**, **B** = −**i** + 2**j** + 5**k**, and **C** = 2**j** − 3**k**, find **C·**(**A** − **B**).

13. Using the definition of the scalar product, find the angles between (a) **A** = 3**i** − 2**j** and **B** = 4**i** − 4**j**; (b) **A** = −2**i** + 4**j** and **B** = 3**i** − 4**j** + 2**k**; (c) **A** = **i** − 2**j** + 2**k** and **B** = 3**j** + 4**k**.

14. Find the scalar product of the vectors in Figure P7.14.

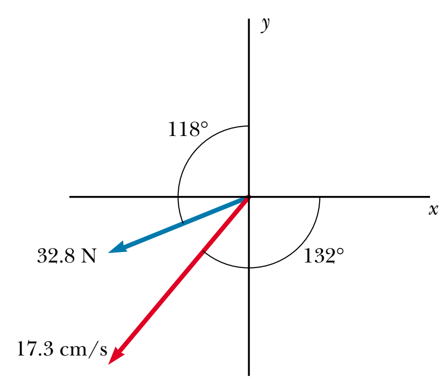


Figure P7.14

Section 7.3 Work Done by a Varying Force

15. The force acting on a particle varies as shown in Figure P7.15. Find the work done by the force as the particle moves (a) from $x = 0$ to $x = 8.00$ m, (b) from $x = 8.00$ m to $x = 10.0$ m, and (c) from $x = 0$ to $x = 10.0$ m.

16. The force acting on a particle is $F_x = (8x - 16)$ N, where x is in meters. (a) Make a plot of this force versus x from $x = 0$ to $x = 3.00$ m. (b) From your graph, find the net work done by this force as the particle moves from $x = 0$ to $x = 3.00$ m.

WEB **17.** A particle is subject to a force F_x that varies with position as in Figure P7.17. Find the work done by the force on the body as it moves (a) from $x = 0$ to $x = 5.00$ m,

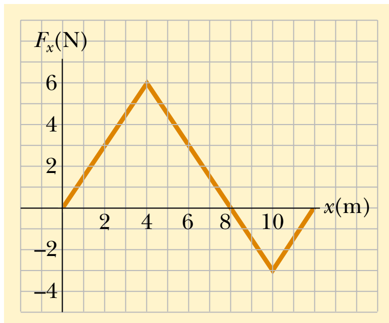


Figure P7.15

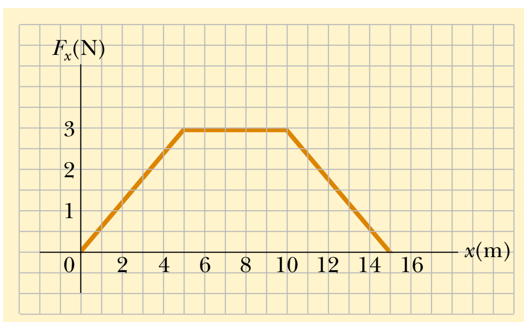


Figure P7.17 Problems 17 and 32.

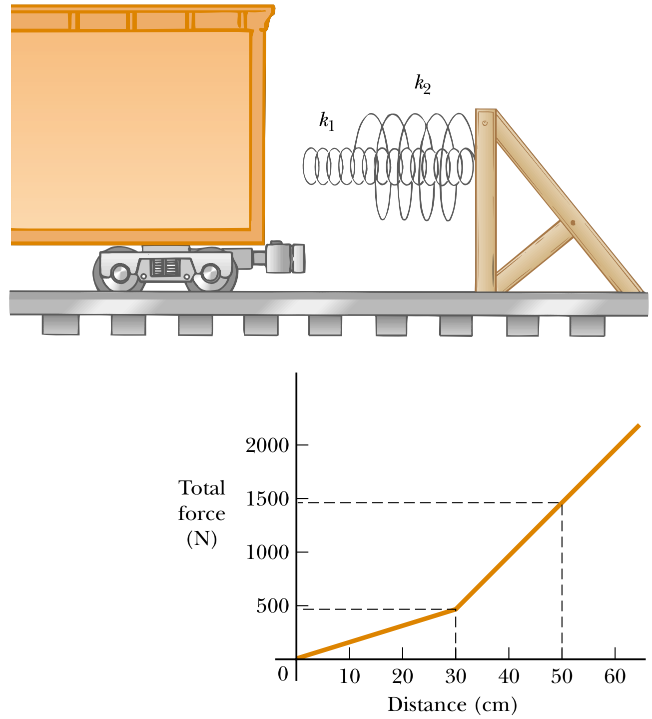


Figure P7.21

(b) from $x = 5.00$ m to $x = 10.0$ m, and (c) from $x = 10.0$ m to $x = 15.0$ m. (d) What is the total work done by the force over the distance $x = 0$ to $x = 15.0$ m?

18. A force $\mathbf{F} = (4x\mathbf{i} + 3y\mathbf{j})$ N acts on an object as it moves in the x direction from the origin to $x = 5.00$ m. Find the work $W = \int \mathbf{F} \cdot d\mathbf{r}$ done on the object by the force.

19. When a 4.00-kg mass is hung vertically on a certain light spring that obeys Hooke's law, the spring stretches 2.50 cm. If the 4.00-kg mass is removed, (a) how far will the spring stretch if a 1.50-kg mass is hung on it and (b) how much work must an external agent do to stretch the same spring 4.00 cm from its unstretched position?

20. An archer pulls her bow string back 0.400 m by exerting a force that increases uniformly from zero to 230 N. (a) What is the equivalent spring constant of the bow? (b) How much work is done by the archer in pulling the bow?

21. A 6 000-kg freight car rolls along rails with negligible friction. The car is brought to rest by a combination of two coiled springs, as illustrated in Figure P7.21. Both springs obey Hooke's law with $k_1 = 1\,600$ N/m and $k_2 = 3\,400$ N/m. After the first spring compresses a distance of 30.0 cm, the second spring (acting with the first) increases the force so that additional compression occurs, as shown in the graph. If the car is brought to rest 50.0 cm after first contacting the two-spring system, find the car's initial speed.

22. A 100-g bullet is fired from a rifle having a barrel 0.600 m long. Assuming the origin is placed where the bullet begins to move, the force (in newtons) exerted on the bullet by the expanding gas is $15\,000 + 10\,000x - 25\,000x^2$, where x is in meters. (a) Determine the work done by the gas on the bullet as the bullet travels the length of the barrel. (b) If the barrel is 1.00 m long, how much work is done and how does this value compare with the work calculated in part (a)?

23. If it takes 4.00 J of work to stretch a Hooke's-law spring 10.0 cm from its unstressed length, determine the extra work required to stretch it an additional 10.0 cm.

24. If it takes work W to stretch a Hooke's-law spring a distance d from its unstressed length, determine the extra work required to stretch it an additional distance d.

25. A small mass m is pulled to the top of a frictionless half-cylinder (of radius R) by a cord that passes over the top of the cylinder, as illustrated in Figure P7.25. (a) If the mass moves at a constant speed, show that $F = mg \cos\theta$. (*Hint:* If the mass moves at a constant speed, the component of its acceleration tangent to the cylinder must be zero at all times.) (b) By directly integrating $W = \int \mathbf{F} \cdot d\mathbf{s}$, find the work done in moving the mass at constant speed from the bottom to the top of the half-

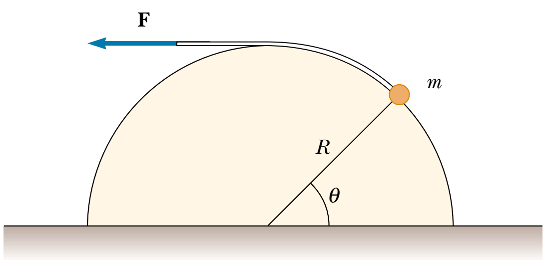


Figure P7.25

cylinder. Here $d\mathbf{s}$ represents an incremental displacement of the small mass.

26. Express the unit of the force constant of a spring in terms of the basic units meter, kilogram, and second.

Section 7.4 Kinetic Energy and the Work–Kinetic Energy Theorem

27. A 0.600-kg particle has a speed of 2.00 m/s at point A and kinetic energy of 7.50 J at point B. What is (a) its kinetic energy at A? (b) its speed at B? (c) the total work done on the particle as it moves from A to B?

28. A 0.300-kg ball has a speed of 15.0 m/s. (a) What is its kinetic energy? (b) If its speed were doubled, what would be its kinetic energy?

29. A 3.00-kg mass has an initial velocity $\mathbf{v}_i = (6.00\mathbf{i} - 2.00\mathbf{j})$ m/s. (a) What is its kinetic energy at this time? (b) Find the total work done on the object if its velocity changes to $(8.00\mathbf{i} + 4.00\mathbf{j})$ m/s. (*Hint:* Remember that $v^2 = \mathbf{v} \cdot \mathbf{v}$.)

30. A mechanic pushes a 2 500-kg car, moving it from rest and making it accelerate from rest to a speed v. He does 5 000 J of work in the process. During this time, the car moves 25.0 m. If friction between the car and the road is negligible, (a) what is the final speed v of the car? (b) What constant horizontal force did he exert on the car?

31. A mechanic pushes a car of mass m, doing work W in making it accelerate from rest. If friction between the car and the road is negligible, (a) what is the final speed of the car? During the time the mechanic pushes the car, the car moves a distance d. (b) What constant horizontal force did the mechanic exert on the car?

32. A 4.00-kg particle is subject to a total force that varies with position, as shown in Figure P7.17. The particle starts from rest at $x = 0$. What is its speed at (a) $x = 5.00$ m, (b) $x = 10.0$ m, (c) $x = 15.0$ m?

33. A 40.0-kg box initially at rest is pushed 5.00 m along a rough, horizontal floor with a constant applied horizontal force of 130 N. If the coefficient of friction between the box and the floor is 0.300, find (a) the work done by the applied force, (b) the energy loss due to friction, (c) the work done by the normal force, (d) the work done by gravity, (e) the change in kinetic energy of the box, and (f) the final speed of the box.

34. You can think of the work–kinetic energy theorem as a second theory of motion, parallel to Newton's laws in describing how outside influences affect the motion of an object. In this problem, work out parts (a) and (b) separately from parts (c) and (d) to compare the predictions of the two theories. In a rifle barrel, a 15.0-g bullet is accelerated from rest to a speed of 780 m/s. (a) Find the work that is done on the bullet. (b) If the rifle barrel is 72.0 cm long, find the magnitude of the average total force that acted on it, as $F = W/(d\cos\theta)$. (c) Find the constant acceleration of a bullet that starts from rest and gains a speed of 780 m/s over a distance of 72.0 cm. (d) Find the total force that acted on it as $\Sigma F = ma$.

35. A crate of mass 10.0 kg is pulled up a rough incline with an initial speed of 1.50 m/s. The pulling force is 100 N parallel to the incline, which makes an angle of 20.0° with the horizontal. The coefficient of kinetic friction is 0.400, and the crate is pulled 5.00 m. (a) How much work is done by gravity? (b) How much energy is lost because of friction? (c) How much work is done by the 100-N force? (d) What is the change in kinetic energy of the crate? (e) What is the speed of the crate after it has been pulled 5.00 m?

36. A block of mass 12.0 kg slides from rest down a frictionless 35.0° incline and is stopped by a strong spring with $k = 3.00 \times 10^4$ N/m. The block slides 3.00 m from the point of release to the point where it comes to rest against the spring. When the block comes to rest, how far has the spring been compressed?

WEB 37. A sled of mass m is given a kick on a frozen pond. The kick imparts to it an initial speed $v_i = 2.00$ m/s. The coefficient of kinetic friction between the sled and the ice is $\mu_k = 0.100$. Utilizing energy considerations, find the distance the sled moves before it stops.

38. A picture tube in a certain television set is 36.0 cm long. The electrical force accelerates an electron in the tube from rest to 1.00% of the speed of light over this distance. Determine (a) the kinetic energy of the electron as it strikes the screen at the end of the tube, (b) the magnitude of the average electrical force acting on the electron over this distance, (c) the magnitude of the average acceleration of the electron over this distance, and (d) the time of flight.

39. A bullet with a mass of 5.00 g and a speed of 600 m/s penetrates a tree to a depth of 4.00 cm. (a) Use work and energy considerations to find the average frictional force that stops the bullet. (b) Assuming that the frictional force is constant, determine how much time elapsed between the moment the bullet entered the tree and the moment it stopped.

40. An Atwood's machine (see Fig. 5.15) supports masses of 0.200 kg and 0.300 kg. The masses are held at rest beside each other and then released. Neglecting friction, what is the speed of each mass the instant it has moved 0.400 m?

41. A 2.00-kg block is attached to a spring of force constant 500 N/m, as shown in Figure 7.10. The block is pulled 5.00 cm to the right of equilibrium and is then released from rest. Find the speed of the block as it passes through equilibrium if (a) the horizontal surface is frictionless and (b) the coefficient of friction between the block and the surface is 0.350.

Section 7.5 Power

42. Make an order-of-magnitude estimate of the power a car engine contributes to speeding up the car to highway speed. For concreteness, consider your own car (if you use one). In your solution, state the physical quantities you take as data and the values you measure or estimate for them. The mass of the vehicle is given in the owner's manual. If you do not wish to consider a car, think about a bus or truck for which you specify the necessary physical quantities.

WEB **43.** A 700-N Marine in basic training climbs a 10.0-m vertical rope at a constant speed in 8.00 s. What is his power output?

44. If a certain horse can maintain 1.00 hp of output for 2.00 h, how many 70.0-kg bundles of shingles can the horse hoist (using some pulley arrangement) to the roof of a house 8.00 m tall, assuming 70.0% efficiency?

45. A certain automobile engine delivers 2.24×10^4 W (30.0 hp) to its wheels when moving at a constant speed of 27.0 m/s ($\approx$ 60 mi/h). What is the resistive force acting on the automobile at that speed?

46. A skier of mass 70.0 kg is pulled up a slope by a motor-driven cable. (a) How much work is required for him to be pulled a distance of 60.0 m up a 30.0° slope (assumed frictionless) at a constant speed of 2.00 m/s? (b) A motor of what power is required to perform this task?

47. A 650-kg elevator starts from rest. It moves upward for 3.00 s with constant acceleration until it reaches its cruising speed of 1.75 m/s. (a) What is the average power of the elevator motor during this period? (b) How does this power compare with its power when it moves at its cruising speed?

48. An energy-efficient lightbulb, taking in 28.0 W of power, can produce the same level of brightness as a conventional bulb operating at 100-W power. The lifetime of the energy-efficient bulb is 10 000 h and its purchase price is \$17.0, whereas the conventional bulb has a lifetime of 750 h and costs \$0.420 per bulb. Determine the total savings obtained through the use of one energy-efficient bulb over its lifetime as opposed to the use of conventional bulbs over the same time period. Assume an energy cost of \$0.080 0 per kilowatt hour.

(Optional)
Section 7.6 Energy and the Automobile

49. A compact car of mass 900 kg has an overall motor efficiency of 15.0%. (That is, 15.0% of the energy supplied by the fuel is delivered to the wheels of the car.) (a) If burning 1 gal of gasoline supplies 1.34×10^8 J of energy, find the amount of gasoline used by the car in accelerating from rest to 55.0 mi/h. Here you may ignore the effects of air resistance and rolling resistance. (b) How many such accelerations will 1 gal provide? (c) The mileage claimed for the car is 38.0 mi/gal at 55 mi/h. What power is delivered to the wheels (to overcome frictional effects) when the car is driven at this speed?

50. Suppose the empty car described in Table 7.2 has a fuel economy of 6.40 km/L (15 mi/gal) when traveling at 26.8 m/s (60 mi/h). Assuming constant efficiency, determine the fuel economy of the car if the total mass of the passengers and the driver is 350 kg.

51. When an air conditioner is added to the car described in Problem 50, the additional output power required to operate the air conditioner is 1.54 kW. If the fuel economy of the car is 6.40 km/L without the air conditioner, what is it when the air conditioner is operating?

(Optional)
Section 7.7 Kinetic Energy at High Speeds

52. An electron moves with a speed of $0.995c$. (a) What is its kinetic energy? (b) If you use the classical expression to calculate its kinetic energy, what percentage error results?

53. A proton in a high-energy accelerator moves with a speed of $c/2$. Using the work–kinetic energy theorem, find the work required to increase its speed to (a) $0.750c$ and (b) $0.995c$.

54. Find the kinetic energy of a 78.0-kg spacecraft launched out of the Solar System with a speed of 106 km/s using (a) the classical equation $K = \frac{1}{2}mv^2$ and (b) the relativistic equation.

ADDITIONAL PROBLEMS

55. A baseball outfielder throws a 0.150-kg baseball at a speed of 40.0 m/s and an initial angle of 30.0°. What is the kinetic energy of the baseball at the highest point of the trajectory?

56. While running, a person dissipates about 0.600 J of mechanical energy per step per kilogram of body mass. If a 60.0-kg runner dissipates a power of 70.0 W during a race, how fast is the person running? Assume a running step is 1.50 m in length.

57. A particle of mass m moves with a constant acceleration $\mathbf{a}$. If the initial position vector and velocity of the particle are $\mathbf{r}_i$ and $\mathbf{v}_i$, respectively, use energy arguments to show that its speed v_f at any time satisfies the equation

$$v_f^{\,2} = v_i^{\,2} + 2\mathbf{a} \cdot (\mathbf{r}_f - \mathbf{r}_i)$$

where $\mathbf{r}_f$ is the position vector of the particle at that same time.

58. The direction of an arbitrary vector $\mathbf{A}$ can be completely specified with the angles α, β, and γ that the vector makes with the x, y, and z axes, respectively. If $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, (a) find expressions for $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ (known as *direction cosines*) and (b) show that these angles satisfy the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. (*Hint:* Take the scalar product of **A** with **i**, **j**, and **k** separately.)

59. A 4.00-kg particle moves along the x axis. Its position varies with time according to $x = t + 2.0t^3$, where x is in meters and t is in seconds. Find (a) the kinetic energy at any time t, (b) the acceleration of the particle and the force acting on it at time t, (c) the power being delivered to the particle at time t, and (d) the work done on the particle in the interval $t = 0$ to $t = 2.00$ s.

60. A traveler at an airport takes an escalator up one floor (Fig. P7.60). The moving staircase would itself carry him upward with vertical velocity component v between entry and exit points separated by height h. However, while the escalator is moving, the hurried traveler climbs the steps of the escalator at a rate of n steps/s. Assume that the height of each step is h_s. (a) Determine the amount of work done by the traveler during his escalator ride, given that his mass is m. (b) Determine the work the escalator motor does on this person.

Figure P7.60 (©Ron Chapple/FPG)

61. When a certain spring is stretched beyond its proportional limit, the restoring force satisfies the equation $F = -kx + \beta x^3$. If $k = 10.0$ N/m and $\beta = 100$ N/m^3, calculate the work done by this force when the spring is stretched 0.100 m.

62. In a control system, an accelerometer consists of a 4.70-g mass sliding on a low-friction horizontal rail. A low-mass spring attaches the mass to a flange at one end of the rail. When subject to a steady acceleration of 0.800g, the mass is to assume a location 0.500 cm away from its equilibrium position. Find the stiffness constant required for the spring.

63. A 2 100-kg pile driver is used to drive a steel I-beam into the ground. The pile driver falls 5.00 m before coming into contact with the beam, and it drives the beam 12.0 cm into the ground before coming to rest. Using energy considerations, calculate the average force the beam exerts on the pile driver while the pile driver is brought to rest.

64. A cyclist and her bicycle have a combined mass of 75.0 kg. She coasts down a road inclined at 2.00° with the horizontal at 4.00 m/s and down a road inclined at 4.00° at 8.00 m/s. She then holds on to a moving vehicle and coasts on a level road. What power must the vehicle expend to maintain her speed at 3.00 m/s? Assume that the force of air resistance is proportional to her speed and that other frictional forces remain constant. (*Warning:* You must *not* attempt this dangerous maneuver.)

65. A single constant force **F** acts on a particle of mass m. The particle starts at rest at $t = 0$. (a) Show that the instantaneous power delivered by the force at any time t is $(F^2/m)t$. (b) If $F = 20.0$ N and $m = 5.00$ kg, what is the power delivered at $t = 3.00$ s?

66. A particle is attached between two identical springs on a horizontal frictionless table. Both springs have spring constant k and are initially unstressed. (a) If the particle is pulled a distance x along a direction perpendicular to the initial configuration of the springs, as in Figure P7.66, show that the force exerted on the particle by the springs is

$$\mathbf{F} = -2kx\left(1 - \frac{L}{\sqrt{x^2 + L^2}}\right)\mathbf{i}$$

(b) Determine the amount of work done by this force in moving the particle from $x = A$ to $x = 0$.

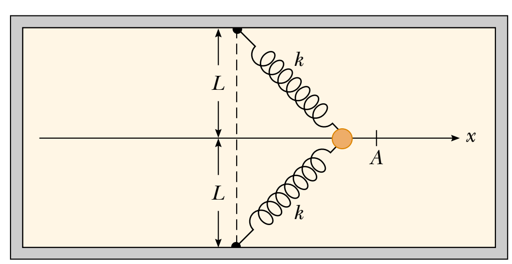


Figure P7.66

67. Review Problem. Two constant forces act on a 5.00-kg object moving in the *xy* plane, as shown in Figure P7.67. Force $\mathbf{F}_1$ is 25.0 N at 35.0°, while $\mathbf{F}_2 = 42.0$ N at 150°. At time $t = 0$, the object is at the origin and has velocity $(4.0\mathbf{i} + 2.5\mathbf{j})$ m/s. (a) Express the two forces in unit–vector notation. Use unit–vector notation for your other answers. (b) Find the total force on the object. (c) Find the object's acceleration. Now, considering the instant $t = 3.00$ s, (d) find the object's velocity, (e) its location, (f) its kinetic energy from $\frac{1}{2}mv_f^2$, and (g) its kinetic energy from $\frac{1}{2}mv_i^2 + \Sigma\mathbf{F}\cdot\mathbf{d}$.

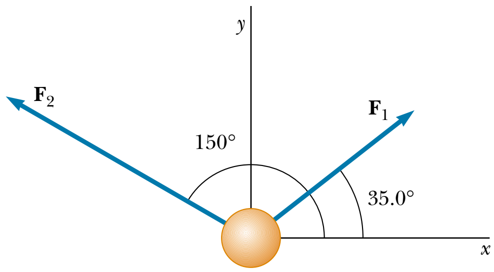


Figure P7.67

68. When different weights are hung on a spring, the spring stretches to different lengths as shown in the following table. (a) Make a graph of the applied force versus the extension of the spring. By least-squares fitting, determine the straight line that best fits the data. (You may not want to use all the data points.) (b) From the slope of the best-fit line, find the spring constant k. (c) If the spring is extended to 105 mm, what force does it exert on the suspended weight?

F (N)	2.0	4.0	6.0	8.0	10	12	14	16	18
L (mm)	15	32	49	64	79	98	112	126	149

69. A 200-g block is pressed against a spring of force constant 1.40 kN/m until the block compresses the spring 10.0 cm. The spring rests at the bottom of a ramp inclined at 60.0° to the horizontal. Using energy considerations, determine how far up the incline the block moves before it stops (a) if there is no friction between the block and the ramp and (b) if the coefficient of kinetic friction is 0.400.

70. A 0.400-kg particle slides around a horizontal track. The track has a smooth, vertical outer wall forming a circle with a radius of 1.50 m. The particle is given an initial speed of 8.00 m/s. After one revolution, its speed has dropped to 6.00 m/s because of friction with the rough floor of the track. (a) Find the energy loss due to friction in one revolution. (b) Calculate the coefficient of kinetic friction. (c) What is the total number of revolutions the particle makes before stopping?

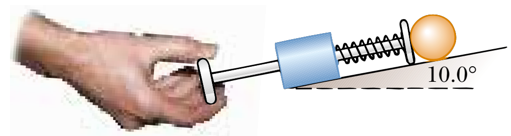


Figure P7.71

WEB **71.** The ball launcher in a pinball machine has a spring that has a force constant of 1.20 N/cm (Fig. P7.71). The surface on which the ball moves is inclined 10.0° with respect to the horizontal. If the spring is initially compressed 5.00 cm, find the launching speed of a 100-g ball when the plunger is released. Friction and the mass of the plunger are negligible.

72. In diatomic molecules, the constituent atoms exert attractive forces on each other at great distances and repulsive forces at short distances. For many molecules, the Lennard–Jones law is a good approximation to the magnitude of these forces:

$$F = F_0\left[2\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right]$$

where r is the center-to-center distance between the atoms in the molecule, σ is a length parameter, and F_0 is the force when $r = \sigma$. For an oxygen molecule, $F_0 = 9.60 \times 10^{-11}$ N and $\sigma = 3.50 \times 10^{-10}$ m. Determine the work done by this force if the atoms are pulled apart from $r = 4.00 \times 10^{-10}$ m to $r = 9.00 \times 10^{-10}$ m.

73. A horizontal string is attached to a 0.250-kg mass lying on a rough, horizontal table. The string passes over a light, frictionless pulley, and a 0.400-kg mass is then attached to its free end. The coefficient of sliding friction between the 0.250-kg mass and the table is 0.200. Using the work–kinetic energy theorem, determine (a) the speed of the masses after each has moved 20.0 m from rest and (b) the mass that must be added to the 0.250-kg mass so that, given an initial velocity, the masses continue to move at a constant speed. (c) What mass must be removed from the 0.400-kg mass so that the same outcome as in part (b) is achieved?

74. Suppose a car is modeled as a cylinder moving with a speed v, as in Figure P7.74. In a time Δt, a column of air

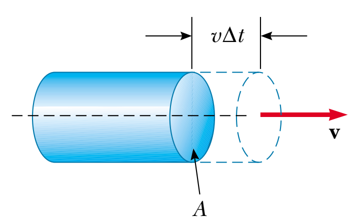


Figure P7.74

of mass Δm must be moved a distance $v\,\Delta t$ and, hence, must be given a kinetic energy $\frac{1}{2}(\Delta m)v^2$. Using this model, show that the power loss due to air resistance is $\frac{1}{2}\rho A v^3$ and that the resistive force is $\frac{1}{2}\rho A v^2$, where ρ is the density of air.

75. A particle moves along the x axis from $x = 12.8$ m to $x = 23.7$ m under the influence of a force

$$F = \frac{375}{x^3 + 3.75\,x}$$

where F is in newtons and x is in meters. Using numerical integration, determine the total work done by this force during this displacement. Your result should be accurate to within 2%.

76. More than 2 300 years ago the Greek teacher Aristotle wrote the first book called *Physics*. The following passage, rephrased with more precise terminology, is from the end of the book's Section Eta:

> Let $\mathcal{P}$ be the power of an agent causing motion; w, the thing moved; d, the distance covered; and t, the time taken. Then (1) a power equal to $\mathcal{P}$ will in a period of time equal to t move $w/2$ a distance $2d$; or (2) it will move $w/2$ the given distance d in time $t/2$. Also, if (3) the given power $\mathcal{P}$ moves the given object w a distance $d/2$ in time $t/2$, then (4) $\mathcal{P}/2$ will move $w/2$ the given distance d in the given time t.

(a) Show that Aristotle's proportions are included in the equation $\mathcal{P}t = bwd$, where b is a proportionality constant. (b) Show that our theory of motion includes this part of Aristotle's theory as one special case. In particular, describe a situation in which it is true, derive the equation representing Aristotle's proportions, and determine the proportionality constant.

ANSWERS TO QUICK QUIZZES

7.1 No. The force does no work on the object because the force is pointed toward the center of the circle and is therefore perpendicular to the motion.

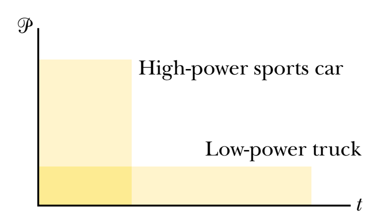
7.2 (a) Assuming the person lifts with a force of magnitude mg, the weight of the box, the work he does during the vertical displacement is mgh because the force is in the direction of the displacement. The work he does during the horizontal displacement is zero because now the force he exerts on the box is perpendicular to the displacement. The net work he does is $mgh + 0 = mgh$.
(b) The work done by the gravitational force on the box as the box is displaced vertically is $-mgh$ because the direction of this force is opposite the direction of the displacement. The work done by the gravitational force is zero during the horizontal displacement because now the direction of this force is perpendicular to the direction of the displacement. The net work done by the gravitational force $-mgh + 0 = -mgh$. The total work done on the box is $+mgh - mgh = 0$.

7.3 No. For example, consider the two vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} - \mathbf{j}$. Their dot product is $\mathbf{A}\cdot\mathbf{B} = 8$, yet both vectors have negative y components.

7.4 Force divided by displacement, which in SI units is newtons per meter (N/m).

7.5 Yes, whenever the frictional force has a component along the direction of motion. Consider a crate sitting on the bed of a truck as the truck accelerates to the east. The static friction force exerted on the crate by the truck acts to the east to give the crate the same acceleration as the truck (assuming that the crate does not slip). Because the crate accelerates, its kinetic energy must increase.

7.6 Because the two vehicles perform the same amount of work, the areas under the two graphs are equal. However, the graph for the low-power truck extends over a longer time interval and does not extend as high on the $\mathcal{P}$ axis as the graph for the sports car does.



PUZZLER

A common scene at a carnival is the Ring-the-Bell attraction, in which the player swings a heavy hammer downward in an attempt to project a mass upward to ring a bell. What is the best strategy to win the game and impress your friends? *(Robert E. Daemmrich/Tony Stone Images)*

chapter

8 Potential Energy and Conservation of Energy

Chapter Outline

In Chapter 7 we introduced the concept of kinetic energy, which is the energy associated with the motion of an object. In this chapter we introduce another form of energy—*potential energy,* which is the energy associated with the arrangement of a system of objects that exert forces on each other. Potential energy can be thought of as stored energy that can either do work or be converted to kinetic energy.

The potential energy concept can be used only when dealing with a special class of forces called *conservative forces.* When only conservative forces act within an isolated system, the kinetic energy gained (or lost) by the system as its members change their relative positions is balanced by an equal loss (or gain) in potential energy. This balancing of the two forms of energy is known as the *principle of conservation of mechanical energy.*

Energy is present in the Universe in various forms, including mechanical, electromagnetic, chemical, and nuclear. Furthermore, one form of energy can be converted to another. For example, when an electric motor is connected to a battery, the chemical energy in the battery is converted to electrical energy in the motor, which in turn is converted to mechanical energy as the motor turns some device. The transformation of energy from one form to another is an essential part of the study of physics, engineering, chemistry, biology, geology, and astronomy.

When energy is changed from one form to another, the total amount present does not change. Conservation of energy means that although the form of energy may change, if an object (or system) loses energy, that same amount of energy appears in another object or in the object's surroundings.

8.1 POTENTIAL ENERGY

An object that possesses kinetic energy can do work on another object—for example,
5.3 a moving hammer driving a nail into a wall. Now we shall introduce another form of energy. This energy, called **potential energy *U*,** is the energy associated with a system of objects.

Before we describe specific forms of potential energy, we must first define a *system,* which consists of two or more objects that exert forces on one another. **If the arrangement of the system changes, then the potential energy of the system changes.** If the system consists of only two particle-like objects that exert forces on each other, then the work done by the force acting on one of the objects causes a transformation of energy between the object's kinetic energy and other forms of the system's energy.

Gravitational Potential Energy

As an object falls toward the Earth, the Earth exerts a gravitational force $m\mathbf{g}$ on the object, with the direction of the force being the same as the direction of the object's motion. The gravitational force does work on the object and thereby increases the object's kinetic energy. Imagine that a brick is dropped from rest directly above a nail in a board lying on the ground. When the brick is released, it falls toward the ground, gaining speed and therefore gaining kinetic energy. The brick–Earth system has potential energy when the brick is at any distance above the ground (that is, it has the *potential* to do work), and this potential energy is converted to kinetic energy as the brick falls. The conversion from potential energy to kinetic energy occurs continuously over the entire fall. When the brick reaches the nail and the board lying on the ground, it does work on the nail, driving it into the board. What determines how much work the brick is able to do on the nail? It is easy to see that the heavier the brick, the farther in it drives the nail; also the higher the brick is before it is released, the more work it does when it strikes the nail.

The product of the magnitude of the gravitational force mg acting on an object and the height y of the object is so important in physics that we give it a name: the **gravitational potential energy.** The symbol for gravitational potential energy is U_g, and so the defining equation for gravitational potential energy is

Gravitational potential energy

$$U_g \equiv mgy \tag{8.1}$$

Gravitational potential energy is the potential energy of the object–Earth system. This potential energy is transformed into kinetic energy of the system by the gravitational force. In this type of system, in which one of the members (the Earth) is much more massive than the other (the object), the massive object can be modeled as stationary, and the kinetic energy of the system can be represented entirely by the kinetic energy of the lighter object. Thus, the kinetic energy of the system is represented by that of the object falling toward the Earth. Also note that Equation 8.1 is valid only for objects near the surface of the Earth, where $\mathbf{g}$ is approximately constant.[1]

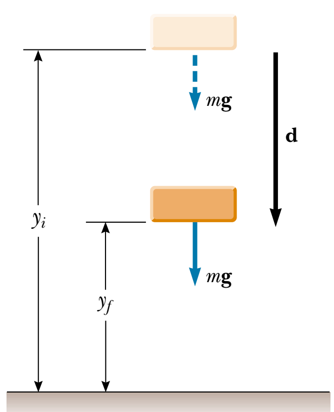


Figure 8.1 The work done on the brick by the gravitational force as the brick falls from a height y_i to a height y_f is equal to $mgy_i - mgy_f$.

Let us now directly relate the work done on an object by the gravitational force to the gravitational potential energy of the object–Earth system. To do this, let us consider a brick of mass m at an initial height y_i above the ground, as shown in Figure 8.1. If we neglect air resistance, then the only force that does work on the brick as it falls is the gravitational force exerted on the brick $m\mathbf{g}$. The work W_g done by the gravitational force as the brick undergoes a downward displacement $\mathbf{d}$ is

$$W_g = (m\mathbf{g}) \cdot \mathbf{d} = (-mg\mathbf{j}) \cdot (y_f - y_i)\,\mathbf{j} = mgy_i - mgy_f$$

where we have used the fact that $\mathbf{j} \cdot \mathbf{j} = 1$ (Eq. 7.4). If an object undergoes both a horizontal and a vertical displacement, so that $\mathbf{d} = (x_f - x_i)\mathbf{i} + (y_f - y_i)\mathbf{j}$, then the work done by the gravitational force is still $mgy_i - mgy_f$ because $-mg\mathbf{j} \cdot (x_f - x_i)\mathbf{i} = 0$. Thus, the work done by the gravitational force depends only on the change in y and not on any change in the horizontal position x.

We just learned that the quantity mgy is the gravitational potential energy of the system U_g, and so we have

$$W_g = U_i - U_f = -(U_f - U_i) = -\Delta U_g \tag{8.2}$$

From this result, we see that the work done on any object by the gravitational force is equal to the negative of the change in the system's gravitational potential energy. Also, this result demonstrates that it is only the *difference* in the gravitational potential energy at the initial and final locations that matters. This means that we are free to place the origin of coordinates in any convenient location. Finally, the work done by the gravitational force on an object as the object falls to the Earth is the same as the work done were the object to start at the same point and slide down an incline to the Earth. Horizontal motion does not affect the value of W_g.

The unit of gravitational potential energy is the same as that of work—the joule. Potential energy, like work and kinetic energy, is a scalar quantity.

[1] The assumption that the force of gravity is constant is a good one as long as the vertical displacement is small compared with the Earth's radius.

Quick Quiz 8.1

Can the gravitational potential energy of a system ever be negative?

EXAMPLE 8.1 The Bowler and the Sore Toe

A bowling ball held by a careless bowler slips from the bowler's hands and drops on the bowler's toe. Choosing floor level as the $y = 0$ point of your coordinate system, estimate the total work done on the ball by the force of gravity as the ball falls. Repeat the calculation, using the top of the bowler's head as the origin of coordinates.

Solution First, we need to estimate a few values. A bowling ball has a mass of approximately 7 kg, and the top of a person's toe is about 0.03 m above the floor. Also, we shall assume the ball falls from a height of 0.5 m. Holding nonsignificant digits until we finish the problem, we calculate the gravitational potential energy of the ball–Earth system just before the ball is released to be $U_i = mgy_i = (7 \text{ kg})(9.80 \text{ m/s}^2)(0.5 \text{ m}) = 34.3 \text{ J}$. A similar calculation for when the ball reaches his toe gives $U_f = mgy_f = (7 \text{ kg})(9.80 \text{ m/s}^2)(0.03 \text{ m}) = 2.06 \text{ J}$. So, the work done by the gravitational force is $W_g = U_i - U_f = 32.24 \text{ J}$. We should probably keep only one digit because of the roughness of our estimates; thus, we estimate that the gravitational force does 30 J of work on the bowling ball as it falls. The system had 30 J of gravitational potential energy relative to the top of the toe before the ball began its fall.

When we use the bowler's head (which we estimate to be 1.50 m above the floor) as our origin of coordinates, we find that $U_i = mgy_i = (7 \text{ kg})(9.80 \text{ m/s}^2)(-1 \text{ m}) = -68.6 \text{ J}$ and that $U_f = mgy_f = (7 \text{ kg})(9.80 \text{ m/s}^2)(-1.47 \text{ m}) = -100.8 \text{ J}$. The work being done by the gravitational force is still $W_g = U_i - U_f = 32.24 \text{ J} \approx$ 30 J.

Elastic Potential Energy

Now consider a system consisting of a block plus a spring, as shown in Figure 8.2. The force that the spring exerts on the block is given by $F_s = -kx$. In the previous chapter, we learned that the work done by the spring force on a block connected to the spring is given by Equation 7.11:

$$W_s = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \tag{8.3}$$

In this situation, the initial and final x coordinates of the block are measured from its equilibrium position, $x = 0$. Again we see that W_s depends only on the initial and final x coordinates of the object and is zero for any closed path. The **elastic potential energy** function associated with the system is defined by

$$U_s \equiv \tfrac{1}{2}kx^2 \tag{8.4}$$

Elastic potential energy stored in a spring

The elastic potential energy of the system can be thought of as the energy stored in the deformed spring (one that is either compressed or stretched from its equilibrium position). To visualize this, consider Figure 8.2, which shows a spring on a frictionless, horizontal surface. When a block is pushed against the spring (Fig. 8.2b) and the spring is compressed a distance x, the elastic potential energy stored in the spring is $kx^2/2$. When the block is released from rest, the spring snaps back to its original length and the stored elastic potential energy is transformed into kinetic energy of the block (Fig. 8.2c). The elastic potential energy stored in the spring is zero whenever the spring is undeformed ($x = 0$). Energy is stored in the spring only when the spring is either stretched or compressed. Furthermore, the elastic potential energy is a maximum when the spring has reached its maximum compression or extension (that is, when $|x|$ is a maximum). Finally, because the elastic potential energy is proportional to x^2, we see that U_s is always positive in a deformed spring.

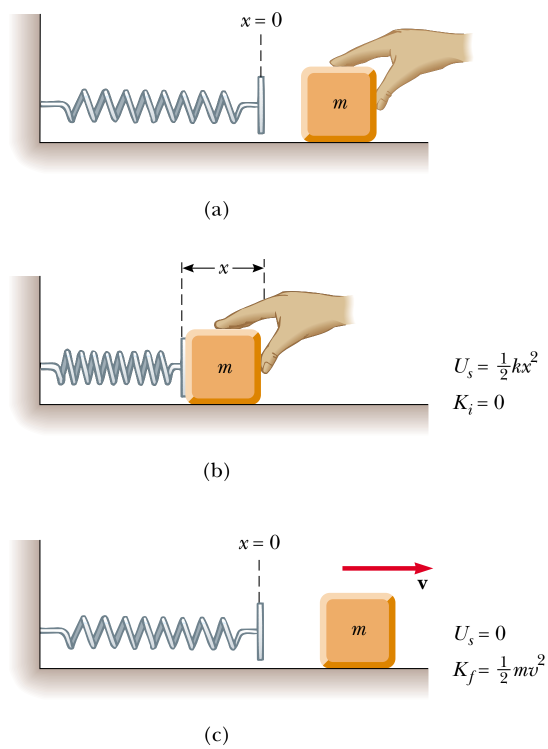


Figure 8.2 (a) An undeformed spring on a frictionless horizontal surface. (b) A block of mass *m* is pushed against the spring, compressing it a distance *x*. (c) When the block is released from rest, the elastic potential energy stored in the spring is transferred to the block in the form of kinetic energy.

8.2 CONSERVATIVE AND NONCONSERVATIVE FORCES

The work done by the gravitational force does not depend on whether an object falls vertically or slides down a sloping incline. All that matters is the change in the object's elevation. On the other hand, the energy loss due to friction on that incline depends on the distance the object slides. In other words, the path makes no difference when we consider the work done by the gravitational force, but it does make a difference when we consider the energy loss due to frictional forces. We can use this varying dependence on path to classify forces as either conservative or nonconservative.

Of the two forces just mentioned, the gravitational force is conservative and the frictional force is nonconservative.

Conservative Forces

Properties of a conservative force

Conservative forces have two important properties:

1. A force is conservative if the work it does on a particle moving between any two points is independent of the path taken by the particle.
2. The work done by a conservative force on a particle moving through any closed path is zero. (A closed path is one in which the beginning and end points are identical.)

The gravitational force is one example of a conservative force, and the force that a spring exerts on any object attached to the spring is another. As we learned in the preceding section, the work done by the gravitational force on an object moving between any two points near the Earth's surface is $W_g = mgy_i - mgy_f$. From this equation we see that W_g depends only on the initial and final y coordinates of the object and hence is independent of the path. Furthermore, W_g is zero when the object moves over any closed path (where $y_i = y_f$).

For the case of the object–spring system, the work W_s done by the spring force is given by $W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$ (Eq. 8.3). Again, we see that the spring force is conservative because W_s depends only on the initial and final x coordinates of the object and is zero for any closed path.

We can associate a potential energy with any conservative force and can do this *only* for conservative forces. In the previous section, the potential energy associated with the gravitational force was defined as $U_g \equiv mgy$. In general, the work W_c done on an object by a conservative force is equal to the initial value of the potential energy associated with the object minus the final value:

$$W_c = U_i - U_f = -\Delta U \quad \textbf{(8.5)}$$

Work done by a conservative force

This equation should look familiar to you. It is the general form of the equation for work done by the gravitational force (Eq. 8.2) and that for the work done by the spring force (Eq. 8.3).

Nonconservative Forces

5.3

Properties of a nonconservative force

A force is nonconservative if it causes a change in mechanical energy *E*, which we define as the sum of kinetic and potential energies. For example, if a book is sent sliding on a horizontal surface that is not frictionless, the force of kinetic friction reduces the book's kinetic energy. As the book slows down, its kinetic energy decreases. As a result of the frictional force, the temperatures of the book and surface increase. The type of energy associated with temperature is *internal energy*, which we will study in detail in Chapter 20. Experience tells us that this internal energy cannot be transferred back to the kinetic energy of the book. In other words, the energy transformation is not reversible. Because the force of kinetic friction changes the mechanical energy of a system, it is a nonconservative force.

From the work–kinetic energy theorem, we see that the work done by a conservative force on an object causes a change in the kinetic energy of the object. The change in kinetic energy depends only on the initial and final positions of the object, and not on the path connecting these points. Let us compare this to the sliding book example, in which the nonconservative force of friction is acting between the book and the surface. According to Equation 7.17a, the change in kinetic energy of the book due to friction is $\Delta K_{\text{friction}} = -f_k d$, where d is the length of the path over which the friction force acts. Imagine that the book slides from A to B over the straight-line path of length d in Figure 8.3. The change in kinetic energy is $-f_k d$. Now, suppose the book slides over the semicircular path from A to B. In this case, the path is longer and, as a result, the change in kinetic energy is greater in magnitude than that in the straight-line case. For this particular path, the change in kinetic energy is $-f_k \pi\, d/2$, since d is the diameter of the semicircle. Thus, we see that for a nonconservative force, the change in kinetic energy depends on the path followed between the initial and final points. If a potential energy is involved, then the change in the total mechanical energy depends on the path followed. We shall return to this point in Section 8.5.

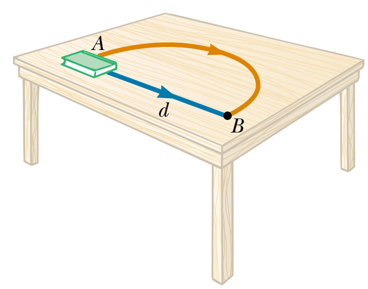


Figure 8.3 The loss in mechanical energy due to the force of kinetic friction depends on the path taken as the book is moved from A to B. The loss in mechanical energy is greater along the red path than along the blue path.

8.3 CONSERVATIVE FORCES AND POTENTIAL ENERGY

In the preceding section we found that the work done on a particle by a conservative force does not depend on the path taken by the particle. The work depends only on the particle's initial and final coordinates. As a consequence, we can define a **potential energy function** U such that the work done by a conservative force equals the decrease in the potential energy of the system. The work done by a conservative force **F** as a particle moves along the x axis is[2]

$$W_c = \int_{x_i}^{x_f} F_x \, dx = -\Delta U \tag{8.6}$$

where F_x is the component of **F** in the direction of the displacement. That is, **the work done by a conservative force equals the negative of the change in the potential energy associated with that force,** where the change in the potential energy is defined as $\Delta U = U_f - U_i$.

We can also express Equation 8.6 as

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x \, dx \tag{8.7}$$

Therefore, ΔU is negative when F_x and dx are in the same direction, as when an object is lowered in a gravitational field or when a spring pushes an object toward equilibrium.

The term *potential energy* implies that the object has the potential, or capability, of either gaining kinetic energy or doing work when it is released from some point under the influence of a conservative force exerted on the object by some other member of the system. It is often convenient to establish some particular location x_i as a reference point and measure all potential energy differences with respect to it. We can then define the potential energy function as

$$U_f(x) = -\int_{x_i}^{x_f} F_x \, dx + U_i \tag{8.8}$$

The value of U_i is often taken to be zero at the reference point. It really does not matter what value we assign to U_i, because any nonzero value merely shifts $U_f(x)$ by a constant amount, and only the *change* in potential energy is physically meaningful.

If the conservative force is known as a function of position, we can use Equation 8.8 to calculate the change in potential energy of a system as an object within the system moves from x_i to x_f. It is interesting to note that in the case of one-dimensional displacement, a force is always conservative if it is a function of position only. This is not necessarily the case for motion involving two- or three-dimensional displacements.

8.4 CONSERVATION OF MECHANICAL ENERGY

5.9 An object held at some height h above the floor has no kinetic energy. However, as we learned earlier, the gravitational potential energy of the object–Earth system is equal to mgh. If the object is dropped, it falls to the floor; as it falls, its speed and thus its kinetic energy increase, while the potential energy of the system decreases. If factors such as air resistance are ignored, whatever potential energy the system loses as the object moves downward appears as kinetic energy of the object. In other words, the sum of the kinetic and potential energies—the total mechanical energy E—remains constant. This is an example of the principle of **conservation**

[2] For a general displacement, the work done in two or three dimensions also equals $U_i - U_f$, where $U = U(x, y, z)$. We write this formally as $W = \int_i^f \mathbf{F} \cdot d\mathbf{s} = U_i - U_f$.

of mechanical energy. For the case of an object in free fall, this principle tells us that any increase (or decrease) in potential energy is accompanied by an equal decrease (or increase) in kinetic energy. Note that **the total mechanical energy of a system remains constant in any isolated system of objects that interact only through conservative forces.**

Because the total mechanical energy E of a system is defined as the sum of the kinetic and potential energies, we can write

$$E \equiv K + U \tag{8.9}$$

Total mechanical energy

We can state the principle of conservation of energy as $E_i = E_f$, and so we have

$$K_i + U_i = K_f + U_f \tag{8.10}$$

The mechanical energy of an isolated system remains constant

It is important to note that Equation 8.10 is valid only when no energy is added to or removed from the system. Furthermore, there must be no nonconservative forces doing work within the system.

Consider the carnival Ring-the-Bell event illustrated at the beginning of the chapter. The participant is trying to convert the initial kinetic energy of the hammer into gravitational potential energy associated with a weight that slides on a vertical track. If the hammer has sufficient kinetic energy, the weight is lifted high enough to reach the bell at the top of the track. To maximize the hammer's kinetic energy, the player must swing the heavy hammer as rapidly as possible. The fast-moving hammer does work on the pivoted target, which in turn does work on the weight. Of course, greasing the track (so as to minimize energy loss due to friction) would also help but is probably not allowed!

If more than one conservative force acts on an object within a system, a potential energy function is associated with each force. In such a case, we can apply the principle of conservation of mechanical energy for the system as

$$K_i + \sum U_i = K_f + \sum U_f \tag{8.11}$$

where the number of terms in the sums equals the number of conservative forces present. For example, if an object connected to a spring oscillates vertically, two conservative forces act on the object: the spring force and the gravitational force.

QuickLab

Dangle a shoe from its lace and use it as a pendulum. Hold it to the side, release it, and note how high it swings at the end of its arc. How does this height compare with its initial height? You may want to check Question 8.3 as part of your investigation.

Twin Falls on the Island of Kauai, Hawaii. The gravitational potential energy of the water–Earth system when the water is at the top of the falls is converted to kinetic energy once that water begins falling. How did the water get to the top of the cliff? In other words, what was the original source of the gravitational potential energy when the water was at the top? (*Hint:* This same source powers nearly everything on the planet.)

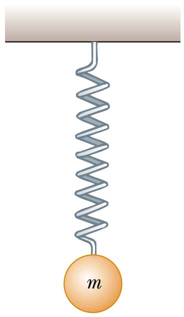


Figure 8.4 A ball connected to a massless spring suspended vertically. What forms of potential energy are associated with the ball–spring–Earth system when the ball is displaced downward?

Quick Quiz 8.2

A ball is connected to a light spring suspended vertically, as shown in Figure 8.4. When displaced downward from its equilibrium position and released, the ball oscillates up and down. If air resistance is neglected, is the total mechanical energy of the system (ball plus spring plus Earth) conserved? How many forms of potential energy are there for this situation?

Quick Quiz 8.3

Three identical balls are thrown from the top of a building, all with the same initial speed. The first is thrown horizontally, the second at some angle above the horizontal, and the third at some angle below the horizontal, as shown in Figure 8.5. Neglecting air resistance, rank the speeds of the balls at the instant each hits the ground.

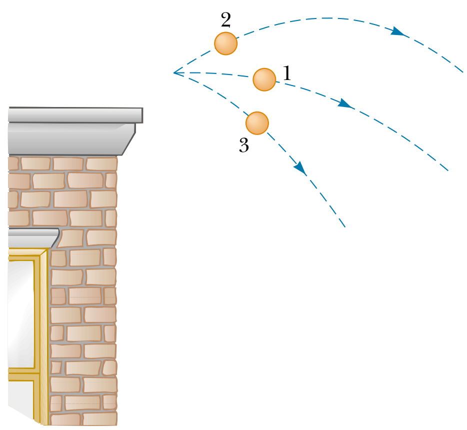


Figure 8.5 Three identical balls are thrown with the same initial speed from the top of a building.

EXAMPLE 8.2 Ball in Free Fall

A ball of mass m is dropped from a height h above the ground, as shown in Figure 8.6. (a) Neglecting air resistance, determine the speed of the ball when it is at a height y above the ground.

Solution Because the ball is in free fall, the only force acting on it is the gravitational force. Therefore, we apply the principle of conservation of mechanical energy to the ball–Earth system. Initially, the system has potential energy but no kinetic energy. As the ball falls, the total mechanical energy remains constant and equal to the initial potential energy of the system.

At the instant the ball is released, its kinetic energy is $K_i = 0$ and the potential energy of the system is $U_i = mgh$. When the ball is at a distance y above the ground, its kinetic energy is $K_f = \frac{1}{2}mv_f^2$ and the potential energy relative to the ground is $U_f = mgy$. Applying Equation 8.10, we obtain

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \tfrac{1}{2}mv_f^2 + mgy$$

$$v_f^2 = 2g(h - y)$$

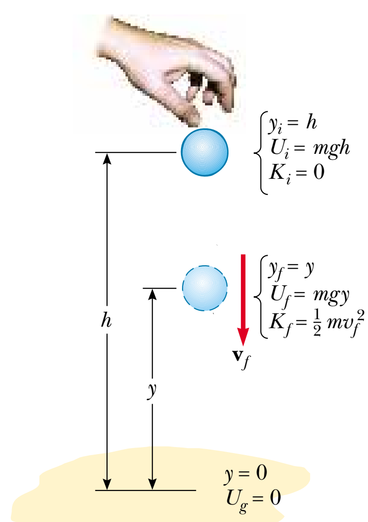


Figure 8.6 A ball is dropped from a height h above the ground. Initially, the total energy of the ball–Earth system is potential energy, equal to mgh relative to the ground. At the elevation y, the total energy is the sum of the kinetic and potential energies.

$$v_f = \sqrt{2g(h - y)}$$

The speed is always positive. If we had been asked to find the ball's velocity, we would use the negative value of the square root as the y component to indicate the downward motion.

(b) Determine the speed of the ball at y if at the instant of release it already has an initial speed v_i at the initial altitude h.

Solution In this case, the initial energy includes kinetic energy equal to $\frac{1}{2}mv_i^2$, and Equation 8.10 gives

$$\tfrac{1}{2}mv_i^2 + mgh = \tfrac{1}{2}mv_f^2 + mgy$$

$$v_f^2 = v_i^2 + 2g(h - y)$$

$$v_f = \sqrt{v_i^2 + 2g(h - y)}$$

This result is consistent with the expression $v_{yf}^2 = v_{yi}^2 - 2g(y_f - y_i)$ from kinematics, where $y_i = h$. Furthermore, this result is valid even if the initial velocity is at an angle to the horizontal (the projectile situation) for two reasons: (1) energy is a scalar, and the kinetic energy depends only on the magnitude of the velocity; and (2) the change in the gravitational potential energy depends only on the change in position in the vertical direction.

EXAMPLE 8.3 The Pendulum

A pendulum consists of a sphere of mass m attached to a light cord of length L, as shown in Figure 8.7. The sphere is released from rest when the cord makes an angle θ_A with the vertical, and the pivot at P is frictionless. (a) Find the speed of the sphere when it is at the lowest point Ⓑ.

Solution The only force that does work on the sphere is the gravitational force. (The force of tension is always perpendicular to each element of the displacement and hence does no work.) Because the gravitational force is conservative, the total mechanical energy of the pendulum–Earth system is constant. (In other words, we can classify this as an "energy conservation" problem.) As the pendulum swings, continuous transformation between potential and kinetic energy occurs. At the instant the pendulum is released, the energy of the system is entirely potential energy. At point Ⓑ the pendulum has kinetic energy, but the system has lost some potential energy. At Ⓒ the system has regained its initial potential energy, and the kinetic energy of the pendulum is again zero.

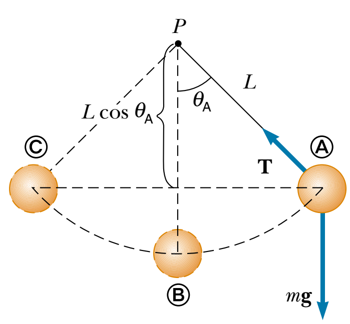


Figure 8.7 If the sphere is released from rest at the angle θ_A it will never swing above this position during its motion. At the start of the motion, position Ⓐ, the energy is entirely potential. This initial potential energy is all transformed into kinetic energy at the lowest elevation Ⓑ. As the sphere continues to move along the arc, the energy again becomes entirely potential energy at Ⓒ.

If we measure the y coordinates of the sphere from the center of rotation, then $y_A = -L\cos\theta_A$ and $y_B = -L$. Therefore, $U_A = -mgL\cos\theta_A$ and $U_B = -mgL$. Applying the principle of conservation of mechanical energy to the system gives

$$K_A + U_A = K_B + U_B$$

$$0 - mgL\cos\theta_A = \tfrac{1}{2}mv_B^2 - mgL$$

$$(1) \qquad v_B = \sqrt{2gL(1 - \cos\theta_A)}$$

(b) What is the tension T_B in the cord at Ⓑ?

Solution Because the force of tension does no work, we cannot determine the tension using the energy method. To find T_B, we can apply Newton's second law to the radial direction. First, recall that the centripetal acceleration of a particle moving in a circle is equal to v^2/r directed toward the center of rotation. Because $r = L$ in this example, we obtain

$$(2) \qquad \sum F_r = T_B - mg = ma_r = m\frac{v_B^2}{L}$$

Substituting (1) into (2) gives the tension at point Ⓑ:

$$(3) \qquad T_B = mg + 2mg(1 - \cos\theta_A)$$

$$= mg(3 - 2\cos\theta_A)$$

From (2) we see that the tension at Ⓑ is greater than the weight of the sphere. Furthermore, (3) gives the expected result that $T_B = mg$ when the initial angle $\theta_A = 0$.

Exercise A pendulum of length 2.00 m and mass 0.500 kg is released from rest when the cord makes an angle of 30.0° with the vertical. Find the speed of the sphere and the tension in the cord when the sphere is at its lowest point.

Answer 2.29 m/s; 6.21 N.

8.5 WORK DONE BY NONCONSERVATIVE FORCES

As we have seen, if the forces acting on objects within a system are conservative, then the mechanical energy of the system remains constant. However, if some of the forces acting on objects within the system are not conservative, then the mechanical energy of the system does not remain constant. Let us examine two types of nonconservative forces: an applied force and the force of kinetic friction.

Work Done by an Applied Force

When you lift a book through some distance by applying a force to it, the force you apply does work W_{app} on the book, while the gravitational force does work W_g on the book. If we treat the book as a particle, then the net work done on the book is related to the change in its kinetic energy as described by the work–kinetic energy theorem given by Equation 7.15:

$$W_{app} + W_g = \Delta K \tag{8.12}$$

Because the gravitational force is conservative, we can use Equation 8.2 to express the work done by the gravitational force in terms of the change in potential energy, or $W_g = -\Delta U$. Substituting this into Equation 8.12 gives

$$W_{app} = \Delta K + \Delta U \tag{8.13}$$

Note that the right side of this equation represents the change in the mechanical energy of the book–Earth system. This result indicates that your applied force transfers energy to the system in the form of kinetic energy of the book and gravitational potential energy of the book–Earth system. Thus, we conclude that if an object is part of a system, then **an applied force can transfer energy into or out of the system.**

Situations Involving Kinetic Friction

Kinetic friction is an example of a nonconservative force. If a book is given some initial velocity on a horizontal surface that is not frictionless, then the force of kinetic friction acting on the book opposes its motion and the book slows down and eventually stops. The force of kinetic friction reduces the kinetic energy of the book by transforming kinetic energy to internal energy of the book and part of the horizontal surface. Only part of the book's kinetic energy is transformed to internal energy in the book. The rest appears as internal energy in the surface. (When you trip and fall while running across a gymnasium floor, not only does the skin on your knees warm up but so does the floor!)

QuickLab

Find a friend and play a game of racquetball. After a long volley, feel the ball and note that it is warm. Why is that?

As the book moves through a distance d, the only force that does work is the force of kinetic friction. This force causes a decrease in the kinetic energy of the book. This decrease was calculated in Chapter 7, leading to Equation 7.17a, which we repeat here:

$$\Delta K_{friction} = -f_k d \tag{8.14}$$

If the book moves on an incline that is not frictionless, a change in the gravitational potential energy of the book–Earth system also occurs, and $-f_k d$ is the amount by which the mechanical energy of the system changes because of the force of kinetic friction. In such cases,

$$\Delta E = \Delta K + \Delta U = -f_k d \tag{8.15}$$

where $E_i + \Delta E = E_f$.

Quick Quiz 8.4

Write down the work–kinetic energy theorem for the general case of two objects that are connected by a spring and acted upon by gravity and some other external applied force. Include the effects of friction as $\Delta E_{\text{friction}}$.

Problem-Solving Hints

Conservation of Energy

We can solve many problems in physics using the principle of conservation of energy. You should incorporate the following procedure when you apply this principle:

- Define your system, which may include two or more interacting particles, as well as springs or other systems in which elastic potential energy can be stored. Choose the initial and final points.
- Identify zero points for potential energy (both gravitational and spring). If there is more than one conservative force, write an expression for the potential energy associated with each force.
- Determine whether any nonconservative forces are present. Remember that if friction or air resistance is present, mechanical energy *is not conserved.*
- If mechanical energy is *conserved,* you can write the total initial energy $E_i = K_i + U_i$ at some point. Then, write an expression for the total final energy $E_f = K_f + U_f$ at the final point that is of interest. Because mechanical energy is *conserved,* you can equate the two total energies and solve for the quantity that is unknown.
- If frictional forces are present (and thus mechanical energy is *not conserved*), first write expressions for the total initial and total final energies. In this case, the difference between the total final mechanical energy and the total initial mechanical energy equals the change in mechanical energy in the system due to friction.

EXAMPLE 8.4 Crate Sliding Down a Ramp

A 3.00-kg crate slides down a ramp. The ramp is 1.00 m in length and inclined at an angle of 30.0°, as shown in Figure 8.8. The crate starts from rest at the top, experiences a constant frictional force of magnitude 5.00 N, and continues to move a short distance on the flat floor after it leaves the ramp. Use energy methods to determine the speed of the crate at the bottom of the ramp.

Solution Because $v_i = 0$, the initial kinetic energy at the top of the ramp is zero. If the y coordinate is measured from the bottom of the ramp (the final position where the potential energy is zero) with the upward direction being positive, then $y_i = 0.500$ m. Therefore, the total mechanical energy of the crate–Earth system at the top is all potential energy:

$$E_i = K_i + U_i = 0 + U_i = mgy_i$$
$$= (3.00 \text{ kg})(9.80 \text{ m/s}^2)(0.500 \text{ m}) = 14.7 \text{ J}$$

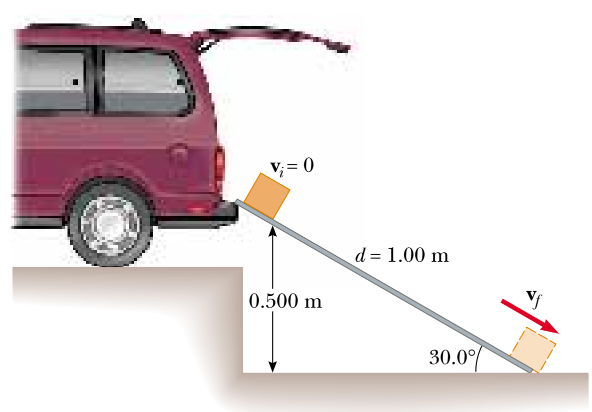


Figure 8.8 A crate slides down a ramp under the influence of gravity. The potential energy decreases while the kinetic energy increases.

When the crate reaches the bottom of the ramp, the potential energy of the system is *zero* because the elevation of the crate is $y_f = 0$. Therefore, the total mechanical energy of the system when the crate reaches the bottom is all kinetic energy:

$$E_f = K_f + U_f = \tfrac{1}{2}mv_f^2 + 0$$

We cannot say that $E_i = E_f$ because a nonconservative force reduces the mechanical energy of the system: the force of kinetic friction acting on the crate. In this case, Equation 8.15 gives $\Delta E = -f_k d$, where d is the displacement along the ramp. (Remember that the forces normal to the ramp do no work on the crate because they are perpendicular to the displacement.) With $f_k = 5.00$ N and $d = 1.00$ m, we have

$$\Delta E = -f_k d = -(5.00\text{ N})(1.00\text{ m}) = -5.00\text{ J}$$

This result indicates that the system loses some mechanical energy because of the presence of the nonconservative frictional force. Applying Equation 8.15 gives

$$E_f - E_i = \tfrac{1}{2}mv_f^2 - mgy_i = -f_k d$$

$$\tfrac{1}{2}mv_f^2 = 14.7\text{ J} - 5.00\text{ J} = 9.70\text{ J}$$

$$v_f^2 = \frac{19.4\text{ J}}{3.00\text{ kg}} = 6.47\text{ m}^2/\text{s}^2$$

$$v_f = \boxed{2.54\text{ m/s}}$$

Exercise Use Newton's second law to find the acceleration of the crate along the ramp, and use the equations of kinematics to determine the final speed of the crate.

Answer 3.23 m/s^2; 2.54 m/s.

Exercise Assuming the ramp to be frictionless, find the final speed of the crate and its acceleration along the ramp.

Answer 3.13 m/s; 4.90 m/s^2.

EXAMPLE 8.5 Motion on a Curved Track

A child of mass m rides on an irregularly curved slide of height $h = 2.00$ m, as shown in Figure 8.9. The child starts from rest at the top. (a) Determine his speed at the bottom, assuming no friction is present.

Solution The normal force **n** does no work on the child because this force is always perpendicular to each element of the displacement. Because there is no friction, the mechanical energy of the child–Earth system is conserved. If we measure the y coordinate in the upward direction from the bottom of the slide, then $y_i = h$, $y_f = 0$, and we obtain

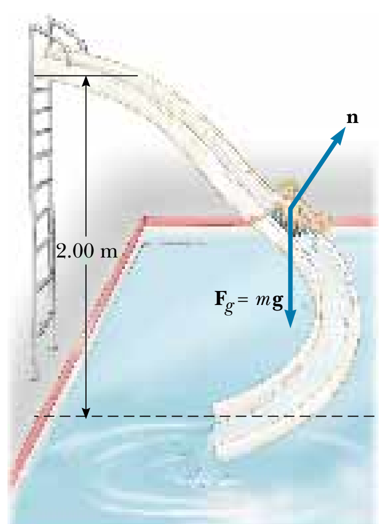


Figure 8.9 If the slide is frictionless, the speed of the child at the bottom depends only on the height of the slide.

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \tfrac{1}{2}mv_f^2 + 0$$

$$v_f = \sqrt{2gh}$$

Note that the result is the same as it would be had the child fallen vertically through a distance h! In this example, $h = 2.00$ m, giving

$$v_f = \sqrt{2gh} = \sqrt{2(9.80\text{ m/s}^2)(2.00\text{ m})} = \boxed{6.26\text{ m/s}}$$

(b) If a force of kinetic friction acts on the child, how much mechanical energy does the system lose? Assume that $v_f = 3.00$ m/s and $m = 20.0$ kg.

Solution In this case, mechanical energy is *not* conserved, and so we must use Equation 8.15 to find the loss of mechanical energy due to friction:

$$\Delta E = E_f - E_i = (K_f + U_f) - (K_i + U_i)$$
$$= (\tfrac{1}{2}mv_f^2 + 0) - (0 + mgh) = \tfrac{1}{2}mv_f^2 - mgh$$
$$= \tfrac{1}{2}(20.0\text{ kg})(3.00\text{ m/s})^2 - (20.0\text{ kg})(9.80\text{ m/s}^2)(2.00\text{ m})$$
$$= \boxed{-302\text{ J}}$$

Again, ΔE is negative because friction is reducing mechanical energy of the system (the final mechanical energy is less than the initial mechanical energy). Because the slide is curved, the normal force changes in magnitude and direction during the motion. Therefore, the frictional force, which is proportional to n, also changes during the motion. Given this changing frictional force, do you think it is possible to determine μ_k from these data?

EXAMPLE 8.6 Let's Go Skiing!

A skier starts from rest at the top of a frictionless incline of height 20.0 m, as shown in Figure 8.10. At the bottom of the incline, she encounters a horizontal surface where the coefficient of kinetic friction between the skis and the snow is 0.210. How far does she travel on the horizontal surface before coming to rest?

Solution First, let us calculate her speed at the bottom of the incline, which we choose as our zero point of potential energy. Because the incline is frictionless, the mechanical energy of the skier–Earth system remains constant, and we find, as we did in the previous example, that

$$v_B = \sqrt{2gh} = \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m})} = 19.8 \text{ m/s}$$

Now we apply Equation 8.15 as the skier moves along the rough horizontal surface from Ⓑ to Ⓒ. The change in mechanical energy along the horizontal is $\Delta E = -f_k d$, where d is the horizontal displacement.

To find the distance the skier travels before coming to rest, we take $K_C = 0$. With $v_B = 19.8$ m/s and the frictional force given by $f_k = \mu_k n = \mu_k mg$, we obtain

$$\Delta E = E_C - E_B = -\mu_k mgd$$

$$(K_C + U_C) - (K_B + U_B) = (0 + 0) - (\tfrac{1}{2}mv_B^2 + 0) = -\mu_k mgd$$

$$d = \frac{v_B^2}{2\mu_k g} = \frac{(19.8 \text{ m/s})^2}{2(0.210)(9.80 \text{ m/s}^2)} = 95.2 \text{ m}$$

Exercise Find the horizontal distance the skier travels before coming to rest if the incline also has a coefficient of kinetic friction equal to 0.210.

Answer 40.3 m.

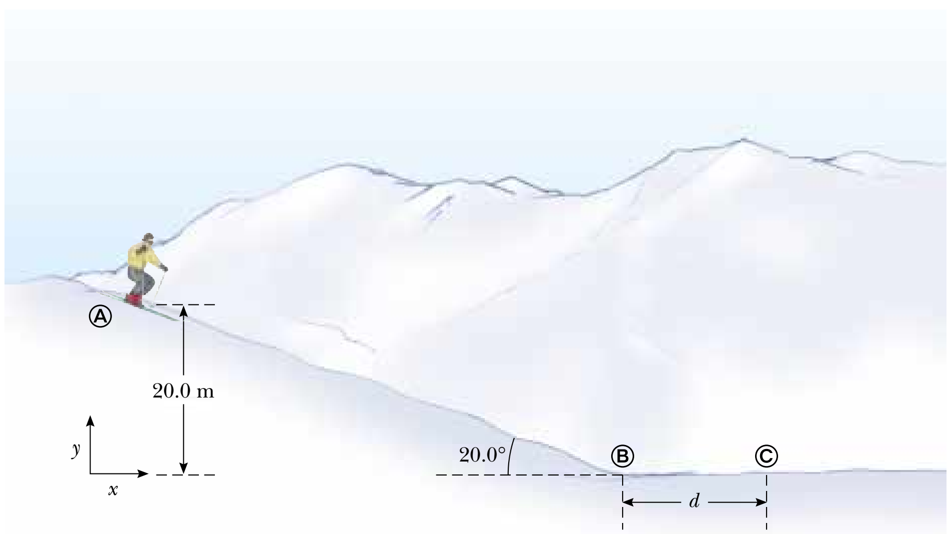


Figure 8.10 The skier slides down the slope and onto a level surface, stopping after a distance d from the bottom of the hill.

EXAMPLE 8.7 The Spring-Loaded Popgun

The launching mechanism of a toy gun consists of a spring of unknown spring constant (Fig. 8.11a). When the spring is compressed 0.120 m, the gun, when fired vertically, is able to launch a 35.0-g projectile to a maximum height of 20.0 m above the position of the projectile before firing. (a) Neglecting all resistive forces, determine the spring constant.

Solution Because the projectile starts from rest, the initial kinetic energy is zero. If we take the zero point for the gravitational potential energy of the projectile–Earth system to be at the lowest position of the projectile x_A, then the initial gravitational potential energy also is zero. The mechanical energy of this system is constant because no nonconservative forces are present.

Initially, the only mechanical energy in the system is the elastic potential energy stored in the spring of the gun, $U_{sA} = kx^2/2$, where the compression of the spring is $x = 0.120$ m. The projectile rises to a maximum height

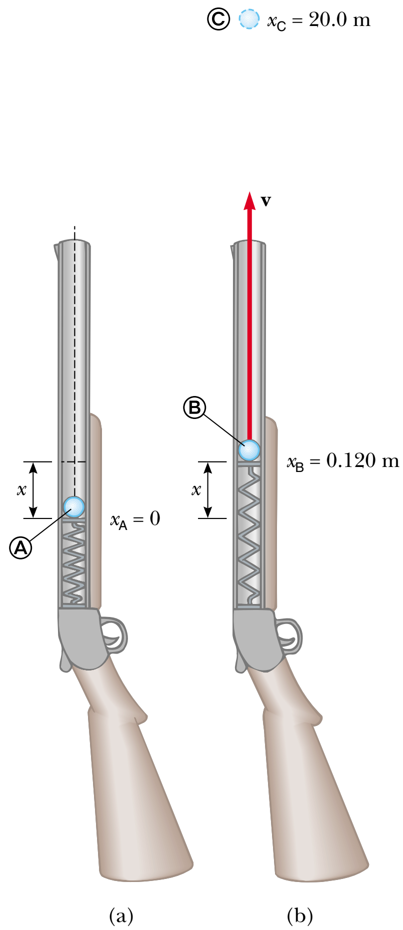


Figure 8.11 A spring-loaded popgun.

$x_C = h = 20.0$ m, and so the final gravitational potential energy when the projectile reaches its peak is mgh. The final kinetic energy of the projectile is zero, and the final elastic potential energy stored in the spring is zero. Because the mechanical energy of the system is constant, we find that

$$E_A = E_C$$
$$K_A + U_{gA} + U_{sA} = K_C + U_{gC} + U_{sC}$$
$$0 + 0 + \tfrac{1}{2}kx^2 = 0 + mgh + 0$$
$$\tfrac{1}{2}k(0.120\ \text{m})^2 = (0.0350\ \text{kg})(9.80\ \text{m/s}^2)(20.0\ \text{m})$$
$$k = \boxed{953\ \text{N/m}}$$

(b) Find the speed of the projectile as it moves through the equilibrium position of the spring (where $x_B = 0.120$ m) as shown in Figure 8.11b.

Solution As already noted, the only mechanical energy in the system at Ⓐ is the elastic potential energy $kx^2/2$. The total energy of the system as the projectile moves through the equilibrium position of the spring comprises the kinetic energy of the projectile $mv_B{}^2/2$, and the gravitational potential energy mgx_B. Hence, the principle of the conservation of mechanical energy in this case gives

$$E_A = E_B$$
$$K_A + U_{gA} + U_{sA} = K_B + U_{gB} + U_{sB}$$
$$0 + 0 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv_B{}^2 + mgx_B + 0$$

Solving for v_B gives

$$v_B = \sqrt{\frac{kx^2}{m} - 2gx_B}$$
$$= \sqrt{\frac{(953\ \text{N/m})(0.120\ \text{m})^2}{0.0350\ \text{kg}} - 2(9.80\ \text{m/s}^2)(0.120\ \text{m})}$$
$$= \boxed{19.7\ \text{m/s}}$$

You should compare the different examples we have presented so far in this chapter. Note how breaking the problem into a sequence of labeled events helps in the analysis.

Exercise What is the speed of the projectile when it is at a height of 10.0 m?

Answer 14.0 m/s.

EXAMPLE 8.8 Block–Spring Collision

A block having a mass of 0.80 kg is given an initial velocity $v_A = 1.2$ m/s to the right and collides with a spring of negligible mass and force constant $k = 50$ N/m, as shown in Figure 8.12. (a) Assuming the surface to be frictionless, calculate the maximum compression of the spring after the collision.

Solution Our system in this example consists of the block and spring. Before the collision, at Ⓐ, the block has kinetic energy and the spring is uncompressed, so that the elastic potential energy stored in the spring is zero. Thus, the total mechanical energy of the system before the collision is just $\frac{1}{2}mv_A{}^2$. After the collision, at Ⓒ, the spring is fully compressed; now the block is at rest and so has zero kinetic energy, while the energy stored in the spring has its maximum value $\frac{1}{2}kx^2 = \frac{1}{2}kx_m{}^2$, where the origin of coordinates $x = 0$ is chosen to be the equilibrium position of the spring and x_m is

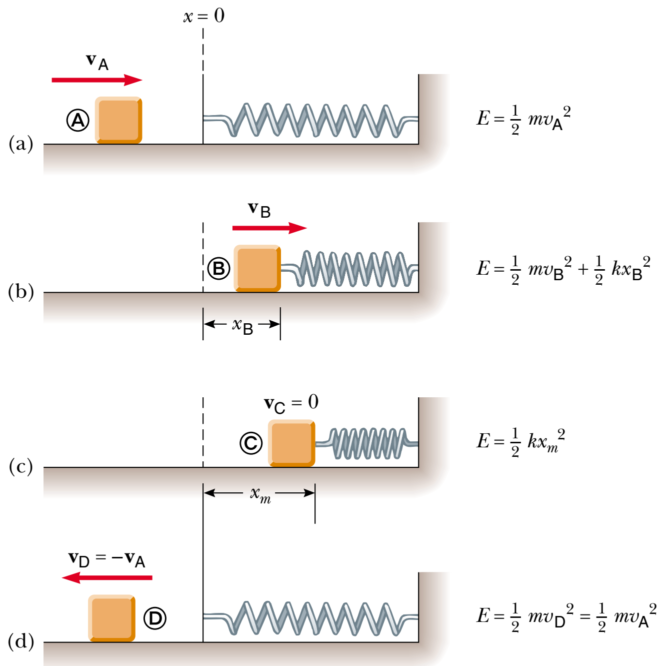


Figure 8.12 A block sliding on a smooth, horizontal surface collides with a light spring. (a) Initially the mechanical energy is all kinetic energy. (b) The mechanical energy is the sum of the kinetic energy of the block and the elastic potential energy in the spring. (c) The energy is entirely potential energy. (d) The energy is transformed back to the kinetic energy of the block. The total energy remains constant throughout the motion.

Multiflash photograph of a pole vault event. How many forms of energy can you identify in this picture?

the maximum compression of the spring, which in this case happens to be x_C. The total mechanical energy of the system is conserved because no nonconservative forces act on objects within the system.

Because mechanical energy is conserved, the kinetic energy of the block before the collision must equal the maximum potential energy stored in the fully compressed spring:

$$E_A = E_C$$

$$K_A + U_{sA} = K_C + U_{sC}$$

$$\tfrac{1}{2}mv_A{}^2 + 0 = 0 + \tfrac{1}{2}kx_m{}^2$$

$$x_m = \sqrt{\frac{m}{k}}\,v_A = \sqrt{\frac{0.80\text{ kg}}{50\text{ N/m}}}\,(1.2\text{ m/s})$$

$$= \boxed{0.15\text{ m}}$$

Note that we have not included U_g terms because no change in vertical position occurred.

(b) Suppose a constant force of kinetic friction acts between the block and the surface, with $\mu_k = 0.50$. If the speed of the block at the moment it collides with the spring is $v_A = 1.2$ m/s, what is the maximum compression in the spring?

Solution In this case, mechanical energy is *not* conserved because a frictional force acts on the block. The magnitude of the frictional force is

$$f_k = \mu_k n = \mu_k mg = 0.50(0.80\text{ kg})(9.80\text{ m/s}^2) = 3.92\text{ N}$$

Therefore, the change in the block's mechanical energy due to friction as the block is displaced from the equilibrium position of the spring (where we have set our origin) to x_B is

$$\Delta E = -f_k x_B = -3.92x_B$$

Substituting this into Equation 8.15 gives

$$\Delta E = E_f - E_i = (0 + \tfrac{1}{2}kx_B{}^2) - (\tfrac{1}{2}mv_A{}^2 + 0) = -f_k x_B$$

$$\tfrac{1}{2}(50)x_B{}^2 - \tfrac{1}{2}(0.80)(1.2)^2 = -3.92x_B$$

$$25x_B{}^2 + 3.92x_B - 0.576 = 0$$

Solving the quadratic equation for x_B gives $x_B = 0.092$ m and $x_B = -0.25$ m. The physically meaningful root is $x_B = \boxed{0.092\text{ m.}}$ The negative root does not apply to this situation because the block must be to the right of the origin (positive value of x) when it comes to rest. Note that 0.092 m is less than the distance obtained in the frictionless case of part (a). This result is what we expect because friction retards the motion of the system.

EXAMPLE 8.9 Connected Blocks in Motion

Two blocks are connected by a light string that passes over a frictionless pulley, as shown in Figure 8.13. The block of mass m_1 lies on a horizontal surface and is connected to a spring of force constant k. The system is released from rest when the spring is unstretched. If the hanging block of mass m_2 falls a distance h before coming to rest, calculate the coefficient of kinetic friction between the block of mass m_1 and the surface.

Solution The key word *rest* appears twice in the problem statement, telling us that the initial and final velocities and kinetic energies are zero. (Also note that because we are concerned only with the beginning and ending points of the motion, we do not need to label events with circled letters as we did in the previous two examples. Simply using i and f is sufficient to keep track of the situation.) In this situation, the system consists of the two blocks, the spring, and the Earth. We need to consider two forms of potential energy: gravitational and elastic. Because the initial and final kinetic energies of the system are zero, $\Delta K = 0$, and we can write

$$(1) \qquad \Delta E = \Delta U_g + \Delta U_s$$

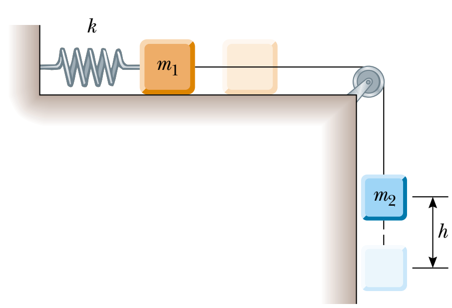


Figure 8.13 As the hanging block moves from its highest elevation to its lowest, the system loses gravitational potential energy but gains elastic potential energy in the spring. Some mechanical energy is lost because of friction between the sliding block and the surface.

where $\Delta U_g = U_{gf} - U_{gi}$ is the change in the system's gravitational potential energy and $\Delta U_s = U_{sf} - U_{si}$ is the change in the system's elastic potential energy. As the hanging block falls a distance h, the horizontally moving block moves the same distance h to the right. Therefore, using Equation 8.15, we find that the loss in energy due to friction between the horizontally sliding block and the surface is

$$(2) \qquad \Delta E = -f_k h = -\mu_k m_1 g h$$

The change in the gravitational potential energy of the system is associated with only the falling block because the vertical coordinate of the horizontally sliding block does not change. Therefore, we obtain

$$(3) \qquad \Delta U_g = U_{gf} - U_{gi} = 0 - m_2 g h$$

where the coordinates have been measured from the lowest position of the falling block.

The change in the elastic potential energy stored in the spring is

$$(4) \qquad \Delta U_s = U_{sf} - U_{si} = \tfrac{1}{2}kh^2 - 0$$

Substituting Equations (2), (3), and (4) into Equation (1) gives

$$-\mu_k m_1 g h = -m_2 g h + \tfrac{1}{2}kh^2$$

$$\mu_k = \frac{m_2 g - \frac{1}{2}kh}{m_1 g}$$

This setup represents a way of measuring the coefficient of kinetic friction between an object and some surface. As you can see from the problem, sometimes it is easier to work with the changes in the various types of energy rather than the actual values. For example, if we wanted to calculate the numerical value of the gravitational potential energy associated with the horizontally sliding block, we would need to specify the height of the horizontal surface relative to the lowest position of the falling block. Fortunately, this is not necessary because the gravitational potential energy associated with the first block does not change.

EXAMPLE 8.10 A Grand Entrance

You are designing apparatus to support an actor of mass 65 kg who is to "fly" down to the stage during the performance of a play. You decide to attach the actor's harness to a 130-kg sandbag by means of a lightweight steel cable running smoothly over two frictionless pulleys, as shown in Figure 8.14a. You need 3.0 m of cable between the harness and the nearest pulley so that the pulley can be hidden behind a curtain. For the apparatus to work successfully, the sandbag must never lift above the floor as the actor swings from above the stage to the floor. Let us call the angle that the actor's cable makes with the vertical θ. What is the maximum value θ can have before the sandbag lifts off the floor?

Solution We need to draw on several concepts to solve this problem. First, we use the principle of the conservation of mechanical energy to find the actor's speed as he hits the floor as a function of θ and the radius R of the circular path through which he swings. Next, we apply Newton's second law to the actor at the bottom of his path to find the cable tension as a function of the given parameters. Finally, we note that the sandbag lifts off the floor when the upward force exerted on it by the cable exceeds the gravitational force acting on it; the normal force is zero when this happens.

Applying conservation of energy to the actor–Earth system gives

$$K_i + U_i = K_f + U_f$$

$$(1) \qquad 0 + m_{\text{actor}}\, g y_i = \tfrac{1}{2} m_{\text{actor}} v_f^{\,2} + 0$$

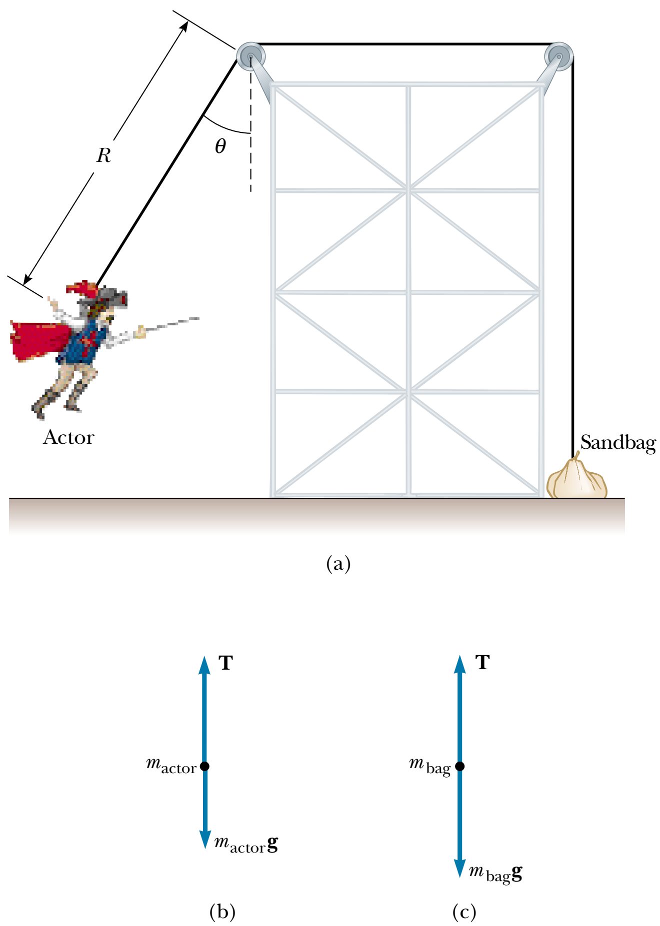


Figure 8.14 (a) An actor uses some clever staging to make his entrance. (b) Free-body diagram for actor at the bottom of the circular path. (c) Free-body diagram for sandbag.

where y_i is the initial height of the actor above the floor and v_f is the speed of the actor at the instant before he lands. (Note that $K_i = 0$ because he starts from rest and that $U_f = 0$ because we set the level of the actor's harness when he is standing on the floor as the zero level of potential energy.) From the geometry in Figure 8.14a, we see that $y_i = R - R\cos\theta = R(1 - \cos\theta)$. Using this relationship in Equation (1), we obtain

$$(2) \qquad v_f^{\,2} = 2gR(1 - \cos\theta)$$

Now we apply Newton's second law to the actor when he is at the bottom of the circular path, using the free-body diagram in Figure 8.14b as a guide:

$$\sum F_y = T - m_{\text{actor}}g = m_{\text{actor}}\frac{v_f^{\,2}}{R}$$

$$(3) \qquad T = m_{\text{actor}}g + m_{\text{actor}}\frac{v_f^{\,2}}{R}$$

A force of the same magnitude as T is transmitted to the sandbag. If it is to be just lifted off the floor, the normal force on it becomes zero, and we require that $T = m_{\text{bag}}g$, as shown in Figure 8.14c. Using this condition together with Equations (2) and (3), we find that

$$m_{\text{bag}}g = m_{\text{actor}}g + m_{\text{actor}}\frac{2gR(1 - \cos\theta)}{R}$$

Solving for θ and substituting in the given parameters, we obtain

$$\cos\theta = \frac{3m_{\text{actor}} - m_{\text{bag}}}{2m_{\text{actor}}} = \frac{3(65\text{ kg}) - 130\text{ kg}}{2(65\text{ kg})} = \frac{1}{2}$$

$$\theta = 60°$$

Notice that we did not need to be concerned with the length R of the cable from the actor's harness to the leftmost pulley. The important point to be made from this problem is that it is sometimes necessary to combine energy considerations with Newton's laws of motion.

Exercise If the initial angle $\theta = 40°$, find the speed of the actor and the tension in the cable just before he reaches the floor. (*Hint:* You cannot ignore the length $R = 3.0$ m in this calculation.)

Answer 3.7 m/s; 940 N.

8.6 RELATIONSHIP BETWEEN CONSERVATIVE FORCES AND POTENTIAL ENERGY

Once again let us consider a particle that is part of a system. Suppose that the particle moves along the x axis, and assume that a conservative force with an x component F_x acts on the particle. Earlier in this chapter, we showed how to determine the change in potential energy of a system when we are given the conservative force. We now show how to find F_x if the potential energy of the system is known.

In Section 8.2 we learned that the work done by the conservative force as its point of application undergoes a displacement Δx equals the negative of the change in the potential energy associated with that force; that is, $W = F_x \Delta x = -\Delta U$. If the point of application of the force undergoes an infinitesimal displacement dx, we can express the infinitesimal change in the potential energy of the system dU as

$$dU = -F_x\,dx$$

Therefore, the conservative force is related to the potential energy function through the relationship[3]

Relationship between force and potential energy

$$F_x = -\frac{dU}{dx} \tag{8.16}$$

That is, **any conservative force acting on an object within a system equals the negative derivative of the potential energy of the system with respect to *x*.**

We can easily check this relationship for the two examples already discussed. In the case of the deformed spring, $U_s = \frac{1}{2}kx^2$, and therefore

$$F_s = -\frac{dU_s}{dx} = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx$$

which corresponds to the restoring force in the spring. Because the gravitational potential energy function is $U_g = mgy$, it follows from Equation 8.16 that $F_g = -mg$ when we differentiate U_g with respect to y instead of x.

We now see that U is an important function because a conservative force can be derived from it. Furthermore, Equation 8.16 should clarify the fact that adding a constant to the potential energy is unimportant because the derivative of a constant is zero.

Quick Quiz 8.5

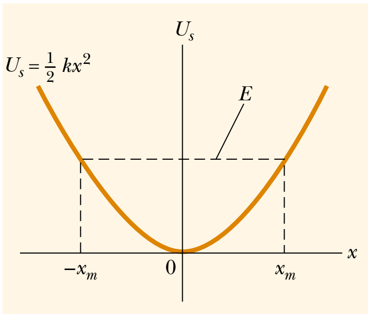
What does the slope of a graph of $U(x)$ versus x represent?

Optional Section

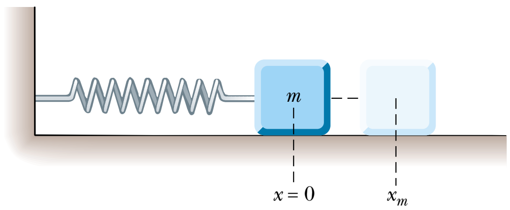
8.7 ENERGY DIAGRAMS AND THE EQUILIBRIUM OF A SYSTEM

The motion of a system can often be understood qualitatively through a graph of its potential energy versus the separation distance between the objects in the system. Consider the potential energy function for a block–spring system, given by $U_s = \frac{1}{2}kx^2$. This function is plotted versus x in Figure 8.15a. (A common mistake is to think that potential energy on the graph represents height. This is clearly not

[3] In three dimensions, the expression is $\mathbf{F} = -\mathbf{i}\frac{\partial U}{\partial x} - \mathbf{j}\frac{\partial U}{\partial y} - \mathbf{k}\frac{\partial U}{\partial z}$, where $\frac{\partial U}{\partial x}$, and so forth, are partial derivatives. In the language of vector calculus, **F** equals the negative of the gradient of the scalar quantity $U(x, y, z)$.



(a)



(b)

Figure 8.15 (a) Potential energy as a function of x for the block–spring system shown in (b). The block oscillates between the turning points, which have the coordinates $x = \pm x_m$. Note that the restoring force exerted by the spring always acts toward $x = 0$, the position of stable equilibrium.

the case here, where the block is only moving horizontally.) The force F_s exerted by the spring on the block is related to U_s through Equation 8.16:

$$F_s = -\frac{dU_s}{dx} = -kx$$

As we saw in Quick Quiz 8.5, the force is equal to the negative of the slope of the U versus x curve. When the block is placed at rest at the equilibrium position of the spring ($x = 0$), where $F_s = 0$, it will remain there unless some external force F_{ext} acts on it. If this external force stretches the spring from equilibrium, x is positive and the slope dU/dx is positive; therefore, the force F_s exerted by the spring is negative, and the block accelerates back toward $x = 0$ when released. If the external force compresses the spring, then x is negative and the slope is negative; therefore, F_s is positive, and again the mass accelerates toward $x = 0$ upon release.

From this analysis, we conclude that the $x = 0$ position for a block–spring system is one of **stable equilibrium.** That is, any movement away from this position results in a force directed back toward $x = 0$. In general, **positions of stable equilibrium correspond to points for which $U(x)$ is a minimum.**

From Figure 8.15 we see that if the block is given an initial displacement x_m and is released from rest, its total energy initially is the potential energy stored in the spring $\frac{1}{2}kx_m^2$. As the block starts to move, the system acquires kinetic energy and loses an equal amount of potential energy. Because the total energy must remain constant, the block oscillates (moves back and forth) between the two points $x = -x_m$ and $x = +x_m$, called the *turning points.* In fact, because no energy is lost (no friction), the block will oscillate between $-x_m$ and $+x_m$ forever. (We discuss these oscillations further in Chapter 13.) From an energy viewpoint, the energy of the system cannot exceed $\frac{1}{2}kx_m^2$; therefore, the block must stop at these points and, because of the spring force, must accelerate toward $x = 0$.

Another simple mechanical system that has a position of stable equilibrium is a ball rolling about in the bottom of a bowl. Anytime the ball is displaced from its lowest position, it tends to return to that position when released.

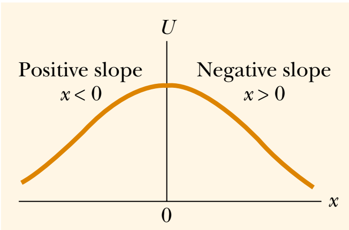


Figure 8.16 A plot of U versus x for a particle that has a position of unstable equilibrium located at $x = 0$. For any finite displacement of the particle, the force on the particle is directed away from $x = 0$.

Now consider a particle moving along the x axis under the influence of a conservative force F_x, where the U versus x curve is as shown in Figure 8.16. Once again, $F_x = 0$ at $x = 0$, and so the particle is in equilibrium at this point. However, this is a position of **unstable equilibrium** for the following reason: Suppose that the particle is displaced to the right $(x > 0)$. Because the slope is negative for $x > 0$, $F_x = -dU/dx$ is positive and the particle accelerates *away from* $x = 0$. If instead the particle is at $x = 0$ and is displaced to the left $(x < 0)$, the force is negative because the slope is positive for $x < 0$, and the particle again accelerates *away from* the equilibrium position. The position $x = 0$ in this situation is one of unstable equilibrium because for any displacement from this point, the force pushes the particle *farther away from* equilibrium. The force pushes the particle toward a position of lower potential energy. A pencil balanced on its point is in a position of unstable equilibrium. If the pencil is displaced slightly from its absolutely vertical position and is then released, it will surely fall over. In general, **positions of unstable equilibrium correspond to points for which $U(x)$ is a maximum.**

Finally, a situation may arise where U is constant over some region and hence $F_x = 0$. This is called a position of **neutral equilibrium.** Small displacements from this position produce neither restoring nor disrupting forces. A ball lying on a flat horizontal surface is an example of an object in neutral equilibrium.

EXAMPLE 8.11 Force and Energy on an Atomic Scale

The potential energy associated with the force between two neutral atoms in a molecule can be modeled by the Lennard–Jones potential energy function:

$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$

where x is the separation of the atoms. The function $U(x)$ contains two parameters σ and ϵ that are determined from experiments. Sample values for the interaction between two atoms in a molecule are $\sigma = 0.263$ nm and $\epsilon = 1.51 \times 10^{-22}$ J. (a) Using a spreadsheet or similar tool, graph this function and find the most likely distance between the two atoms.

Solution We expect to find stable equilibrium when the two atoms are separated by some equilibrium distance and the potential energy of the system of two atoms (the molecule) is a minimum. One can minimize the function $U(x)$ by taking its derivative and setting it equal to zero:

$$\frac{dU(x)}{dx} = 4\epsilon\frac{d}{dx}\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right] = 0$$

$$= 4\epsilon\left[\frac{-12\sigma^{12}}{x^{13}} - \frac{-6\sigma^{6}}{x^{7}}\right] = 0$$

Solving for x—the equilibrium separation of the two atoms in the molecule—and inserting the given information yield

$$x = 2.95 \times 10^{-10}\text{ m.}$$

We graph the Lennard–Jones function on both sides of this critical value to create our energy diagram, as shown in Figure 8.17a. Notice how $U(x)$ is extremely large when the atoms are very close together, is a minimum when the atoms are at their critical separation, and then increases again as the atoms move apart. When $U(x)$ is a minimum, the atoms are in stable equlibrium; this indicates that this is the most likely separation between them.

(b) Determine $F_x(x)$—the force that one atom exerts on the other in the molecule as a function of separation—and argue that the way this force behaves is physically plausible when the atoms are close together and far apart.

Solution Because the atoms combine to form a molecule, we reason that the force must be attractive when the atoms are far apart. On the other hand, the force must be repulsive when the two atoms get very close together. Otherwise, the molecule would collapse in on itself. Thus, the force must change sign at the critical separation, similar to the way spring forces switch sign in the change from extension to compression. Applying Equation 8.16 to the Lennard–Jones potential energy function gives

$$F_x = -\frac{dU(x)}{dx} = -4\epsilon\frac{d}{dx}\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$

$$= 4\epsilon\left[\frac{12\sigma^{12}}{x^{13}} - \frac{6\sigma^{6}}{x^{7}}\right]$$

This result is graphed in Figure 8.17b. As expected, the force is positive (repulsive) at small atomic separations, zero when the atoms are at the position of stable equilibrium [recall how we found the minimum of $U(x)$], and negative (attractive) at greater separations. Note that the force approaches zero as the separation between the atoms becomes very great.

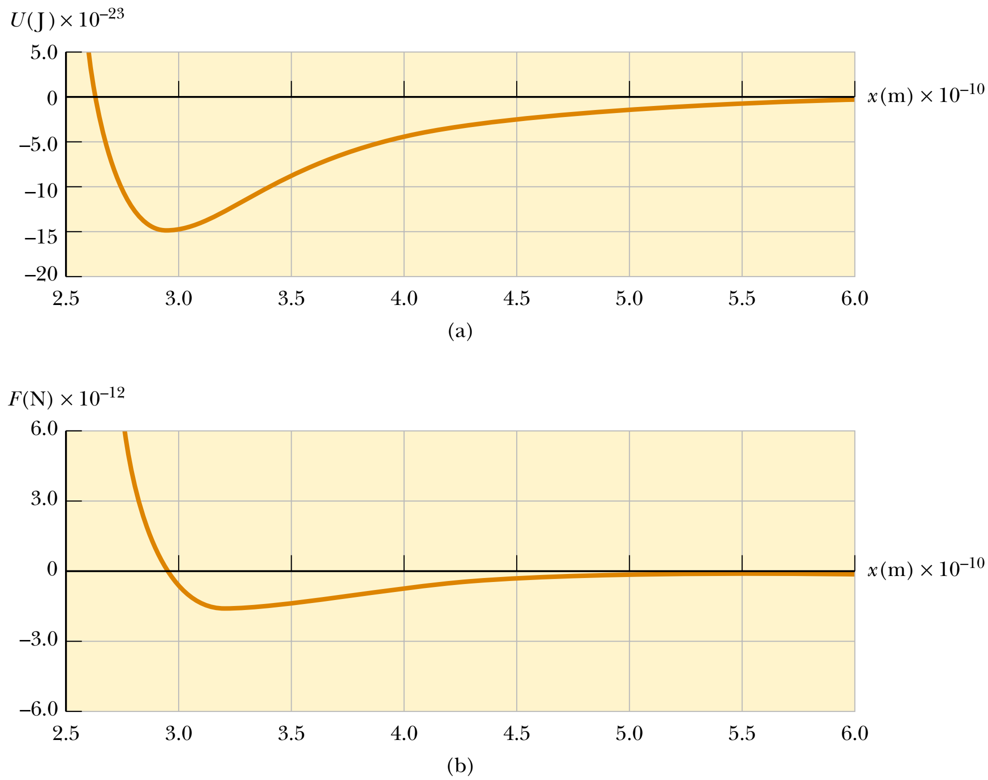


Figure 8.17 (a) Potential energy curve associated with a molecule. The distance x is the separation between the two atoms making up the molecule. (b) Force exerted on one atom by the other.

8.8 CONSERVATION OF ENERGY IN GENERAL

We have seen that the total mechanical energy of a system is constant when only conservative forces act within the system. Furthermore, we can associate a potential energy function with each conservative force. On the other hand, as we saw in Section 8.5, mechanical energy is lost when nonconservative forces such as friction are present.

In our study of thermodynamics later in this course, we shall find that mechanical energy can be transformed into energy stored *inside* the various objects that make up the system. This form of energy is called *internal energy*. For example, when a block slides over a rough surface, the mechanical energy lost because of friction is transformed into internal energy that is stored temporarily inside the block and inside the surface, as evidenced by a measurable increase in the temperature of both block and surface. We shall see that on a submicroscopic scale, this internal energy is associated with the vibration of atoms about their equilibrium positions. Such internal atomic motion involves both kinetic and potential energy. Therefore, if we include in our energy expression this increase in the internal energy of the objects that make up the system, then energy is conserved.

This is just one example of how you can analyze an isolated system and always find that the total amount of energy it contains does not change, as long as you account for all forms of energy. That is, **energy can never be created or destroyed. Energy may be transformed from one form to another, but the**

Total energy is always conserved

total energy of an isolated system is always constant. From a universal point of view, we can say that the **total energy of the Universe is constant.** If one part of the Universe gains energy in some form, then another part must lose an equal amount of energy. No violation of this principle has ever been found.

Optional Section

8.9 MASS–ENERGY EQUIVALENCE

This chapter has been concerned with the important principle of energy conservation and its application to various physical phenomena. Another important principle, **conservation of mass,** states that **in any physical or chemical process, mass is neither created nor destroyed.** That is, the mass before the process equals the mass after the process.

For centuries, scientists believed that energy and mass were two quantities that were separately conserved. However, in 1905 Einstein made the brilliant discovery that the mass of any system is a measure of the energy of that system. Hence, energy and mass are related concepts. The relationship between the two is given by Einstein's most famous formula:

$$E_R = mc^2 \qquad \textbf{(8.17)}$$

where c is the speed of light and E_R is the energy equivalent of a mass m. The subscript R on the energy refers to the **rest energy** of an object of mass m—that is, the energy of the object when its speed is $v = 0$.

The rest energy associated with even a small amount of matter is enormous. For example, the rest energy of 1 kg of any substance is

$$E_R = mc^2 = (1\text{ kg})(3 \times 10^8\text{ m/s})^2 = 9 \times 10^{16}\text{ J}$$

This is equivalent to the energy content of about 15 million barrels of crude oil—about one day's consumption in the United States! If this energy could easily be released as useful work, our energy resources would be unlimited.

In reality, only a small fraction of the energy contained in a material sample can be released through chemical or nuclear processes. The effects are greatest in nuclear reactions, in which fractional changes in energy, and hence mass, of approximately 10^{-3} are routinely observed. A good example is the enormous amount of energy released when the uranium-235 nucleus splits into two smaller nuclei. This happens because the sum of the masses of the product nuclei is slightly less than the mass of the original ^{235}U nucleus. The awesome nature of the energy released in such reactions is vividly demonstrated in the explosion of a nuclear weapon.

Equation 8.17 indicates that *energy has mass.* Whenever the energy of an object changes in any way, its mass changes as well. If ΔE is the change in energy of an object, then its change in mass is

$$\Delta m = \frac{\Delta E}{c^2} \qquad \textbf{(8.18)}$$

Anytime energy ΔE in any form is supplied to an object, the change in the mass of the object is $\Delta m = \Delta E/c^2$. However, because c^2 is so large, the changes in mass in any ordinary mechanical experiment or chemical reaction are too small to be detected.

EXAMPLE 8.12 Here Comes the Sun

The Sun converts an enormous amount of matter to energy. Each second, 4.19×10^9 kg—approximately the capacity of 400 average-sized cargo ships—is changed to energy. What is the power output of the Sun?

Solution We find the energy liberated per second by means of a straightforward conversion:

$$E_R = (4.19 \times 10^9 \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 3.77 \times 10^{26} \text{ J}$$

We then apply the definition of power:

$$\mathcal{P} = \frac{3.77 \times 10^{26} \text{ J}}{1.00 \text{ s}} = 3.77 \times 10^{26} \text{ W}$$

The Sun radiates uniformly in all directions, and so only a very tiny fraction of its total output is collected by the Earth. Nonetheless this amount is sufficient to supply energy to nearly everything on the Earth. (Nuclear and geothermal energy are the only alternatives.) Plants absorb solar energy and convert it to chemical potential energy (energy stored in the plant's molecules). When an animal eats the plant, this chemical potential energy can be turned into kinetic and other forms of energy. You are reading this book with solar-powered eyes!

Optional Section

8.10 QUANTIZATION OF ENERGY

Certain physical quantities such as electric charge are *quantized;* that is, the quantities have discrete values rather than continuous values. The quantized nature of energy is especially important in the atomic and subatomic world. As an example, let us consider the energy levels of the hydrogen atom (which consists of an electron orbiting around a proton). The atom can occupy only certain energy levels, called *quantum states,* as shown in Figure 8.18a. The atom cannot have any energy values lying between these quantum states. The lowest energy level E_1 is called the

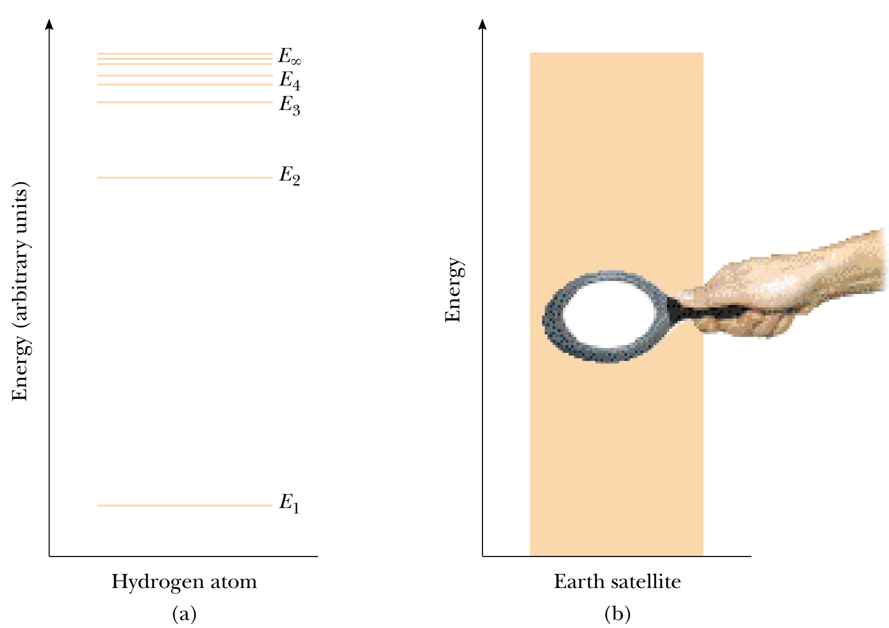


Figure 8.18 Energy-level diagrams: (a) Quantum states of the hydrogen atom. The lowest state E_1 is the ground state. (b) The energy levels of an Earth satellite are also quantized but are so close together that they cannot be distinguished from one another.

ground state of the atom. The ground state corresponds to the state that an isolated atom usually occupies. The atom can move to higher energy states by absorbing energy from some external source or by colliding with other atoms. The highest energy on the scale shown in Figure 8.18a, E_∞, corresponds to the energy of the atom when the electron is completely removed from the proton. The energy difference $E_\infty - E_1$ is called the **ionization energy.** Note that the energy levels get closer together at the high end of the scale.

Next, consider a satellite in orbit about the Earth. If you were asked to describe the possible energies that the satellite could have, it would be reasonable (but incorrect) to say that it could have any arbitrary energy value. Just like that of the hydrogen atom, however, **the energy of the satellite is quantized.** If you were to construct an energy level diagram for the satellite showing its allowed energies, the levels would be so close to one another, as shown in Figure 8.18b, that it would be difficult to discern that they were not continuous. In other words, we have no way of experiencing quantization of energy in the macroscopic world; hence, we can ignore it in describing everyday experiences.

SUMMARY

If a particle of mass m is at a distance y above the Earth's surface, the **gravitational potential energy** of the particle–Earth system is

$$U_g = mgy \quad \textbf{(8.1)}$$

The **elastic potential energy** stored in a spring of force constant k is

$$U_s \equiv \tfrac{1}{2}kx^2 \quad \textbf{(8.4)}$$

You should be able to apply these two equations in a variety of situations to determine the potential an object has to perform work.

A force is **conservative** if the work it does on a particle moving between two points is independent of the path the particle takes between the two points. Furthermore, a force is conservative if the work it does on a particle is zero when the particle moves through an arbitrary closed path and returns to its initial position. A force that does not meet these criteria is said to be **nonconservative.**

A **potential energy** function U can be associated only with a conservative force. If a conservative force **F** acts on a particle that moves along the x axis from x_i to x_f, then the change in the potential energy of the system equals the negative of the work done by that force:

$$U_f - U_i = -\int_{x_i}^{x_f} F_x\,dx \quad \textbf{(8.7)}$$

You should be able to use calculus to find the potential energy associated with a conservative force and vice versa.

The **total mechanical energy of a system** is defined as the sum of the kinetic energy and the potential energy:

$$E \equiv K + U \quad \textbf{(8.9)}$$

If no external forces do work on a system and if no nonconservative forces are acting on objects inside the system, then the total mechanical energy of the system is constant:

$$K_i + U_i = K_f + U_f \quad \textbf{(8.10)}$$

If nonconservative forces (such as friction) act on objects inside a system, then mechanical energy is not conserved. In these situations, the difference between the total final mechanical energy and the total initial mechanical energy of the system equals the energy transferred to or from the system by the nonconservative forces.

QUESTIONS

1. Many mountain roads are constructed so that they spiral around a mountain rather than go straight up the slope. Discuss this design from the viewpoint of energy and power.
2. A ball is thrown straight up into the air. At what position is its kinetic energy a maximum? At what position is the gravitational potential energy a maximum?
3. A bowling ball is suspended from the ceiling of a lecture hall by a strong cord. The bowling ball is drawn away from its equilibrium position and released from rest at the tip of the student's nose as in Figure Q8.3. If the student remains stationary, explain why she will not be struck by the ball on its return swing. Would the student be safe if she pushed the ball as she released it?

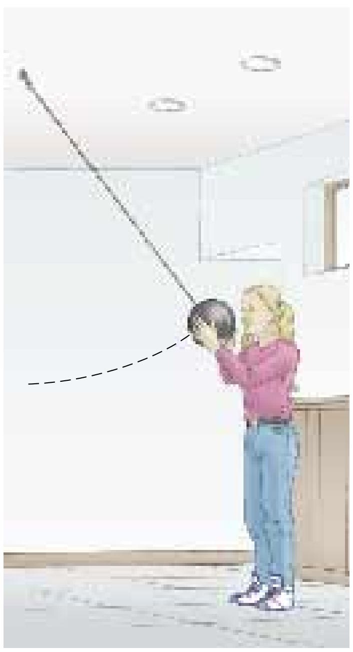


Figure Q8.3

4. One person drops a ball from the top of a building, while another person at the bottom observes its motion. Will these two people agree on the value of the potential energy of the ball–Earth system? on its change in potential energy? on the kinetic energy of the ball?
5. When a person runs in a track event at constant velocity, is any work done? (*Note:* Although the runner moves with constant velocity, the legs and arms accelerate.) How does air resistance enter into the picture? Does the center of mass of the runner move horizontally?
6. Our body muscles exert forces when we lift, push, run, jump, and so forth. Are these forces conservative?
7. If three conservative forces and one nonconservative force act on a system, how many potential energy terms appear in the equation that describes this system?
8. Consider a ball fixed to one end of a rigid rod whose other end pivots on a horizontal axis so that the rod can rotate in a vertical plane. What are the positions of stable and unstable equilibrium?
9. Is it physically possible to have a situation where $E - U < 0$?
10. What would the curve of U versus x look like if a particle were in a region of neutral equilibrium?
11. Explain the energy transformations that occur during (a) the pole vault, (b) the shot put, (c) the high jump. What is the source of energy in each case?
12. Discuss some of the energy transformations that occur during the operation of an automobile.
13. If only one external force acts on a particle, does it necessarily change the particle's (a) kinetic energy? (b) velocity?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 8.1 Potential Energy
Section 8.2 Conservative and Nonconservative Forces

1. A 1 000-kg roller coaster is initially at the top of a rise, at point A. It then moves 135 ft, at an angle of 40.0° below the horizontal, to a lower point B. (a) Choose point B to be the zero level for gravitational potential energy. Find the potential energy of the roller coaster–Earth system at points A and B and the change in its potential energy as the coaster moves. (b) Repeat part (a), setting the zero reference level at point A.

2. A 40.0-N child is in a swing that is attached to ropes 2.00 m long. Find the gravitational potential energy of the child–Earth system relative to the child's lowest position when (a) the ropes are horizontal, (b) the ropes make a 30.0° angle with the vertical, and (c) the child is at the bottom of the circular arc.

3. A 4.00-kg particle moves from the origin to position *C*, which has coordinates $x = 5.00$ m and $y = 5.00$ m (Fig. P8.3). One force on it is the force of gravity acting in the negative *y* direction. Using Equation 7.2, calculate the work done by gravity as the particle moves from *O* to *C* along (a) *OAC*, (b) *OBC*, and (c) *OC*. Your results should all be identical. Why?

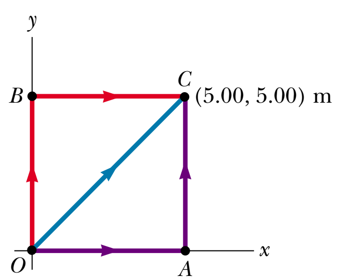


Figure P8.3 Problems 3, 4, and 5.

4. (a) Suppose that a constant force acts on an object. The force does not vary with time, nor with the position or velocity of the object. Start with the general definition for work done by a force

$$W = \int_i^f \mathbf{F} \cdot d\mathbf{s}$$

and show that the force is conservative. (b) As a special case, suppose that the force $\mathbf{F} = (3\mathbf{i} + 4\mathbf{j})$ N acts on a particle that moves from *O* to *C* in Figure P8.3. Calculate the work done by **F** if the particle moves along each one of the three paths *OAC*, *OBC*, and *OC*. (Your three answers should be identical.)

5. A force acting on a particle moving in the *xy* plane is given by $\mathbf{F} = (2y\mathbf{i} + x^2\mathbf{j})$ N, where *x* and *y* are in meters. The particle moves from the origin to a final position having coordinates $x = 5.00$ m and $y = 5.00$ m, as in Figure P8.3. Calculate the work done by **F** along (a) *OAC*, (b) *OBC*, (c) *OC*. (d) Is **F** conservative or nonconservative? Explain.

Section 8.3 Conservative Forces and Potential Energy
Section 8.4 Conservation of Mechanical Energy

6. At time t_i, the kinetic energy of a particle in a system is 30.0 J and the potential energy of the system is 10.0 J. At some later time t_f, the kinetic energy of the particle is 18.0 J. (a) If only conservative forces act on the particle, what are the potential energy and the total energy at time t_f? (b) If the potential energy of the system at time t_f is 5.00 J, are any nonconservative forces acting on the particle? Explain.

WEB **7.** A single conservative force acts on a 5.00-kg particle. The equation $F_x = (2x + 4)$ N, where *x* is in meters, describes this force. As the particle moves along the *x* axis from $x = 1.00$ m to $x = 5.00$ m, calculate (a) the work done by this force, (b) the change in the potential energy of the system, and (c) the kinetic energy of the particle at $x = 5.00$ m if its speed at $x = 1.00$ m is 3.00 m/s.

8. A single constant force $\mathbf{F} = (3\mathbf{i} + 5\mathbf{j})$ N acts on a 4.00-kg particle. (a) Calculate the work done by this force if the particle moves from the origin to the point having the vector position $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j})$ m. Does this result depend on the path? Explain. (b) What is the speed of the particle at **r** if its speed at the origin is 4.00 m/s? (c) What is the change in the potential energy of the system?

9. A single conservative force acting on a particle varies as $\mathbf{F} = (-Ax + Bx^2)\mathbf{i}$ N, where *A* and *B* are constants and *x* is in meters. (a) Calculate the potential energy function $U(x)$ associated with this force, taking $U = 0$ at $x = 0$. (b) Find the change in potential energy and change in kinetic energy as the particle moves from $x = 2.00$ m to $x = 3.00$ m.

10. A particle of mass 0.500 kg is shot from *P* as shown in Figure P8.10. The particle has an initial velocity $\mathbf{v}_i$ with a horizontal component of 30.0 m/s. The particle rises to a maximum height of 20.0 m above *P*. Using the law of conservation of energy, determine (a) the vertical component of $\mathbf{v}_i$, (b) the work done by the gravitational force on the particle during its motion from *P* to *B*, and (c) the horizontal and the vertical components of the velocity vector when the particle reaches *B*.

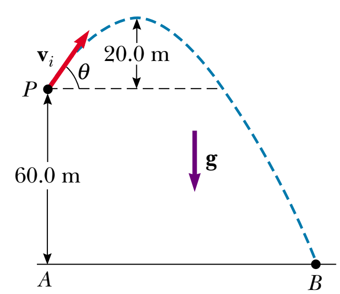


Figure P8.10

11. A 3.00-kg mass starts from rest and slides a distance *d* down a frictionless 30.0° incline. While sliding, it comes into contact with an unstressed spring of negligible mass, as shown in Figure P8.11. The mass slides an additional 0.200 m as it is brought momentarily to rest by compression of the spring ($k = 400$ N/m). Find the initial separation *d* between the mass and the spring.

12. A mass m starts from rest and slides a distance d down a frictionless incline of angle θ. While sliding, it contacts an unstressed spring of negligible mass, as shown in Figure P8.11. The mass slides an additional distance x as it is brought momentarily to rest by compression of the spring (of force constant k). Find the initial separation d between the mass and the spring.

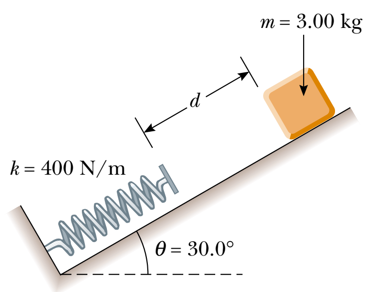


Figure P8.11 Problems 11 and 12.

13. A particle of mass m = 5.00 kg is released from point Ⓐ and slides on the frictionless track shown in Figure P8.13. Determine (a) the particle's speed at points Ⓑ and Ⓒ and (b) the net work done by the force of gravity in moving the particle from Ⓐ to Ⓒ.

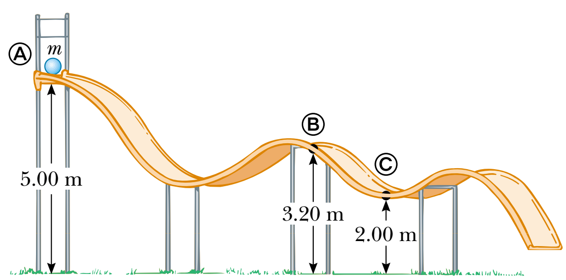


Figure P8.13

14. A simple, 2.00-m-long pendulum is released from rest when the support string is at an angle of 25.0° from the vertical. What is the speed of the suspended mass at the bottom of the swing?

15. A bead slides without friction around a loop-the-loop (Fig. P8.15). If the bead is released from a height h = 3.50R, what is its speed at point A? How great is the normal force on it if its mass is 5.00 g?

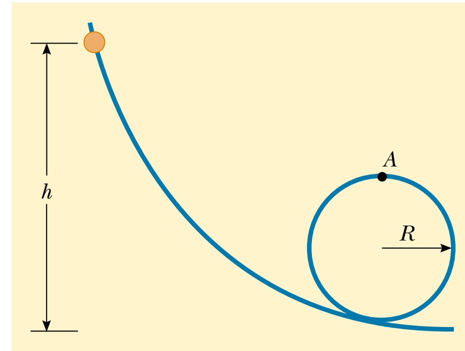


Figure P8.15

16. A 120-g mass is attached to the bottom end of an unstressed spring. The spring is hanging vertically and has a spring constant of 40.0 N/m. The mass is dropped. (a) What is its maximum speed? (b) How far does it drop before coming to rest momentarily?

17. A block of mass 0.250 kg is placed on top of a light vertical spring of constant k = 5 000 N/m and is pushed downward so that the spring is compressed 0.100 m. After the block is released, it travels upward and then leaves the spring. To what maximum height above the point of release does it rise?

18. Dave Johnson, the bronze medalist at the 1992 Olympic decathlon in Barcelona, leaves the ground for his high jump with a vertical velocity component of 6.00 m/s. How far up does his center of gravity move as he makes the jump?

19. A 0.400-kg ball is thrown straight up into the air and reaches a maximum altitude of 20.0 m. Taking its initial position as the point of zero potential energy and using energy methods, find (a) its initial speed, (b) its total mechanical energy, and (c) the ratio of its kinetic energy to the potential energy of the ball–Earth system when the ball is at an altitude of 10.0 m.

20. In the dangerous "sport" of bungee-jumping, a daring student jumps from a balloon with a specially designed

Figure P8.20 Bungee-jumping. *(Gamma)* elastic cord attached to his ankles, as shown in Figure P8.20. The unstretched length of the cord is 25.0 m, the student weighs 700 N, and the balloon is 36.0 m above the surface of a river below. Assuming that Hooke's law describes the cord, calculate the required force constant if the student is to stop safely 4.00 m above the river.

21. Two masses are connected by a light string passing over a light frictionless pulley, as shown in Figure P8.21. The 5.00-kg mass is released from rest. Using the law of conservation of energy, (a) determine the speed of the 3.00-kg mass just as the 5.00-kg mass hits the ground and (b) find the maximum height to which the 3.00-kg mass rises.

22. Two masses are connected by a light string passing over a light frictionless pulley, as shown in Figure P8.21. The mass m_1 (which is greater than m_2) is released from rest. Using the law of conservation of energy, (a) determine the speed of m_2 just as m_1 hits the ground in terms of m_1, m_2, and h, and (b) find the maximum height to which m_2 rises.

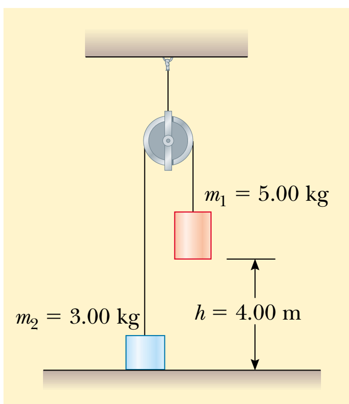


Figure P8.21 Problems 21 and 22.

23. A 20.0-kg cannon ball is fired from a cannon with a muzzle speed of 1 000 m/s at an angle of 37.0° with the horizontal. A second ball is fired at an angle of 90.0°. Use the law of conservation of mechanical energy to find (a) the maximum height reached by each ball and (b) the total mechanical energy at the maximum height for each ball. Let $y = 0$ at the cannon.

24. A 2.00-kg ball is attached to the bottom end of a length of 10-lb (44.5-N) fishing line. The top end of the fishing line is held stationary. The ball is released from rest while the line is taut and horizontal ($\theta = 90.0°$). At what angle θ (measured from the vertical) will the fishing line break?

25. The circus apparatus known as the *trapeze* consists of a bar suspended by two parallel ropes, each of length ℓ. The trapeze allows circus performers to swing in a vertical circular arc (Fig. P8.25). Suppose a performer with mass m and holding the bar steps off an elevated platform, starting from rest with the ropes at an angle of θ_i with respect to the vertical. Suppose the size of the performer's body is small compared with the length ℓ, that she does not pump the trapeze to swing higher, and that air resistance is negligible. (a) Show that when the ropes make an angle of θ with respect to the vertical, the performer must exert a force

$$F = mg\,(3\cos\theta - 2\cos\theta_i)$$

in order to hang on. (b) Determine the angle θ_i at which the force required to hang on at the bottom of the swing is twice the performer's weight.

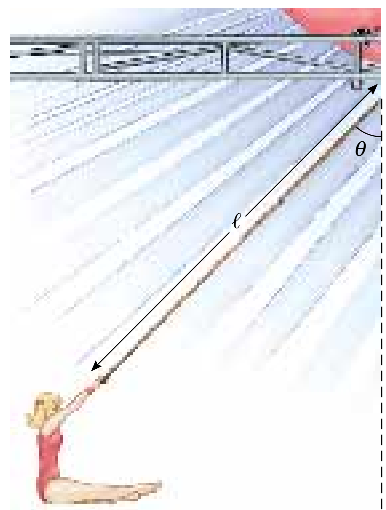


Figure P8.25

26. After its release at the top of the first rise, a roller-coaster car moves freely with negligible friction. The roller coaster shown in Figure P8.26 has a circular loop of radius 20.0 m. The car barely makes it around the loop: At the top of the loop, the riders are upside down and feel weightless. (a) Find the speed of the roller coaster car at the top of the loop (position 3). Find the speed of the roller coaster car (b) at position 1 and (c) at position 2. (d) Find the difference in height between positions 1 and 4 if the speed at position 4 is 10.0 m/s.

27. A light rigid rod is 77.0 cm long. Its top end is pivoted on a low-friction horizontal axle. The rod hangs straight down at rest, with a small massive ball attached to its bottom end. You strike the ball, suddenly giving it a horizontal velocity so that it swings around in a full circle. What minimum speed at the bottom is required to make the ball go over the top of the circle?

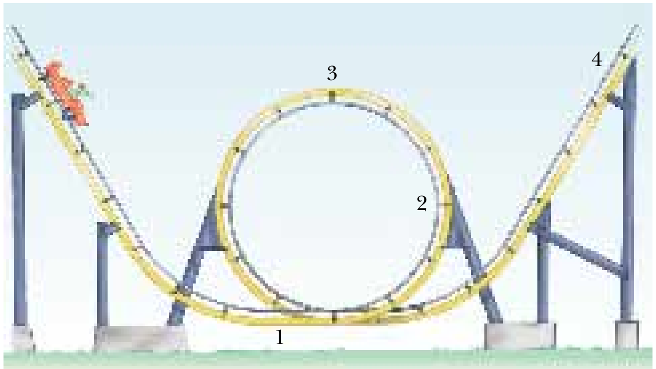


Figure P8.26

Section 8.5 Work Done by Nonconservative Forces

28. A 70.0-kg diver steps off a 10.0-m tower and drops straight down into the water. If he comes to rest 5.00 m beneath the surface of the water, determine the average resistance force that the water exerts on the diver.

29. A force F_x, shown as a function of distance in Figure P8.29, acts on a 5.00-kg mass. If the particle starts from rest at $x = 0$ m, determine the speed of the particle at $x = 2.00$, 4.00, and 6.00 m.

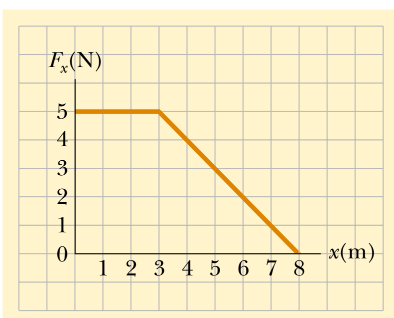


Figure P8.29

30. A softball pitcher swings a ball of mass 0.250 kg around a vertical circular path of radius 60.0 cm before releasing it from her hand. The pitcher maintains a component of force on the ball of constant magnitude 30.0 N in the direction of motion around the complete path. The speed of the ball at the top of the circle is 15.0 m/s. If the ball is released at the bottom of the circle, what is its speed upon release?

WEB **31.** The coefficient of friction between the 3.00-kg block and the surface in Figure P8.31 is 0.400. The system starts from rest. What is the speed of the 5.00-kg ball when it has fallen 1.50 m?

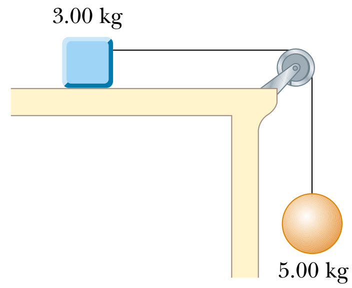


Figure P8.31

32. A 2 000-kg car starts from rest and coasts down from the top of a 5.00-m-long driveway that is sloped at an angle of 20.0° with the horizontal. If an average friction force of 4 000 N impedes the motion of the car, find the speed of the car at the bottom of the driveway.

33. A 5.00-kg block is set into motion up an inclined plane with an initial speed of 8.00 m/s (Fig. P8.33). The block comes to rest after traveling 3.00 m along the plane, which is inclined at an angle of 30.0° to the horizontal. For this motion determine (a) the change in the block's kinetic energy, (b) the change in the potential energy, and (c) the frictional force exerted on it (assumed to be constant). (d) What is the coefficient of kinetic friction?

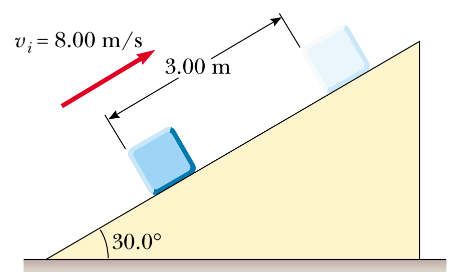


Figure P8.33

34. A boy in a wheelchair (total mass, 47.0 kg) wins a race with a skateboarder. He has a speed of 1.40 m/s at the crest of a slope 2.60 m high and 12.4 m long. At the bottom of the slope, his speed is 6.20 m/s. If air resistance and rolling resistance can be modeled as a constant frictional force of 41.0 N, find the work he did in pushing forward on his wheels during the downhill ride.

35. A parachutist of mass 50.0 kg jumps out of a balloon at a height of 1 000 m and lands on the ground with a speed of 5.00 m/s. How much energy was lost to air friction during this jump?

36. An 80.0-kg sky diver jumps out of a balloon at an altitude of 1 000 m and opens the parachute at an altitude of 200.0 m. (a) Assuming that the total retarding force on the diver is constant at 50.0 N with the parachute closed and constant at 3 600 N with the parachute open, what is the speed of the diver when he lands on the ground? (b) Do you think the sky diver will get hurt? Explain. (c) At what height should the parachute be opened so that the final speed of the sky diver when he hits the ground is 5.00 m/s? (d) How realistic is the assumption that the total retarding force is constant? Explain.

37. A toy cannon uses a spring to project a 5.30-g soft rubber ball. The spring is originally compressed by 5.00 cm and has a stiffness constant of 8.00 N/m. When it is fired, the ball moves 15.0 cm through the barrel of the cannon, and there is a constant frictional force of 0.032 0 N between the barrel and the ball. (a) With what speed does the projectile leave the barrel of the cannon? (b) At what point does the ball have maximum speed? (c) What is this maximum speed?

38. A 1.50-kg mass is held 1.20 m above a relaxed, massless vertical spring with a spring constant of 320 N/m. The mass is dropped onto the spring. (a) How far does it compress the spring? (b) How far would it compress the spring if the same experiment were performed on the Moon, where $g = 1.63\ \text{m/s}^2$? (c) Repeat part (a), but this time assume that a constant air-resistance force of 0.700 N acts on the mass during its motion.

39. A 3.00-kg block starts at a height $h = 60.0$ cm on a plane that has an inclination angle of 30.0°, as shown in Figure P8.39. Upon reaching the bottom, the block slides along a horizontal surface. If the coefficient of friction on both surfaces is $\mu_k = 0.200$, how far does the block slide on the horizontal surface before coming to rest? (*Hint:* Divide the path into two straight-line parts.)

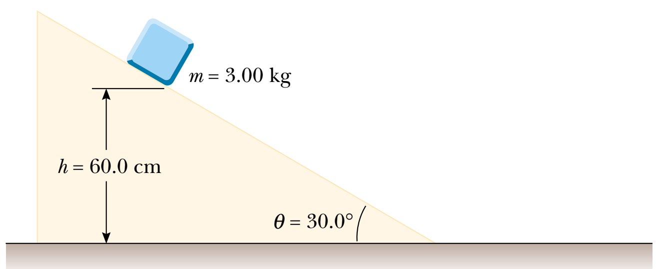


Figure P8.39

40. A 75.0-kg sky diver is falling with a terminal speed of 60.0 m/s. Determine the rate at which he is losing mechanical energy.

Section 8.6 Relationship Between Conservative Forces and Potential Energy

WEB **41.** The potential energy of a two-particle system separated by a distance r is given by $U(r) = A/r$, where A is a constant. Find the radial force $\mathbf{F}_r$ that each particle exerts on the other.

42. A potential energy function for a two-dimensional force is of the form $U = 3x^3y - 7x$. Find the force that acts at the point (x, y).

(Optional)

Section 8.7 Energy Diagrams and the Equilibrium of a System

43. A particle moves along a line where the potential energy depends on its position r, as graphed in Figure P8.43. In the limit as r increases without bound, $U(r)$ approaches $+1$ J. (a) Identify each equilibrium position for this particle. Indicate whether each is a point of stable, unstable, or neutral equilibrium. (b) The particle will be bound if its total energy is in what range? Now suppose the particle has energy -3 J. Determine (c) the range of positions where it can be found, (d) its maximum kinetic energy, (e) the location at which it has maximum kinetic energy, and (f) its *binding energy*—that is, the additional energy that it would have to be given in order for it to move out to $r \to \infty$.

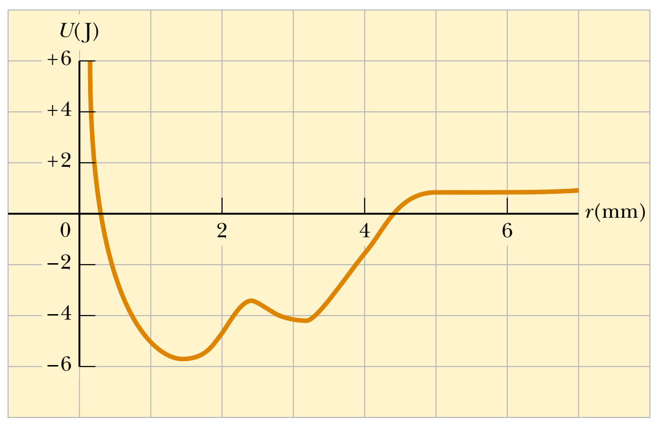


Figure P8.43

44. A right circular cone can be balanced on a horizontal surface in three different ways. Sketch these three equilibrium configurations and identify them as positions of stable, unstable, or neutral equilibrium.

45. For the potential energy curve shown in Figure P8.45, (a) determine whether the force F_x is positive, negative, or zero at the five points indicated. (b) Indicate points of stable, unstable, and neutral equilibrium. (c) Sketch the curve for F_x versus x from $x = 0$ to $x = 9.5$ m.

46. A hollow pipe has one or two weights attached to its inner surface, as shown in Figure P8.46. Characterize each configuration as being stable, unstable, or neutral equilibrium and explain each of your choices ("CM" indicates center of mass).

47. A particle of mass m is attached between two identical springs on a horizontal frictionless tabletop. The

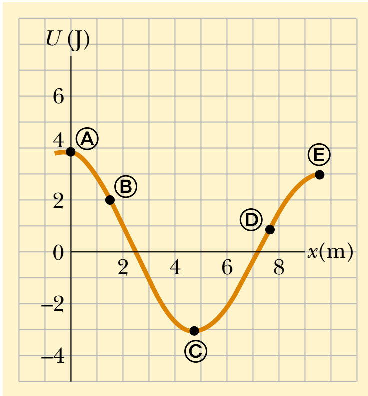


Figure P8.45

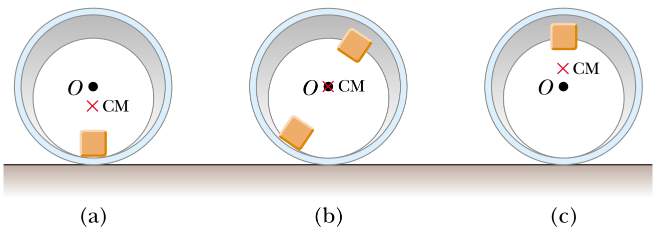


Figure P8.46

springs have spring constant k, and each is initially unstressed. (a) If the mass is pulled a distance x along a direction perpendicular to the initial configuration of the springs, as in Figure P8.47, show that the potential energy of the system is

$$U(x) = kx^2 + 2kL\left(L - \sqrt{x^2 + L^2}\right)$$

(*Hint:* See Problem 66 in Chapter 7.) (b) Make a plot of $U(x)$ versus x and identify all equilibrium points. Assume that $L = 1.20$ m and $k = 40.0$ N/m. (c) If the mass is pulled 0.500 m to the right and then released, what is its speed when it reaches the equilibrium point $x = 0$?

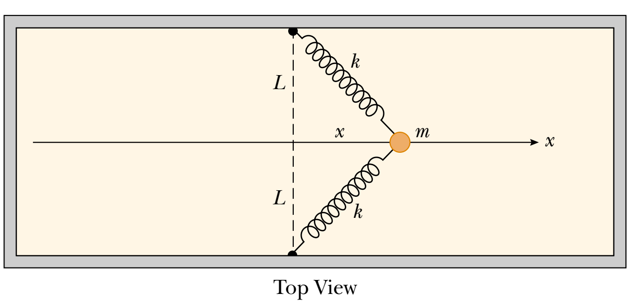


Figure P8.47

(Optional)

Section 8.9 Mass–Energy Equivalence

48. Find the energy equivalents of (a) an electron of mass 9.11×10^{-31} kg, (b) a uranium atom with a mass of 4.00×10^{-25} kg, (c) a paper clip of mass 2.00 g, and (d) the Earth (of mass 5.99×10^{24} kg).

49. The expression for the kinetic energy of a particle moving with speed v is given by Equation 7.19, which can be written as $K = \gamma mc^2 - mc^2$, where $\gamma = [1 - (v/c)^2]^{-1/2}$. The term γmc^2 is the total energy of the particle, and the term mc^2 is its rest energy. A proton moves with a speed of $0.990c$, where c is the speed of light. Find (a) its rest energy, (b) its total energy, and (c) its kinetic energy.

ADDITIONAL PROBLEMS

50. A block slides down a curved frictionless track and then up an inclined plane as in Figure P8.50. The coefficient of kinetic friction between the block and the incline is μ_k. Use energy methods to show that the maximum height reached by the block is

$$y_{max} = \frac{h}{1 + \mu_k \cot \theta}$$

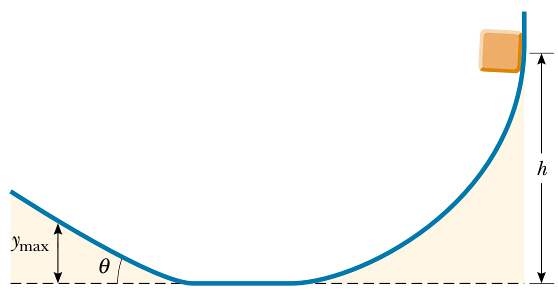


Figure P8.50

51. Close to the center of a campus is a tall silo topped with a hemispherical cap. The cap is frictionless when wet. Someone has somehow balanced a pumpkin at the highest point. The line from the center of curvature of the cap to the pumpkin makes an angle $\theta_i = 0°$ with the vertical. On a rainy night, a breath of wind makes the pumpkin start sliding downward from rest. It loses contact with the cap when the line from the center of the hemisphere to the pumpkin makes a certain angle with the vertical; what is this angle?

52. A 200-g particle is released from rest at point Ⓐ along the horizontal diameter on the inside of a frictionless, hemispherical bowl of radius $R = 30.0$ cm (Fig. P8.52). Calculate (a) the gravitational potential energy when the particle is at point Ⓐ relative to point Ⓑ, (b) the kinetic energy of the particle at point Ⓑ, (c) its speed at point Ⓑ, and (d) its kinetic energy and the potential energy at point Ⓒ.

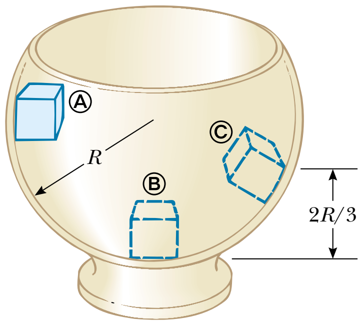


Figure P8.52 Problems 52 and 53.

WEB **53.** The particle described in Problem 52 (Fig. P8.52) is released from rest at Ⓐ, and the surface of the bowl is rough. The speed of the particle at Ⓑ is 1.50 m/s. (a) What is its kinetic energy at Ⓑ? (b) How much energy is lost owing to friction as the particle moves from Ⓐ to Ⓑ? (c) Is it possible to determine μ from these results in any simple manner? Explain.

54. **Review Problem.** The mass of a car is 1 500 kg. The shape of the body is such that its aerodynamic drag coefficient is $D = 0.330$ and the frontal area is 2.50 m^2. Assuming that the drag force is proportional to v^2 and neglecting other sources of friction, calculate the power the car requires to maintain a speed of 100 km/h as it climbs a long hill sloping at 3.20°.

55. Make an order-of-magnitude estimate of your power output as you climb stairs. In your solution, state the physical quantities you take as data and the values you measure or estimate for them. Do you consider your peak power or your sustainable power?

56. A child's pogo stick (Fig. P8.56) stores energy in a spring ($k = 2.50 \times 10^4$ N/m). At position Ⓐ ($x_A = -0.100$ m), the spring compression is a maximum and the child is momentarily at rest. At position Ⓑ ($x_B = 0$), the spring is relaxed and the child is moving upward. At position Ⓒ, the child is again momentarily at rest at the top of the jump. Assuming that the combined mass of the child and the pogo stick is 25.0 kg, (a) calculate the total energy of the system if both potential energies are zero at $x = 0$, (b) determine x_C, (c) calculate the speed of the child at $x = 0$, (d) determine the value of x for which the kinetic energy of the system is a maximum, and (e) calculate the child's maximum upward speed.

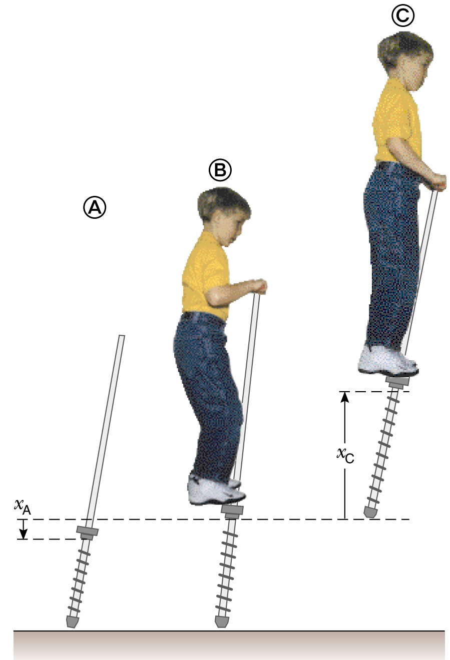


Figure P8.56

57. A 10.0-kg block is released from point Ⓐ in Figure P8.57. The track is frictionless except for the portion between Ⓑ and Ⓒ, which has a length of 6.00 m. The block travels down the track, hits a spring of force constant $k = 2\,250$ N/m, and compresses the spring 0.300 m from its equilibrium position before coming to rest momentarily. Determine the coefficient of kinetic friction between the block and the rough surface between Ⓑ and Ⓒ.

58. A 2.00-kg block situated on a rough incline is connected to a spring of negligible mass having a spring constant of 100 N/m (Fig. P8.58). The pulley is frictionless. The block is released from rest when the spring is unstretched. The block moves 20.0 cm down the incline before coming to rest. Find the coefficient of kinetic friction between block and incline.

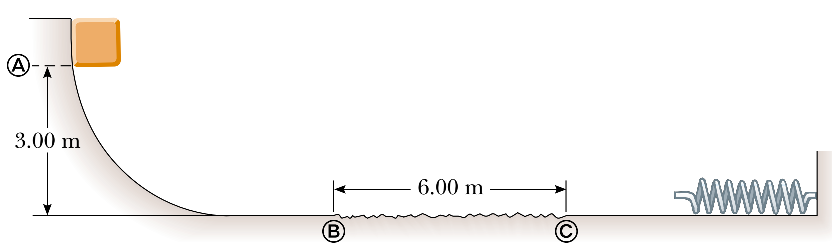


Figure P8.57

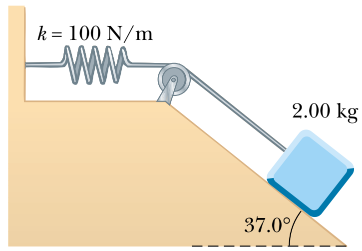


Figure P8.58 Problems 58 and 59.

59. **Review Problem.** Suppose the incline is frictionless for the system described in Problem 58 (see Fig. P8.58). The block is released from rest with the spring initially unstretched. (a) How far does it move down the incline before coming to rest? (b) What is its acceleration at its lowest point? Is the acceleration constant? (c) Describe the energy transformations that occur during the descent.

60. The potential energy function for a system is given by $U(x) = -x^3 + 2x^2 + 3x$. (a) Determine the force F_x as a function of x. (b) For what values of x is the force equal to zero? (c) Plot $U(x)$ versus x and F_x versus x, and indicate points of stable and unstable equilibrium.

61. A 20.0-kg block is connected to a 30.0-kg block by a string that passes over a frictionless pulley. The 30.0-kg block is connected to a spring that has negligible mass and a force constant of 250 N/m, as shown in Figure P8.61. The spring is unstretched when the system is as shown in the figure, and the incline is frictionless. The 20.0-kg block is pulled 20.0 cm down the incline (so that the 30.0-kg block is 40.0 cm above the floor) and is released from rest. Find the speed of each block when the 30.0-kg block is 20.0 cm above the floor (that is, when the spring is unstretched).

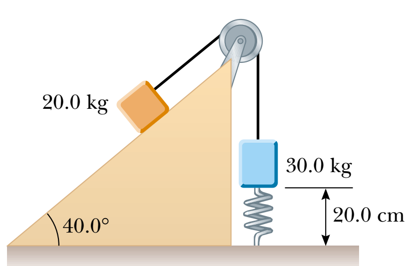


Figure P8.61

62. A 1.00-kg mass slides to the right on a surface having a coefficient of friction $\mu = 0.250$ (Fig. P8.62). The mass has a speed of $v_i = 3.00$ m/s when it makes contact with a light spring that has a spring constant $k = 50.0$ N/m. The mass comes to rest after the spring has been compressed a distance d. The mass is then forced toward the left by the spring and continues to move in that direction beyond the spring's unstretched position. Finally, the mass comes to rest at a distance D to the left of the unstretched spring. Find (a) the distance of compression d, (b) the speed v of the mass at the unstretched position when the mass is moving to the left, and (c) the distance D between the unstretched spring and the point at which the mass comes to rest.

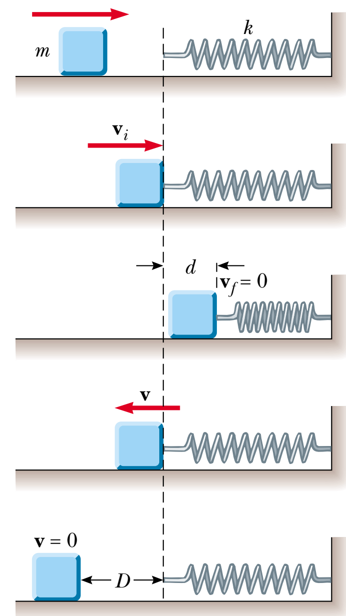


Figure P8.62

WEB **63.** A block of mass 0.500 kg is pushed against a horizontal spring of negligible mass until the spring is compressed a distance Δx (Fig. P8.63). The spring constant is 450 N/m. When it is released, the block travels along a frictionless, horizontal surface to point B, at the bottom of a vertical circular track of radius $R = 1.00$ m, and continues to move up the track. The speed of the block at the bottom of the track is $v_B = 12.0$ m/s, and the block experiences an average frictional force of 7.00 N while sliding up the track. (a) What is Δx? (b) What speed do you predict for the block at the top of the track? (c) Does the block actually reach the top of the track, or does it fall off before reaching the top?

64. A uniform chain of length 8.00 m initially lies stretched out on a horizontal table. (a) If the coefficient of static friction between the chain and the table is 0.600, show that the chain will begin to slide off the table if at least 3.00 m of it hangs over the edge of the table. (b) Determine the speed of the chain as all of it leaves the table, given that the coefficient of kinetic friction between the chain and the table is 0.400.

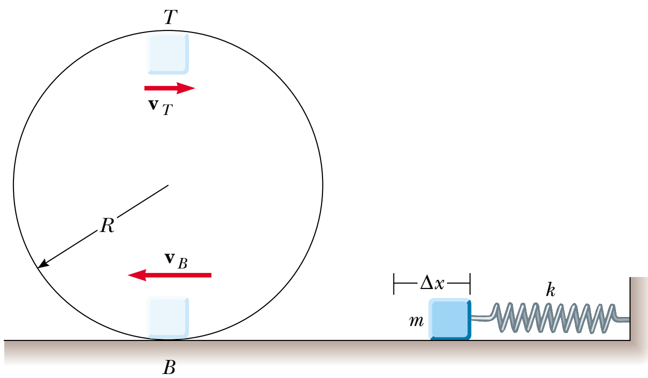


Figure P8.63

65. An object of mass m is suspended from a post on top of a cart by a string of length L as in Figure P8.65a. The cart and object are initially moving to the right at constant speed v_i. The cart comes to rest after colliding and sticking to a bumper as in Figure P8.65b, and the suspended object swings through an angle θ. (a) Show that the speed is $v_i = \sqrt{2gL(1 - \cos\theta)}$. (b) If $L = 1.20$ m and $\theta = 35.0°$, find the initial speed of the cart. (*Hint:* The force exerted by the string on the object does no work on the object.)

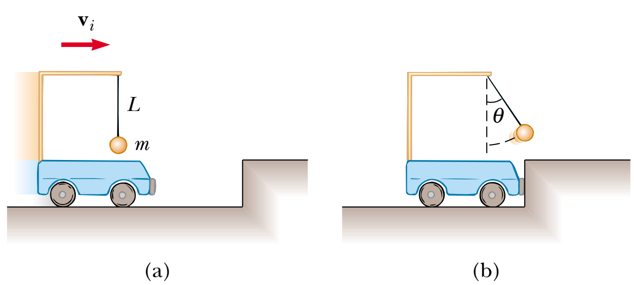


Figure P8.65

66. A child slides without friction from a height h along a curved water slide (Fig. P8.66). She is launched from a height $h/5$ into the pool. Determine her maximum airborne height y in terms of h and θ.

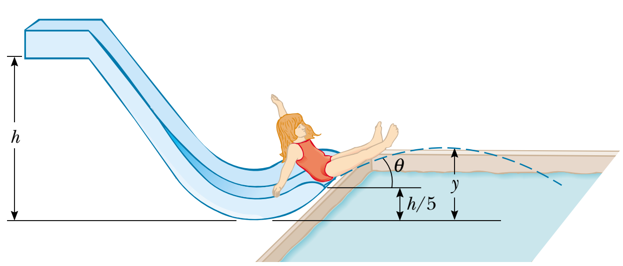


Figure P8.66

67. A ball having mass m is connected by a strong string of length L to a pivot point and held in place in a vertical position. A wind exerting constant force of magnitude F is blowing from left to right as in Figure P8.67a. (a) If the ball is released from rest, show that the maximum height H it reaches, as measured from its initial height, is

$$H = \frac{2L}{1 + (mg/F)^2}$$

Check that the above formula is valid both when $0 \le H \le L$ and when $L \le H \le 2L$. (*Hint:* First determine the potential energy associated with the *constant* wind force.) (b) Compute the value of H using the values $m = 2.00$ kg, $L = 2.00$ m, and $F = 14.7$ N. (c) Using these same values, determine the *equilibrium* height of the ball. (d) Could the equilibrium height ever be greater than L? Explain.

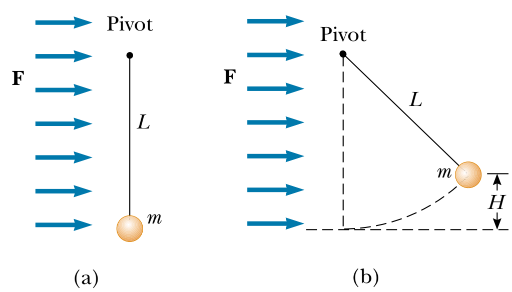


Figure P8.67

68. A ball is tied to one end of a string. The other end of the string is fixed. The ball is set in motion around a vertical circle without friction. At the top of the circle, the ball has a speed of $v_i = \sqrt{Rg}$, as shown in Figure P8.68. At what angle θ should the string be cut so that the ball will travel through the center of the circle?

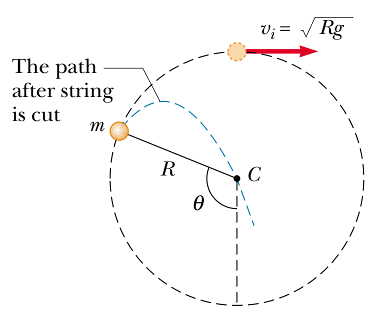


Figure P8.68

69. A ball at the end of a string whirls around in a vertical circle. If the ball's total energy remains constant, show that the tension in the string at the bottom is greater than the tension at the top by a value six times the weight of the ball.

70. A pendulum comprising a string of length L and a sphere swings in the vertical plane. The string hits a peg located a distance d below the point of suspension (Fig. P8.70). (a) Show that if the sphere is released from a height below that of the peg, it will return to this height after striking the peg. (b) Show that if the pendulum is released from the horizontal position ($\theta = 90°$) and is to swing in a complete circle centered on the peg, then the minimum value of d must be $3L/5$.

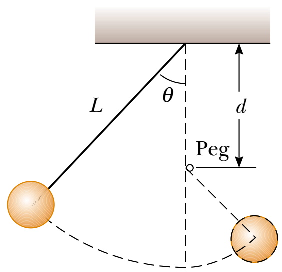


Figure P8.70

71. Jane, whose mass is 50.0 kg, needs to swing across a river (having width D) filled with man-eating crocodiles to save Tarzan from danger. However, she must swing into a wind exerting constant horizontal force **F** on a vine having length L and initially making an angle θ with the vertical (Fig. P8.71). Taking $D = 50.0$ m, $F = 110$ N, $L = 40.0$ m, and $\theta = 50.0°$, (a) with what minimum speed must Jane begin her swing to just make it to the other side? (*Hint:* First determine the potential energy associated with the wind force.) (b) Once the rescue is complete, Tarzan and Jane must swing back across the river. With what minimum speed must they begin their swing? Assume that Tarzan has a mass of 80.0 kg.

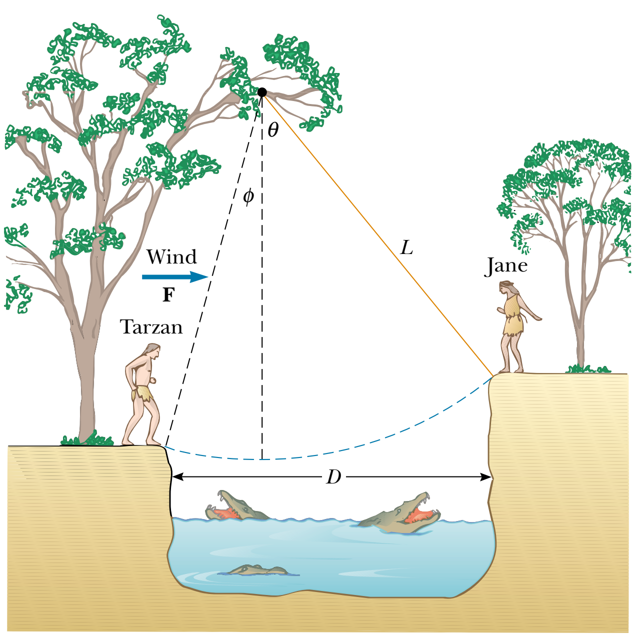


Figure P8.71

72. A child starts from rest and slides down the frictionless slide shown in Figure P8.72. In terms of R and H, at what height h will he lose contact with the section of radius R?

Figure P8.72

73. A 5.00-kg block free to move on a horizontal, frictionless surface is attached to one end of a light horizontal spring. The other end of the spring is fixed. The spring is compressed 0.100 m from equilibrium and is then released. The speed of the block is 1.20 m/s when it passes the equilibrium position of the spring. The same experiment is now repeated with the frictionless surface replaced by a surface for which $\mu_k = 0.300$. Determine the speed of the block at the equilibrium position of the spring.

74. A 50.0-kg block and a 100-kg block are connected by a string as in Figure P8.74. The pulley is frictionless and of negligible mass. The coefficient of kinetic friction between the 50.0-kg block and the incline is $\mu_k = 0.250$. Determine the change in the kinetic energy of the 50.0-kg block as it moves from Ⓐ to Ⓑ, a distance of 20.0 m.

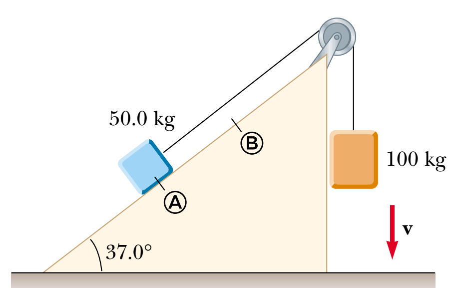


Figure P8.74

Answers to Quick Quizzes

8.1 Yes, because we are free to choose any point whatsoever as our origin of coordinates, which is the $U_g = 0$ point. If the object is below the origin of coordinates that we choose, then $U_g < 0$ for the object–Earth system.

8.2 Yes, the total mechanical energy of the system is conserved because the only forces acting are conservative: the force of gravity and the spring force. There are two forms of potential energy: (1) gravitational potential energy and (2) elastic potential energy stored in the spring.

8.3 The first and third balls speed up after they are thrown, while the second ball initially slows down but then speeds up after reaching its peak. The paths of all three balls are parabolas, and the balls take different times to reach the ground because they have different initial velocities. However, all three balls have the same speed at the moment they hit the ground because all start with the same kinetic energy and undergo the same change in gravitational potential energy. In other words, $E_{\text{total}} = \frac{1}{2}mv^2 + mgh$ is the same for all three balls at the start of the motion.

8.4 Designate one object as No. 1 and the other as No. 2. The external force does work W_{app} on the system. If $W_{\text{app}} > 0$, then the system energy increases. If $W_{\text{app}} < 0$, then the system energy decreases. The effect of friction is to decrease the total system energy. Equation 8.15 then becomes

$$\begin{aligned}\Delta E &= W_{\text{app}} - \Delta E_{\text{friction}} \\ &= \Delta K + \Delta U \\ &= [K_{1f} + K_{2f}) - (K_{1i} + K_{2i})] \\ &\quad + [(U_{g1f} + U_{g2f} + U_{sf}) - (U_{g1i} + U_{g2i} + U_{si})]\end{aligned}$$

You may find it easier to think of this equation with its terms in a different order, saying

total initial energy + net change = total final energy

$$\begin{aligned}K_{1i} + K_{2i} + U_{g1i} + U_{g2i} + U_{si} + W_{\text{app}} - f_k d = \\ K_{1f} + K_{2f} + U_{g1f} + U_{g2f} + U_{sf}\end{aligned}$$

8.5 The slope of a $U(x)$-versus-x graph is by definition $dU(x)/dx$. From Equation 8.16, we see that this expression is equal to the negative of the x component of the conservative force acting on an object that is part of the system.

PUZZLER

Airbags have saved countless lives by reducing the forces exerted on vehicle occupants during collisions. How can airbags change the force needed to bring a person from a high speed to a complete stop? Why are they usually safer than seat belts alone? *(Courtesy of Saab)*

chapter 9

Linear Momentum and Collisions

Chapter Outline

Consider what happens when a golf ball is struck by a club. The ball is given a very large initial velocity as a result of the collision; consequently, it is able to travel more than 100 m through the air. The ball experiences a large acceleration. Furthermore, because the ball experiences this acceleration over a very short time interval, the average force exerted on it during the collision is very great. According to Newton's third law, the ball exerts on the club a reaction force that is equal in magnitude to and opposite in direction to the force exerted by the club on the ball. This reaction force causes the club to accelerate. Because the club is much more massive than the ball, however, the acceleration of the club is much less than the acceleration of the ball.

One of the main objectives of this chapter is to enable you to understand and analyze such events. As a first step, we introduce the concept of *momentum,* which is useful for describing objects in motion and as an alternate and more general means of applying Newton's laws. For example, a very massive football player is often said to have a great deal of momentum as he runs down the field. A much less massive player, such as a halfback, can have equal or greater momentum if his speed is greater than that of the more massive player. This follows from the fact that momentum is defined as the product of mass and velocity. The concept of momentum leads us to a second conservation law, that of conservation of momentum. This law is especially useful for treating problems that involve collisions between objects and for analyzing rocket propulsion. The concept of the center of mass of a system of particles also is introduced, and we shall see that the motion of a system of particles can be described by the motion of one representative particle located at the center of mass.

9.1 LINEAR MOMENTUM AND ITS CONSERVATION

In the preceding two chapters we studied situations too complex to analyze easily with Newton's laws. In fact, Newton himself used a form of his second law slightly different from $\Sigma \mathbf{F} = m\mathbf{a}$ (Eq. 5.2)—a form that is considerably easier to apply in complicated circumstances. Physicists use this form to study everything from subatomic particles to rocket propulsion. In studying situations such as these, it is often useful to know both something about the object and something about its motion. We start by defining a new term that incorporates this information:

Definition of linear momentum of a particle

The **linear momentum** of a particle of mass m moving with a velocity $\mathbf{v}$ is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v} \qquad \textbf{(9.1)}$$

6.2

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity $\mathbf{v}$. Its direction is along $\mathbf{v}$, it has dimensions ML/T, and its SI unit is kg · m/s.

If a particle is moving in an arbitrary direction, $\mathbf{p}$ must have three components, and Equation 9.1 is equivalent to the component equations

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \qquad \textbf{(9.2)}$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product $m\mathbf{v}$

quantity of motion; this is perhaps a more graphic description than our present-day word *momentum,* which comes from the Latin word for movement.

Quick Quiz 9.1

Two objects have equal kinetic energies. How do the magnitudes of their momenta compare? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) not enough information to tell.

Newton's second law for a particle

Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle: **The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle:**

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} \qquad (9.3)$$

In addition to situations in which the velocity vector varies with time, we can use Equation 9.3 to study phenomena in which the mass changes. The real value of Equation 9.3 as a tool for analysis, however, stems from the fact that when the net force acting on a particle is zero, the time derivative of the momentum of the particle is zero, and therefore its linear momentum[1] is constant. Of course, if the particle is *isolated,* then by necessity $\Sigma \mathbf{F} = 0$ and $\mathbf{p}$ remains unchanged. This means that $\mathbf{p}$ is conserved. Just as the law of conservation of energy is useful in solving complex motion problems, the law of conservation of momentum can greatly simplify the analysis of other types of complicated motion.

Conservation of Momentum for a Two-Particle System

6.2 Consider two particles 1 and 2 that can interact with each other but are isolated from their surroundings (Fig. 9.1). That is, the particles may exert a force on each other, but no external forces are present. It is important to note the impact of Newton's third law on this analysis. If an internal force from particle 1 (for example, a gravitational force) acts on particle 2, then there must be a second internal force—equal in magnitude but opposite in direction—that particle 2 exerts on particle 1.

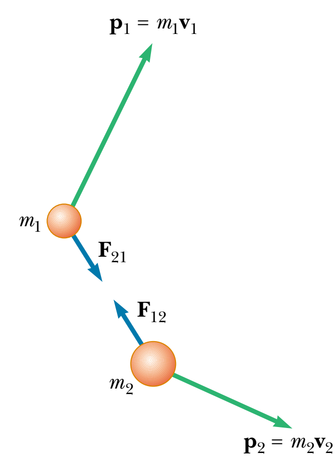


Figure 9.1 At some instant, the momentum of particle 1 is $\mathbf{p}_1 = m_1\mathbf{v}_1$ and the momentum of particle 2 is $\mathbf{p}_2 = m_2\mathbf{v}_2$. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The total momentum of the system $\mathbf{p}_{\text{tot}}$ is equal to the vector sum $\mathbf{p}_1 + \mathbf{p}_2$.

Suppose that at some instant, the momentum of particle 1 is $\mathbf{p}_1$ and that of particle 2 is $\mathbf{p}_2$. Applying Newton's second law to each particle, we can write

$$\mathbf{F}_{21} = \frac{d\mathbf{p}_1}{dt} \qquad \text{and} \qquad \mathbf{F}_{12} = \frac{d\mathbf{p}_2}{dt}$$

where $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1 and $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2. Newton's third law tells us that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal in magnitude and opposite in direction. That is, they form an action–reaction pair $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can express this condition as

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

or as

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

[1]In this chapter, the terms *momentum* and *linear momentum* have the same meaning. Later, in Chapter 11, we shall use the term *angular momentum* when dealing with rotational motion.

Because the time derivative of the total momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2$ is *zero,* we conclude that the *total* momentum of the system must remain constant:

$$\mathbf{p}_{\text{tot}} = \sum_{\text{system}} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \qquad \textbf{(9.4)}$$

or, equivalently,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \qquad \textbf{(9.5)}$$

where $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ are the initial values and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ the final values of the momentum during the time interval dt over which the reaction pair interacts. Equation 9.5 in component form demonstrates that the total momenta in the x, y, and z directions are all independently conserved:

$$\sum_{\text{system}} p_{ix} = \sum_{\text{system}} p_{fx} \qquad \sum_{\text{system}} p_{iy} = \sum_{\text{system}} p_{fy} \qquad \sum_{\text{system}} p_{iz} = \sum_{\text{system}} p_{fz} \qquad \textbf{(9.6)}$$

This result, known as the **law of conservation of linear momentum,** can be extended to any number of particles in an isolated system. It is considered one of the most important laws of mechanics. We can state it as follows:

Conservation of momentum

Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

This law tells us that **the total momentum of an isolated system at all times equals its initial momentum.**

Notice that we have made no statement concerning the nature of the forces acting on the particles of the system. The only requirement is that the forces must be *internal* to the system.

Quick Quiz 9.2

Your physical education teacher throws a baseball to you at a certain speed, and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. You are given the following choices: You can have the medicine ball thrown with (a) the same speed as the baseball, (b) the same momentum, or (c) the same kinetic energy. Rank these choices from easiest to hardest to catch.

EXAMPLE 9.1 The Floating Astronaut

A SkyLab astronaut discovered that while concentrating on writing some notes, he had gradually floated to the middle of an open area in the spacecraft. Not wanting to wait until he floated to the opposite side, he asked his colleagues for a push. Laughing at his predicament, they decided not to help, and so he had to take off his uniform and throw it in one direction so that he would be propelled in the opposite direction. Estimate his resulting velocity.

Figure 9.2 A hapless astronaut has discarded his uniform to get somewhere.

Solution We begin by making some reasonable guesses of relevant data. Let us assume we have a 70-kg astronaut who threw his 1-kg uniform at a speed of 20 m/s. For convenience, we set the positive direction of the x axis to be the direction of the throw (Fig. 9.2). Let us also assume that the x axis is tangent to the circular path of the spacecraft.

We take the system to consist of the astronaut and the uniform. Because of the gravitational force (which keeps the astronaut, his uniform, and the entire spacecraft in orbit), the system is not really isolated. However, this force is directed perpendicular to the motion of the system. Therefore, momentum is constant in the x direction because there are no external forces in this direction.

The total momentum of the system before the throw is zero ($m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = 0$). Therefore, the total momentum after the throw must be zero; that is,

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = 0$$

With $m_1 = 70$ kg, $\mathbf{v}_{2f} = 20\mathbf{i}$ m/s, and $m_2 = 1$ kg, solving for $\mathbf{v}_{1f}$, we find the recoil velocity of the astronaut to be

$$\mathbf{v}_{1f} = -\frac{m_2}{m_1}\mathbf{v}_{2f} = -\left(\frac{1\text{ kg}}{70\text{ kg}}\right)(20\mathbf{i}\text{ m/s}) = \boxed{-0.3\mathbf{i}\text{ m/s}}$$

The negative sign for $\mathbf{v}_{1f}$ indicates that the astronaut is moving to the left after the throw, in the direction opposite the direction of motion of the uniform, in accordance with Newton's third law. Because the astronaut is much more massive than his uniform, his acceleration and consequent velocity are much smaller than the acceleration and velocity of the uniform.

EXAMPLE 9.2 Breakup of a Kaon at Rest

One type of nuclear particle, called the *neutral kaon* (K^0), breaks up into a pair of other particles called *pions* (π^+ and π^-) that are oppositely charged but equal in mass, as illustrated in Figure 9.3. Assuming the kaon is initially at rest, prove that the two pions must have momenta that are equal in magnitude and opposite in direction.

Solution The breakup of the kaon can be written

$$K^0 \longrightarrow \pi^+ + \pi^-$$

If we let $\mathbf{p}^+$ be the momentum of the positive pion and $\mathbf{p}^-$ the momentum of the negative pion, the final momentum of the system consisting of the two pions can be written

$$\mathbf{p}_f = \mathbf{p}^+ + \mathbf{p}^-$$

Because the kaon is at rest before the breakup, we know that $\mathbf{p}_i = 0$. Because momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f = 0$, so that $\mathbf{p}^+ + \mathbf{p}^- = 0$, or

$$\mathbf{p}^+ = -\mathbf{p}^-$$

The important point behind this problem is that even though it deals with objects that are very different from those in the preceding example, the physics is identical: Linear momentum is conserved in an isolated system.

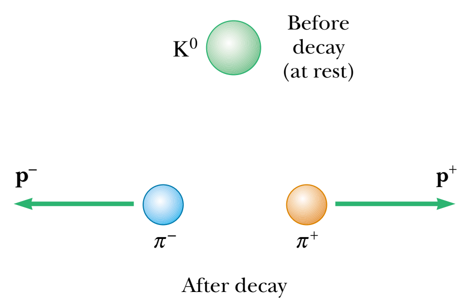


Figure 9.3 A kaon at rest breaks up spontaneously into a pair of oppositely charged pions. The pions move apart with momenta that are equal in magnitude but opposite in direction.

9.2 IMPULSE AND MOMENTUM

6.3 & 6.4

As we have seen, the momentum of a particle changes if a net force acts on the particle. Knowing the change in momentum caused by a force is useful in solving some types of problems. To begin building a better understanding of this important concept, let us assume that a single force $\mathbf{F}$ acts on a particle and that this force may vary with time. According to Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, or

$$d\mathbf{p} = \mathbf{F}\,dt \qquad \textbf{(9.7)}$$

We can integrate[2] this expression to find the change in the momentum of a particle when the force acts over some time interval. If the momentum of the particle

[2]Note that here we are integrating force with respect to time. Compare this with our efforts in Chapter 7, where we integrated force with respect to position to express the work done by the force.

changes from $\mathbf{p}_i$ at time t_i to $\mathbf{p}_f$ at time t_f, integrating Equation 9.7 gives

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\, dt \tag{9.8}$$

To evaluate the integral, we need to know how the force varies with time. The quantity on the right side of this equation is called the **impulse** of the force **F** acting on a particle over the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

Impulse of a force

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt = \Delta \mathbf{p} \tag{9.9}$$

Impulse–momentum theorem

The impulse of the force **F** acting on a particle equals the change in the momentum of the particle caused by that force.

This statement, known as the **impulse–momentum theorem,**[3] is equivalent to Newton's second law. From this definition, we see that impulse is a vector quantity having a magnitude equal to the area under the force–time curve, as described in Figure 9.4a. In this figure, it is assumed that the force varies in time in the general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum—that is, ML/T. Note that impulse is *not* a property of a particle; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

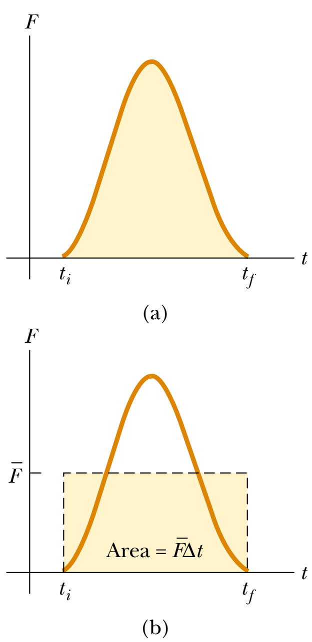


Figure 9.4 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force versus time curve. (b) In the time interval Δt, the time-averaged force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\overline{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} \mathbf{F}\, dt \tag{9.10}$$

where $\Delta t = t_f - t_i$. (This is an application of the mean value theorem of calculus.) Therefore, we can express Equation 9.9 as

$$\mathbf{I} \equiv \overline{\mathbf{F}}\, \Delta t \tag{9.11}$$

This time-averaged force, described in Figure 9.4b, can be thought of as the constant force that would give to the particle in the time interval Δt the same impulse that the time-varying force gives over this same interval.

In principle, if **F** is known as a function of time, the impulse can be calculated from Equation 9.9. The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\overline{\mathbf{F}} = \mathbf{F}$ and Equation 9.11 becomes

$$\mathbf{I} = \mathbf{F}\, \Delta t \tag{9.12}$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions in which the duration of the

[3]Although we assumed that only a single force acts on the particle, the impulse–momentum theorem is valid when several forces act; in this case, we replace **F** in Equation 9.9 with $\Sigma\mathbf{F}$.

During the brief time the club is in contact with the ball, the ball gains momentum as a result of the collision, and the club loses the same amount of momentum.

collision is very short. When this approximation is made, we refer to the force as an *impulsive force.* For example, when a baseball is struck with a bat, the time of the collision is about 0.01 s and the average force that the bat exerts on the ball in this time is typically several thousand newtons. Because this is much greater than the magnitude of the gravitational force, the impulse approximation justifies our ignoring the weight of the ball and bat. When we use this approximation, it is important to remember that $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta *immediately* before and after the collision, respectively. Therefore, in any situation in which it is proper to use the impulse approximation, the particle moves very little during the collision.

QuickLab

If you can find someone willing, play catch with an egg. What is the best way to move your hands so that the egg does not break when you change its momentum to zero?

Quick Quiz 9.3

Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. When a force is applied to object 1, it accelerates through a distance *d*. The force is removed from object 1 and is applied to object 2. At the moment when object 2 has accelerated through the same distance *d*, which statements are true? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) $K_1 < K_2$, (e) $K_1 = K_2$, (f) $K_1 > K_2$.

EXAMPLE 9.3 Teeing Off

A golf ball of mass 50 g is struck with a club (Fig. 9.5). The force exerted on the ball by the club varies from zero, at the instant before contact, up to some maximum value (at which the ball is deformed) and then back to zero when the ball leaves the club. Thus, the force–time curve is qualitatively described by Figure 9.4. Assuming that the ball travels 200 m, estimate the magnitude of the impulse caused by the collision.

Solution Let us use Ⓐ to denote the moment when the club first contacts the ball, Ⓑ to denote the moment when the club loses contact with the ball as the ball starts on its trajectory, and Ⓒ to denote its landing. Neglecting air resistance, we can use Equation 4.14 for the range of a projectile:

$$R = x_C = \frac{{v_B}^2}{g} \sin 2\theta_B$$

Let us assume that the launch angle θ_B is 45°, the angle that provides the maximum range for any given launch velocity. This assumption gives $\sin 2\theta_B = 1$, and the launch velocity of the ball is

$$v_B = \sqrt{x_C g} = \sqrt{(200\text{ m})(9.80\text{ m/s}^2)} = 44\text{ m/s}$$

Considering the time interval for the collision, $v_i = v_A = 0$ and $v_f = v_B$ for the ball. Hence, the magnitude of the impulse imparted to the ball is

$$I = \Delta p = mv_B - mv_A = (50 \times 10^{-3}\text{ kg})(44\text{ m/s}) - 0$$
$$= 2.2\text{ kg}\cdot\text{m/s}$$

Exercise If the club is in contact with the ball for a time of 4.5×10^{-4} s, estimate the magnitude of the average force exerted by the club on the ball.

Answer 4.9×10^3 N, a value that is extremely large when compared with the weight of the ball, 0.49 N.

Figure 9.5 A golf ball being struck by a club. *(© Harold E. Edgerton/ Courtesy of Palm Press, Inc.)*

EXAMPLE 9.4 How Good Are the Bumpers?

In a particular crash test, an automobile of mass 1 500 kg collides with a wall, as shown in Figure 9.6. The initial and final velocities of the automobile are $\mathbf{v}_i = -15.0\mathbf{i}$ m/s and $\mathbf{v}_f = 2.60\mathbf{i}$ m/s, respectively. If the collision lasts for 0.150 s, find the impulse caused by the collision and the average force exerted on the automobile.

Solution Let us assume that the force exerted on the car by the wall is large compared with other forces on the car so that we can apply the impulse approximation. Furthermore, we note that the force of gravity and the normal force exerted by the road on the car are perpendicular to the motion and therefore do not affect the horizontal momentum.

The initial and final momenta of the automobile are

$$\mathbf{p}_i = m\mathbf{v}_i = (1\,500\text{ kg})(-15.0\mathbf{i}\text{ m/s}) = -2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$
$$\mathbf{p}_f = m\mathbf{v}_f = (1\,500\text{ kg})(2.60\mathbf{i}\text{ m/s}) = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

Hence, the impulse is

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s})$$
$$\mathbf{I} = 2.64 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

The average force exerted on the automobile is

$$\overline{\mathbf{F}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4\,\mathbf{i}\text{ kg}\cdot\text{m/s}}{0.150\text{ s}} = 1.76 \times 10^5\mathbf{i}\text{ N}$$

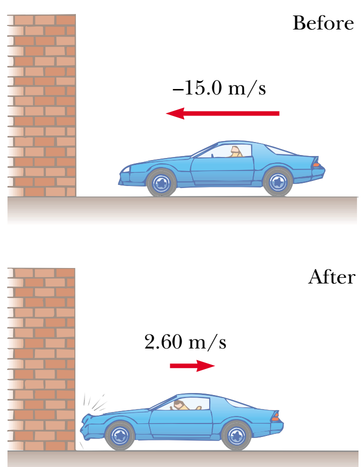


Figure 9.6 (a) This car's momentum changes as a result of its collision with the wall. (b) In a crash test, much of the car's initial kinetic energy is transformed into energy used to damage the car.

Note that the magnitude of this force is large compared with the weight of the car ($mg = 1.47 \times 10^4$ N), which justifies our initial assumption. Of note in this problem is how the signs of the velocities indicated the reversal of directions. What would the mathematics be describing if both the initial and final velocities had the same sign?

Quick Quiz 9.4

Rank an automobile dashboard, seatbelt, and airbag in terms of (a) the impulse and (b) the average force they deliver to a front-seat passenger during a collision.

9.3 COLLISIONS

6.5 & 6.6 In this section we use the law of conservation of linear momentum to describe what happens when two particles collide. We use the term **collision** to represent the event of two particles' coming together for a short time and thereby producing impulsive forces on each other. **These forces are assumed to be much greater than any external forces present.**

A collision may entail physical contact between two macroscopic objects, as described in Figure 9.7a, but the notion of what we mean by collision must be generalized because "physical contact" on a submicroscopic scale is ill-defined and hence meaningless. To understand this, consider a collision on an atomic scale (Fig. 9.7b), such as the collision of a proton with an alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they never come into physical contact with each other; instead, they repel each other because of the strong electrostatic force between them at close separations. When two particles 1 and 2 of masses m_1 and m_2 collide as shown in Figure 9.7, the impulsive forces may vary in time in complicated ways, one of which is described in Figure 9.8. If $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1, and if we assume that no external forces act on the particles, then the change in momentum of particle 1 due to the collision is given by Equation 9.8:

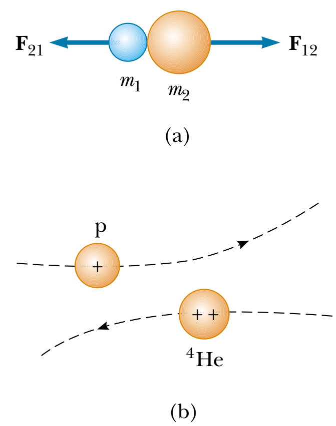


Figure 9.7 (a) The collision between two objects as the result of direct contact. (b) The "collision" between two charged particles.

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{21}\, dt$$

Likewise, if $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2, then the change in momentum of particle 2 is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_{12}\, dt$$

From Newton's third law, we conclude that

$$\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$$

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$$

Because the total momentum of the system is $\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2$, we conclude that the *change* in the momentum of the system due to the collision is zero:

$$\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

This is precisely what we expect because no external forces are acting on the system (see Section 9.2). Because the impulsive forces are internal, they do not change the total momentum of the system (only external forces can do that).

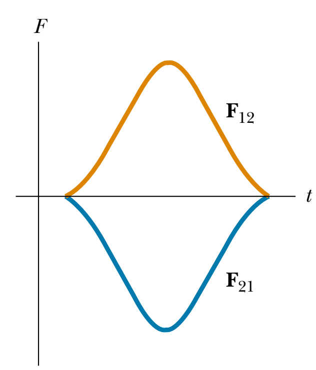


Figure 9.8 The impulse force as a function of time for the two colliding particles described in Figure 9.7a. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Momentum is conserved for any collision

Therefore, we conclude that **the total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.**

EXAMPLE 9.5 Carry Collision Insurance!

A car of mass 1800 kg stopped at a traffic light is struck from the rear by a 900-kg car, and the two become entangled. If the smaller car was moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution We can guess that the final speed is less than 20.0 m/s, the initial speed of the smaller car. The total momentum of the system (the two cars) before the collision must equal the total momentum immediately after the collision because momentum is conserved in any type of collision. The magnitude of the total momentum before the collision is equal to that of the smaller car because the larger car is initially at rest:

$$p_i = m_1 v_{1i} = (900\text{ kg})(20.0\text{ m/s}) = 1.80 \times 10^4\text{ kg}\cdot\text{m/s}$$

After the collision, the magnitude of the momentum of the entangled cars is

$$p_f = (m_1 + m_2)v_f = (2\,700\text{ kg})v_f$$

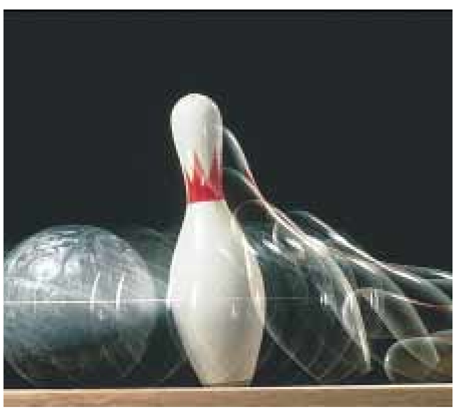
Equating the momentum before to the momentum after and solving for v_f, the final velocity of the entangled cars, we have

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{1.80 \times 10^4\text{ kg}\cdot\text{m/s}}{2\,700\text{ kg}} = \boxed{6.67\text{ m/s}}$$

The direction of the final velocity is the same as the velocity of the initially moving car.

Exercise What would be the final speed if the two cars each had a mass of 900 kg?

Answer 10.0 m/s.



When the bowling ball and pin collide, part of the ball's momentum is transferred to the pin. Consequently, the pin acquires momentum and kinetic energy, and the ball loses momentum and kinetic energy. However, the total momentum of the system (ball and pin) remains constant.

Quick Quiz 9.5

As a ball falls toward the Earth, the ball's momentum increases because its speed increases. Does this mean that momentum is not conserved in this situation?

Quick Quiz 9.6

A skater is using very low-friction rollerblades. A friend throws a Frisbee straight at her. In which case does the Frisbee impart the greatest impulse to the skater: (a) she catches the Frisbee and holds it, (b) she catches it momentarily but drops it, (c) she catches it and at once throws it back to her friend?

9.4 ELASTIC AND INELASTIC COLLISIONS IN ONE DIMENSION

As we have seen, momentum is conserved in any collision in which external forces are negligible. In contrast, kinetic energy may or may not be constant, depending on the type of collision. In fact, whether or not kinetic energy is the same before and after the collision is used to classify collisions as being either elastic or inelastic.

Elastic collision

An **elastic collision** between two objects is one in which *total kinetic energy (as well as total momentum) is the same before and after the collision.* Billiard-ball collisions and the collisions of air molecules with the walls of a container at ordinary temperatures are approximately elastic. Truly elastic collisions do occur, however, between atomic and subatomic particles. Collisions between certain objects in the macroscopic world, such as billiard-ball collisions, are only approximately elastic because some deformation and loss of kinetic energy take place.

An **inelastic collision** is one in which *total kinetic energy is not the same before and after the collision (even though momentum is constant).* Inelastic collisions are of two types. When the colliding objects stick together after the collision, as happens when a meteorite collides with the Earth, the collision is called **perfectly inelastic.** When the colliding objects do not stick together, but some kinetic energy is lost, as in the case of a rubber ball colliding with a hard surface, the collision is called **inelastic** (with no modifying adverb). For example, when a rubber ball collides with a hard surface, the collision is inelastic because some of the kinetic energy of the ball is lost when the ball is deformed while it is in contact with the surface.

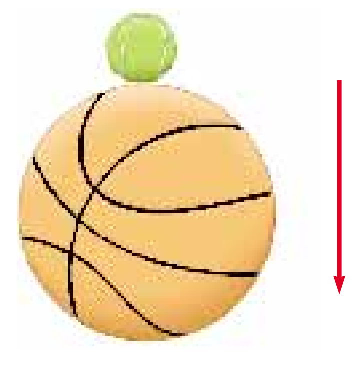
Inelastic collision

In most collisions, kinetic energy is *not* the same before and after the collision because some of it is converted to internal energy, to elastic potential energy when the objects are deformed, and to rotational energy. Elastic and perfectly inelastic collisions are limiting cases; most collisions fall somewhere between them.

In the remainder of this section, we treat collisions in one dimension and consider the two extreme cases—perfectly inelastic and elastic collisions. The important distinction between these two types of collisions is that **momentum is constant in all collisions, but kinetic energy is constant only in elastic collisions.**

QuickLab

Hold a Ping-Pong ball or tennis ball on top of a basketball. Drop them both at the same time so that the basketball hits the floor, bounces up, and hits the smaller falling ball. What happens and why?



Perfectly Inelastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ along a straight line, as shown in Figure 9.9. The two particles collide head-on, stick together, and then move with some common velocity $\mathbf{v}_f$ after the collision. Because momentum is conserved in any collision, we can say that the total momentum before the collision equals the total momentum of the composite system after the collision:

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \quad \textbf{(9.13)}$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \quad \textbf{(9.14)}$$

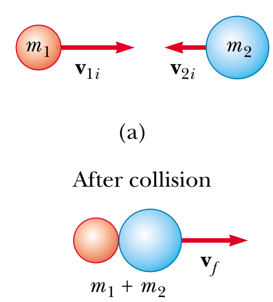


Figure 9.9 Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before collision and (b) after collision.

Quick Quiz 9.7

Which is worse, crashing into a brick wall at 40 mi/h or crashing head-on into an oncoming car that is identical to yours and also moving at 40 mi/h?

Elastic Collisions

6.6 Now consider two particles that undergo an elastic head-on collision (Fig. 9.10). In this case, both momentum and kinetic energy are conserved; therefore, we have

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \quad \textbf{(9.15)}$$

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \quad \textbf{(9.16)}$$

Because all velocities in Figure 9.10 are either to the left or the right, they can be represented by the corresponding speeds along with algebraic signs indicating direction. We shall indicate v as positive if a particle moves to the right and negative

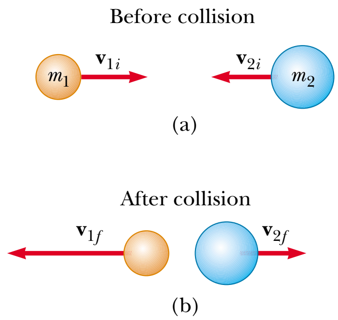


Figure 9.10 Schematic representation of an elastic head-on collision between two particles: (a) before collision and (b) after collision.

if it moves to the left. As has been seen in earlier chapters, it is common practice to call these values "speed" even though this term technically refers to the magnitude of the velocity vector, which does not have an algebraic sign.

In a typical problem involving elastic collisions, there are two unknown quantities, and Equations 9.15 and 9.16 can be solved simultaneously to find these. An alternative approach, however—one that involves a little mathematical manipulation of Equation 9.16—often simplifies this process. To see how, let us cancel the factor $\frac{1}{2}$ in Equation 9.16 and rewrite it as

$$m_1({v_{1i}}^2 - {v_{1f}}^2) = m_2({v_{2f}}^2 - {v_{2i}}^2)$$

and then factor both sides:

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \tag{9.17}$$

Next, let us separate the terms containing m_1 and m_2 in Equation 9.15 to get

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \tag{9.18}$$

To obtain our final result, we divide Equation 9.17 by Equation 9.18 and get

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \tag{9.19}$$

This equation, in combination with Equation 9.15, can be used to solve problems dealing with elastic collisions. According to Equation 9.19, the relative speed of the two particles before the collision $v_{1i} - v_{2i}$ equals the negative of their relative speed after the collision, $-(v_{1f} - v_{2f})$.

Suppose that the masses and initial velocities of both particles are known. Equations 9.15 and 9.19 can be solved for the final speeds in terms of the initial speeds because there are two equations and two unknowns:

Elastic collision: relationships between final and initial velocities

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \tag{9.20}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} \tag{9.21}$$

It is important to remember that the appropriate signs for v_{1i} and v_{2i} must be included in Equations 9.20 and 9.21. For example, if particle 2 is moving to the left initially, then v_{2i} is negative.

Let us consider some special cases: If $m_1 = m_2$, then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, the particles exchange speeds if they have equal masses. This is approximately what one observes in head-on billiard ball collisions—the cue ball stops, and the struck ball moves away from the collision with the same speed that the cue ball had.

If particle 2 is initially at rest, then $v_{2i} = 0$ and Equations 9.20 and 9.21 become

Elastic collision: particle 2 initially at rest

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} \tag{9.22}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} \tag{9.23}$$

If m_1 is much greater than m_2 and $v_{2i} = 0$, we see from Equations 9.22 and 9.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. That is, when a very heavy particle collides head-on with a very light one that is initially at rest, the heavy particle continues its motion unaltered after the collision, and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle. An example of such a collision would be that of a moving heavy atom, such as uranium, with a light atom, such as hydrogen.

If m_2 is much greater than m_1 and particle 2 is initially at rest, then $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx v_{2i} = 0$. That is, when a very light particle collides head-on with a very heavy particle that is initially at rest, the light particle has its velocity reversed and the heavy one remains approximately at rest.

EXAMPLE 9.6 The Ballistic Pendulum

The ballistic pendulum (Fig. 9.11) is a system used to measure the speed of a fast-moving projectile, such as a bullet. The bullet is fired into a large block of wood suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. The collision is perfectly inelastic, and because momentum is conserved, Equation 9.14 gives the speed of the system right after the collision, when we assume the impulse approximation. If we call the bullet particle 1 and the block particle 2, the total kinetic energy right after the collision is

$$(1) \qquad K_f = \tfrac{1}{2}(m_1 + m_2)v_f^2$$

With $v_{2i} = 0$, Equation 9.14 becomes

$$(2) \qquad v_f = \frac{m_1 v_{1i}}{m_1 + m_2}$$

Substituting this value of v_f into (1) gives

$$K_f = \frac{m_1^2 v_{1i}^2}{2(m_1 + m_2)}$$

Note that this kinetic energy immediately after the collision is less than the initial kinetic energy of the bullet. In all the energy changes that take place *after* the collision, however, the total amount of mechanical energy remains constant; thus, we can say that after the collision, the kinetic energy of the block and bullet at the bottom is transformed to potential energy at the height h:

$$\frac{m_1^2 v_{1i}^2}{2(m_1 + m_2)} = (m_1 + m_2)gh$$

Solving for v_{1i}, we obtain

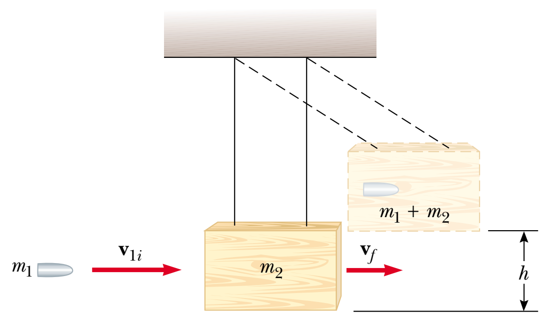
$$v_{1i} = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}$$

This expression tells us that it is possible to obtain the initial speed of the bullet by measuring h and the two masses.

Because the collision is perfectly inelastic, some mechanical energy is converted to internal energy and it would be *incorrect* to equate the initial kinetic energy of the incoming bullet to the final gravitational potential energy of the bullet–block combination.

Exercise In a ballistic pendulum experiment, suppose that $h =$ 5.00 cm, $m_1 = 5.00$ g, and $m_2 = 1.00$ kg. Find (a) the initial speed of the bullet and (b) the loss in mechanical energy due to the collision.

Answer 199 m/s; 98.5 J.



(a)

(b)

Figure 9.11 (a) Diagram of a ballistic pendulum. Note that $\mathbf{v}_{1i}$ is the velocity of the bullet just before the collision and $\mathbf{v}_f = \mathbf{v}_{1f} = \mathbf{v}_{2f}$ is the velocity of the bullet + block system just after the perfectly inelastic collision. (b) Multiflash photograph of a ballistic pendulum used in the laboratory.

EXAMPLE 9.7 A Two-Body Collision with a Spring

A block of mass $m_1 = 1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track collides with a spring attached to a second block of mass $m_2 = 2.10$ kg initially moving to the left with a speed of 2.50 m/s, as shown in Figure 9.12a. The spring constant is 600 N/m. (a) At the instant block 1 is moving to the right with a speed of 3.00 m/s, as in Figure 9.12b, determine the velocity of block 2.

Solution First, note that the initial velocity of block 2 is -2.50 m/s because its direction is to the left. Because momentum is conserved for the system of two blocks, we have

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$(1.60 \text{ kg})(4.00 \text{ m/s}) + (2.10 \text{ kg})(-2.50 \text{ m/s}) = (1.60 \text{ kg})(3.00 \text{ m/s}) + (2.10 \text{ kg})v_{2f}$$

$$v_{2f} = \boxed{-1.74 \text{ m/s}}$$

The negative value for v_{2f} means that block 2 is still moving to the left at the instant we are considering.

(b) Determine the distance the spring is compressed at that instant.

Solution To determine the distance that the spring is compressed, shown as x in Figure 9.12b, we can use the concept of conservation of mechanical energy because no friction or other nonconservative forces are acting on the system. Thus, we have

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 + \tfrac{1}{2}kx^2$$

Substituting the given values and the result to part (a) into this expression gives

$$x = \boxed{0.173 \text{ m}}$$

It is important to note that we needed to use the principles of both conservation of momentum and conservation of mechanical energy to solve the two parts of this problem.

Exercise Find the velocity of block 1 and the compression in the spring at the instant that block 2 is at rest.

Answer 0.719 m/s to the right; 0.251 m.

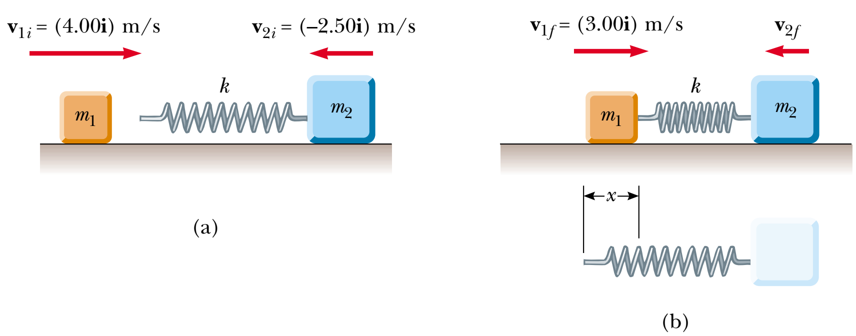


Figure 9.12

EXAMPLE 9.8 Slowing Down Neutrons by Collisions

In a nuclear reactor, neutrons are produced when a $^{235}_{92}$U atom splits in a process called *fission*. These neutrons are moving at about 10^7 m/s and must be slowed down to about 10^3 m/s before they take part in another fission event. They are slowed down by being passed through a solid or liquid material called a *moderator*. The slowing-down process involves elastic collisions. Let us show that a neutron can lose most of its kinetic energy if it collides elastically with a moderator containing light nuclei, such as deuterium (in "heavy water," D_2O) or carbon (in graphite).

Solution Let us assume that the moderator nucleus of mass m_m is at rest initially and that a neutron of mass m_n and initial speed v_{ni} collides with it head-on.

Because these are elastic collisions, the first thing we do is recognize that both momentum and kinetic energy are constant. Therefore, Equations 9.22 and 9.23 can be applied to the head-on collision of a neutron with a moderator nucleus. We can represent this process by a drawing such as Figure 9.10.

The initial kinetic energy of the neutron is

$$K_{ni} = \tfrac{1}{2} m_n v_{ni}^2$$

After the collision, the neutron has kinetic energy $\tfrac{1}{2} m_n v_{nf}^2$, and we can substitute into this the value for v_{nf} given by Equation 9.22:

$$K_{nf} = \tfrac{1}{2} m_n v_{nf}^2 = \frac{m_n}{2}\left(\frac{m_n - m_m}{m_n + m_m}\right)^2 v_{ni}^2$$

Therefore, the fraction f_n of the initial kinetic energy possessed by the neutron after the collision is

$$(1) \qquad f_n = \frac{K_{nf}}{K_{ni}} = \left(\frac{m_n - m_m}{m_n + m_m}\right)^2$$

From this result, we see that the final kinetic energy of the neutron is small when m_m is close to m_n and zero when $m_n = m_m$.

We can use Equation 9.23, which gives the final speed of the particle that was initially at rest, to calculate the kinetic energy of the moderator nucleus after the collision:

$$K_{mf} = \tfrac{1}{2} m_m v_{mf}^2 = \frac{2 m_n^2 m_m}{(m_n + m_m)^2} v_{ni}^2$$

Hence, the fraction f_m of the initial kinetic energy transferred to the moderator nucleus is

$$(2) \qquad f_m = \frac{K_{mf}}{K_{ni}} = \frac{4 m_n m_m}{(m_n + m_m)^2}$$

Because the total kinetic energy of the system is conserved, (2) can also be obtained from (1) with the condition that $f_n + f_m = 1$, so that $f_m = 1 - f_n$.

Suppose that heavy water is used for the moderator. For collisions of the neutrons with deuterium nuclei in D_2O $(m_m = 2m_n)$, $f_n = 1/9$ and $f_m = 8/9$. That is, 89% of the neutron's kinetic energy is transferred to the deuterium nucleus. In practice, the moderator efficiency is reduced because head-on collisions are very unlikely.

How do the results differ when graphite (^{12}C, as found in pencil lead) is used as the moderator?

Quick Quiz 9.8

An ingenious device that illustrates conservation of momentum and kinetic energy is shown in Figure 9.13a. It consists of five identical hard balls supported by strings of equal lengths. When ball 1 is pulled out and released, after the almost-elastic collision between it and ball 2, ball 5 moves out, as shown in Figure 9.13b. If balls 1 and 2 are pulled out and released, balls 4 and 5 swing out, and so forth. Is it ever possible that, when ball 1 is released, balls 4 and 5 will swing out on the opposite side and travel with half the speed of ball 1, as in Figure 9.13c?

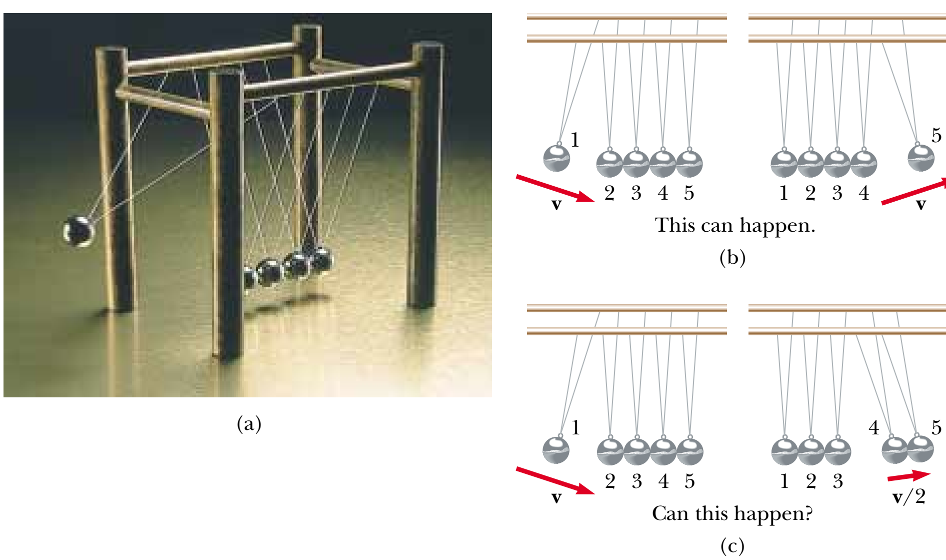


Figure 9.13 An executive stress reliever.

9.5 TWO-DIMENSIONAL COLLISIONS

In Sections 9.1 and 9.3, we showed that the momentum of a system of two particles is constant when the system is isolated. For any collision of two particles, this result implies that the momentum in each of the directions x, y, and z is constant. However, an important subset of collisions takes place in a plane. The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Let us consider a two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2, where particle 2 is initially at rest, as shown in Figure 9.14. After the collision, particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle ϕ with respect to the horizontal. This is called a *glancing* collision. Applying the law of conservation of momentum in component form, and noting that the initial y component of the momentum of the two-particle system is zero, we obtain

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi \qquad \textbf{(9.24)}$$

$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi \qquad \textbf{(9.25)}$$

where the minus sign in Equation 9.25 comes from the fact that after the collision, particle 2 has a y component of velocity that is downward. We now have two independent equations. As long as no more than two of the seven quantities in Equations 9.24 and 9.25 are unknown, we can solve the problem.

If the collision is elastic, we can also use Equation 9.16 (conservation of kinetic energy), with $v_{2i} = 0$, to give

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \qquad \textbf{(9.26)}$$

Knowing the initial speed of particle 1 and both masses, we are left with four unknowns (v_{1f}, v_{2f}, θ, ϕ). Because we have only three equations, one of the four remaining quantities must be given if we are to determine the motion after the collision from conservation principles alone.

If the collision is inelastic, kinetic energy is *not* conserved and Equation 9.26 does *not* apply.

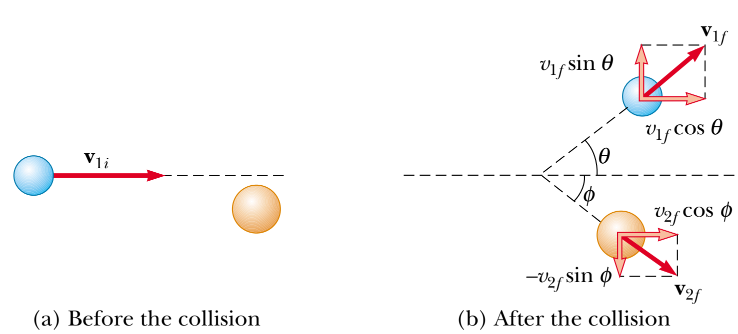


Figure 9.14 An elastic glancing collision between two particles.

Problem-Solving Hints

Collisions

The following procedure is recommended when dealing with problems involving collisions between two objects:

- Set up a coordinate system and define your velocities with respect to that system. It is usually convenient to have the *x* axis coincide with one of the initial velocities.
- In your sketch of the coordinate system, draw and label all velocity vectors and include all the given information.
- Write expressions for the *x* and *y* components of the momentum of each object before and after the collision. Remember to include the appropriate signs for the components of the velocity vectors.
- Write expressions for the total momentum in the *x* direction before and after the collision and equate the two. Repeat this procedure for the total momentum in the *y* direction. These steps follow from the fact that, because the momentum of the *system* is conserved in any collision, the total momentum along any direction must also be constant. Remember, it is the momentum of the *system* that is constant, not the momenta of the individual objects.
- If the collision is inelastic, kinetic energy is not conserved, and additional information is probably required. If the collision is perfectly inelastic, the final velocities of the two objects are equal. Solve the momentum equations for the unknown quantities.
- If the collision is elastic, kinetic energy is conserved, and you can equate the total kinetic energy before the collision to the total kinetic energy after the collision to get an additional relationship between the velocities.

EXAMPLE 9.9 Collision at an Intersection

A 1 500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2 500-kg van traveling north at a speed of 20.0 m/s, as shown in Figure 9.15. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).

Solution Let us choose east to be along the positive *x* direction and north to be along the positive *y* direction. Before the collision, the only object having momentum in the *x* direction is the car. Thus, the magnitude of the total initial momentum of the system (car plus van) in the *x* direction is

$$\sum p_{xi} = (1\,500 \text{ kg})(25.0 \text{ m/s}) = 3.75 \times 10^4 \text{ kg}\cdot\text{m/s}$$

Let us assume that the wreckage moves at an angle θ and speed v_f after the collision. The magnitude of the total momentum in the *x* direction after the collision is

$$\sum p_{xf} = (4\,000 \text{ kg})v_f \cos\theta$$

Because the total momentum in the *x* direction is constant, we can equate these two equations to obtain

$$(1) \qquad 3.75 \times 10^4 \text{ kg}\cdot\text{m/s} = (4\,000 \text{ kg})v_f \cos\theta$$

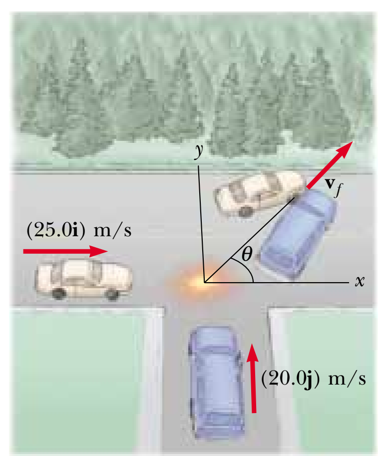


Figure 9.15 An eastbound car colliding with a northbound van.

Similarly, the total initial momentum of the system in the *y* direction is that of the van, and the magnitude of this momentum is (2 500 kg)(20.0 m/s). Applying conservation of momentum to the y direction, we have

$$\sum p_{yi} = \sum p_{yf}$$

$$(2\,500 \text{ kg})(20.0 \text{ m/s}) = (4\,000 \text{ kg})v_f \sin\theta$$

$$(2) \qquad 5.00 \times 10^4 \text{ kg}\cdot\text{m/s} = (4\,000 \text{ kg})v_f \sin\theta$$

If we divide (2) by (1), we get

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = \boxed{53.1°}$$

When this angle is substituted into (2), the value of v_f is

$$v_f = \frac{5.00 \times 10^4 \text{ kg}\cdot\text{m/s}}{(4\,000 \text{ kg})\sin 53.1°} = \boxed{15.6 \text{ m/s}}$$

It might be instructive for you to draw the momentum vectors of each vehicle before the collision and the two vehicles together after the collision.

EXAMPLE 9.10 Proton–Proton Collision

Proton 1 collides elastically with proton 2 that is initially at rest. Proton 1 has an initial speed of 3.50×10^5 m/s and makes a glancing collision with proton 2, as was shown in Figure 9.14. After the collision, proton 1 moves at an angle of 37.0° to the horizontal axis, and proton 2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

Solution Because both particles are protons, we know that $m_1 = m_2$. We also know that $\theta = 37.0°$ and $v_{1i} = 3.50 \times 10^5$ m/s. Equations 9.24, 9.25, and 9.26 become

$$v_{1f}\cos 37.0° + v_{2f}\cos\phi = 3.50 \times 10^5 \text{ m/s}$$

$$v_{1f}\sin 37.0° - v_{2f}\sin\phi = 0$$

$$v_{1f}^2 + v_{2f}^2 = (3.50 \times 10^5 \text{ m/s})^2$$

Solving these three equations with three unknowns simultaneously gives

$$v_{1f} = \boxed{2.80 \times 10^5 \text{ m/s}}$$

$$v_{2f} = \boxed{2.11 \times 10^5 \text{ m/s}}$$

$$\phi = \boxed{53.0°}$$

Note that $\theta + \phi = 90°$. This result is not accidental. **Whenever two equal masses collide elastically in a glancing collision and one of them is initially at rest, their final velocities are always at right angles to each other.** The next example illustrates this point in more detail.

EXAMPLE 9.11 Billiard Ball Collision

In a game of billiards, a player wishes to sink a target ball 2 in the corner pocket, as shown in Figure 9.16. If the angle to the corner pocket is 35°, at what angle θ is the cue ball 1 deflected? Assume that friction and rotational motion are unimportant and that the collision is elastic.

Solution Because the target ball is initially at rest, conservation of energy (Eq. 9.16) gives

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2$$

But $m_1 = m_2$, so that

$$(1) \qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

Applying conservation of momentum to the two-dimensional collision gives

$$(2) \qquad \mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}$$

Note that because $m_1 = m_2$, the masses also cancel in (2). If we square both sides of (2) and use the definition of the dot

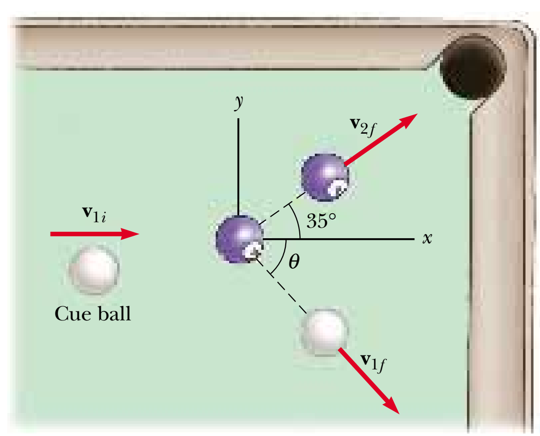


Figure 9.16

product of two vectors from Section 7.2, we get

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}) = v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

Because the angle between $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ is $\theta + 35°$, $\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} = v_{1f} v_{2f} \cos(\theta + 35°)$, and so

$$(3) \qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos(\theta + 35°)$$

Subtracting (1) from (3) gives

$$0 = 2v_{1f}v_{2f}\cos(\theta + 35°)$$

$$0 = \cos(\theta + 35°)$$

$$\theta + 35° = 90° \qquad \text{or} \qquad \theta = \boxed{55°}$$

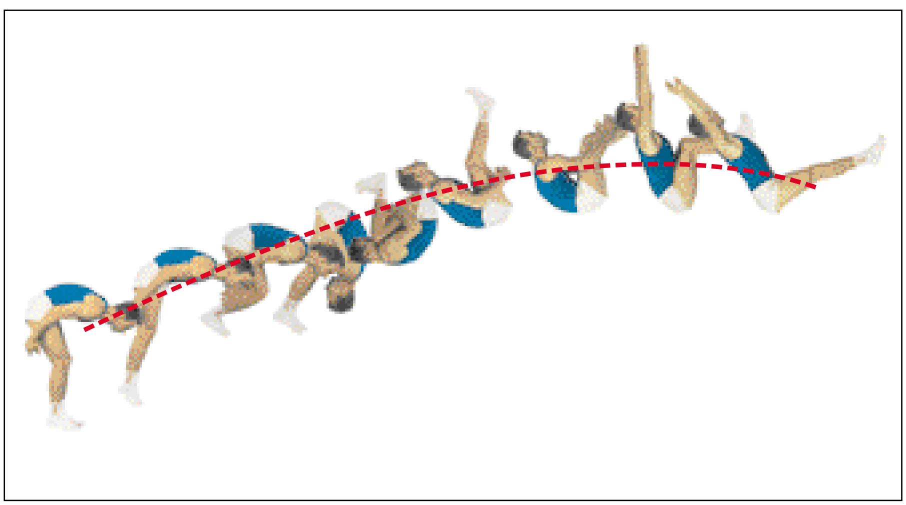
This result shows that whenever two equal masses undergo a glancing elastic collision and one of them is initially at rest, they move at right angles to each other after the collision. The same physics describes two very different situations, protons in Example 9.10 and billiard balls in this example.

9.6 THE CENTER OF MASS

6.7

In this section we describe the overall motion of a mechanical system in terms of a special point called the **center of mass** of the system. The mechanical system can be either a system of particles, such as a collection of atoms in a container, or an extended object, such as a gymnast leaping through the air. We shall see that the center of mass of the system moves as if all the mass of the system were concentrated at that point. Furthermore, if the resultant external force on the system is $\Sigma\mathbf{F}_{\text{ext}}$ and the total mass of the system is M, the center of mass moves with an acceleration given by $\mathbf{a} = \Sigma\mathbf{F}_{\text{ext}}/M$. That is, the system moves as if the resultant external force were applied to a single particle of mass M located at the center of mass. This behavior is independent of other motion, such as rotation or vibration of the system. This result was implicitly assumed in earlier chapters because many examples referred to the motion of extended objects that were treated as particles.

Consider a mechanical system consisting of a pair of particles that have different masses and are connected by a light, rigid rod (Fig. 9.17). One can describe the position of the center of mass of a system as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the



This multiflash photograph shows that as the acrobat executes a somersault, his center of mass follows a parabolic path, the same path that a particle would follow.

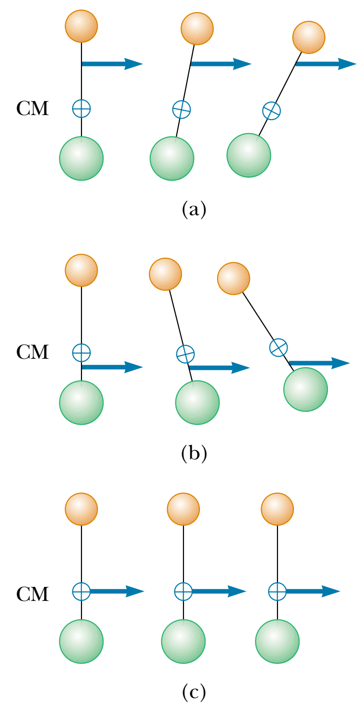


Figure 9.17 Two particles of unequal mass are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the less massive particle and the center of mass. (b) The system rotates counterclockwise when a force is applied between the more massive particle and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.

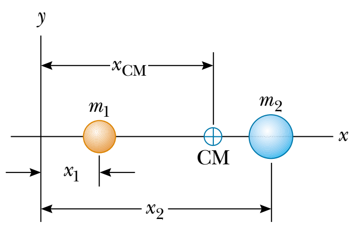


Figure 9.18 The center of mass of two particles of unequal mass on the x axis is located at x_{CM}, a point between the particles, closer to the one having the larger mass.

particles and is closer to the particle having the larger mass. If a single force is applied at some point on the rod somewhere between the center of mass and the less massive particle, the system rotates clockwise (see Fig. 9.17a). If the force is applied at a point on the rod somewhere between the center of mass and the more massive particle, the system rotates counterclockwise (see Fig. 9.17b). If the force is applied at the center of mass, the system moves in the direction of **F** without rotating (see Fig. 9.17c). Thus, the center of mass can be easily located.

The center of mass of the pair of particles described in Figure 9.18 is located on the x axis and lies somewhere between the particles. Its x coordinate is

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \textbf{(9.27)}$$

For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = \frac{2}{3}d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.

We can extend this concept to a system of many particles in three dimensions. The x coordinate of the center of mass of n particles is defined to be

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} \quad \textbf{(9.28)}$$

where x_i is the x coordinate of the ith particle. For convenience, we express the total mass as $M \equiv \sum_i m_i$, where the sum runs over all n particles. The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{CM} \equiv \frac{\sum_i m_i y_i}{M} \quad \text{and} \quad z_{CM} \equiv \frac{\sum_i m_i z_i}{M} \quad \textbf{(9.29)}$$

The center of mass can also be located by its position vector, $\mathbf{r}_{CM}$. The cartesian coordinates of this vector are x_{CM}, y_{CM}, and z_{CM}, defined in Equations 9.28 and 9.29. Therefore,

Vector position of the center of mass for a system of particles

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k}$$
$$= \frac{\sum_i m_i x_i \mathbf{i} + \sum_i m_i y_i \mathbf{j} + \sum_i m_i z_i \mathbf{k}}{M}$$
$$\mathbf{r}_{CM} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \quad \textbf{(9.30)}$$

where $\mathbf{r}_i$ is the position vector of the ith particle, defined by

$$\mathbf{r}_i \equiv x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$$

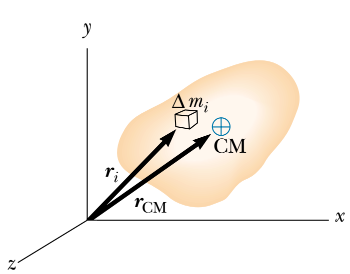


Figure 9.19 An extended object can be considered a distribution of small elements of mass Δm_i. The center of mass is located at the vector position $\mathbf{r}_{CM}$, which has coordinates x_{CM}, y_{CM}, and z_{CM}.

Although locating the center of mass for an extended object is somewhat more cumbersome than locating the center of mass of a system of particles, the basic ideas we have discussed still apply. We can think of an extended object as a system containing a large number of particles (Fig. 9.19). The particle separation is very small, and so the object can be considered to have a continuous mass distribution. By dividing the object into elements of mass Δm_i, with coordinates x_i, y_i, z_i, we see that the x coordinate of the center of mass is approximately

$$x_{CM} \approx \frac{\sum_i x_i \Delta m_i}{M}$$

with similar expressions for y_{CM} and z_{CM}. If we let the number of elements n approach infinity, then x_{CM} is given precisely. In this limit, we replace the sum by an integral and Δm_i by the differential element dm:

$$x_{\rm CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M}\int x\, dm \quad \textbf{(9.31)}$$

Likewise, for $y_{\rm CM}$ and $z_{\rm CM}$ we obtain

$$y_{\rm CM} = \frac{1}{M}\int y\, dm \quad \text{and} \quad z_{\rm CM} = \frac{1}{M}\int z\, dm \quad \textbf{(9.32)}$$

We can express the vector position of the center of mass of an extended object in the form

$$\mathbf{r}_{\rm CM} = \frac{1}{M}\int \mathbf{r}\, dm \quad \textbf{(9.33)}$$

which is equivalent to the three expressions given by Equations 9.31 and 9.32.

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.[4] For example, the center of mass of a rod lies in the rod, midway between its ends. The center of mass of a sphere or a cube lies at its geometric center.

One can determine the center of mass of an irregularly shaped object by suspending the object first from one point and then from another. In Figure 9.20, a wrench is hung from point *A*, and a vertical line *AB* (which can be established with a plumb bob) is drawn when the wrench has stopped swinging. The wrench is then hung from point *C*, and a second vertical line *CD* is drawn. The center of mass is halfway through the thickness of the wrench, under the intersection of these two lines. In general, if the wrench is hung freely from any point, the vertical line through this point must pass through the center of mass.

Because an extended object is a continuous distribution of mass, each small mass element is acted upon by the force of gravity. The net effect of all these forces is equivalent to the effect of a single force, $M\mathbf{g}$, acting through a special point, called the **center of gravity.** If $\mathbf{g}$ is constant over the mass distribution, then the center of gravity coincides with the center of mass. If an extended object is pivoted at its center of gravity, it balances in any orientation.

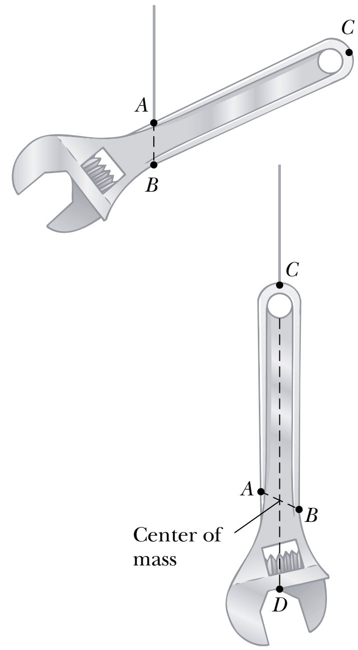


Figure 9.20 An experimental technique for determining the center of mass of a wrench. The wrench is hung freely first from point *A* and then from point *C*. The intersection of the two lines *AB* and *CD* locates the center of mass.

QuickLab

Cut a triangle from a piece of cardboard and draw a set of adjacent strips inside it, parallel to one of the sides. Put a dot at the approximate location of the center of mass of each strip and then draw a straight line through the dots and into the angle opposite your starting side. The center of mass for the triangle must lie on this bisector of the angle. Repeat these steps for the other two sides. The three angle bisectors you have drawn will intersect at the center of mass of the triangle. If you poke a hole anywhere in the triangle and hang the cardboard from a string attached at that hole, the center of mass will be vertically aligned with the hole.

Quick Quiz 9.9

If a baseball bat is cut at the location of its center of mass as shown in Figure 9.21, do the two pieces have the same mass?

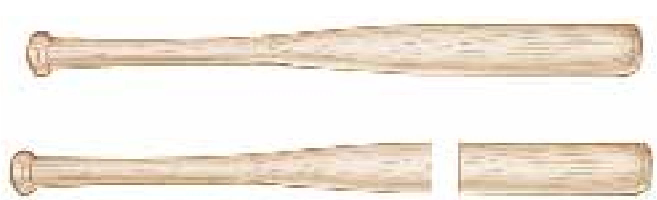


Figure 9.21 A baseball bat cut at the location of its center of mass.

[4]This statement is valid only for objects that have a uniform mass per unit volume.

EXAMPLE 9.12 The Center of Mass of Three Particles

A system consists of three particles located as shown in Figure 9.22a. Find the center of mass of the system.

Solution We set up the problem by labeling the masses of the particles as shown in the figure, with $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg. Using the basic defining equations for the coordinates of the center of mass and noting that $z_{CM} = 0$, we obtain

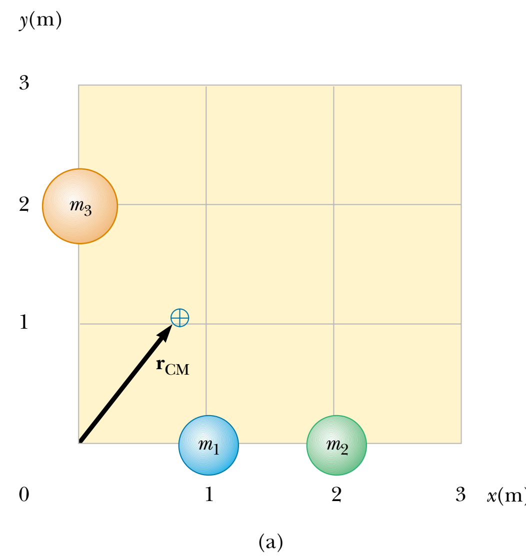
$$x_{CM} = \frac{\sum_i m_i x_i}{M} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$
$$= \frac{(1.0 \text{ kg})(1.0 \text{ m}) + (1.0 \text{ kg})(2.0 \text{ m}) + (2.0 \text{ kg})(0 \text{ m})}{1.0 \text{ kg} + 1.0 \text{ kg} + 2.0 \text{ kg}}$$
$$= \frac{3.0 \text{ kg}\cdot\text{m}}{4.0 \text{ kg}} = 0.75 \text{ m}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{M} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$
$$= \frac{(1.0 \text{ kg})(0) + (1.0 \text{ kg})(0) + (2.0 \text{ kg})(2.0 \text{ m})}{4.0 \text{ kg}}$$
$$= \frac{4.0 \text{ kg}\cdot\text{m}}{4.0 \text{ kg}} = 1.0 \text{ m}$$

The position vector to the center of mass measured from the origin is therefore

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} = 0.75\mathbf{i} \text{ m} + 1.0\,\mathbf{j} \text{ m}$$

We can verify this result graphically by adding together $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$ and dividing the vector sum by M, the total mass. This is shown in Figure 9.22b.



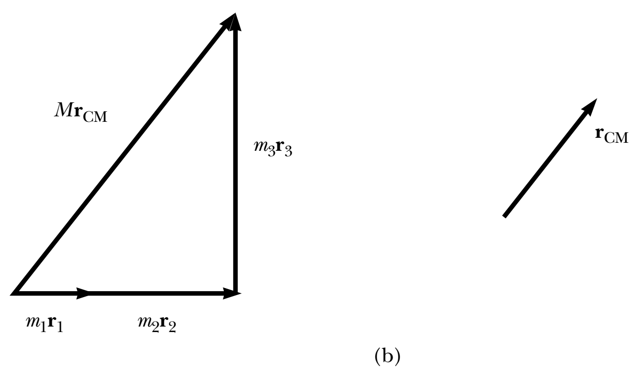


Figure 9.22 (a) Two 1-kg masses and a single 2-kg mass are located as shown. The vector indicates the location of the system's center of mass. (b) The vector sum of $m_i\mathbf{r}_i$.

EXAMPLE 9.13 The Center of Mass of a Rod

(a) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

Solution The rod is shown aligned along the x axis in Figure 9.23, so that $y_{CM} = z_{CM} = 0$. Furthermore, if we call the mass per unit length λ (this quantity is called the *linear mass density*), then $\lambda = M/L$ for the uniform rod we assume here. If we divide the rod into elements of length dx, then the mass of each element is $dm = \lambda\, dx$. For an arbitrary element located a distance x from the origin, Equation 9.31 gives

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\lambda}{M}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda L^2}{2M}$$

Because $\lambda = M/L$, this reduces to

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \frac{L}{2}$$

One can also use symmetry arguments to obtain the same result.

(b) Suppose a rod is *nonuniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Solution In this case, we replace dm by λdx where λ is not constant. Therefore, x_{CM} is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$

$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

We can eliminate α by noting that the total mass of the rod is related to α through the relationship

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

Substituting this into the expression for x_{CM} gives

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\frac{2}{3}L}$$

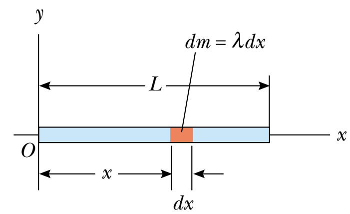


Figure 9.23 The center of mass of a uniform rod of length L is located at $x_{CM} = L/2$.

EXAMPLE 9.14 The Center of Mass of a Right Triangle

An object of mass M is in the shape of a right triangle whose dimensions are shown in Figure 9.24. Locate the coordinates of the center of mass, assuming the object has a uniform mass per unit area.

Solution By inspection we can estimate that the x coordinate of the center of mass must be past the center of the base, that is, greater than $a/2$, because the largest part of the triangle lies beyond that point. A similar argument indicates that its y coordinate must be less than $b/2$. To evaluate the x coordinate, we divide the triangle into narrow strips of width dx and height y as in Figure 9.24. The mass dm of each strip is

$$dm = \frac{\text{total mass of object}}{\text{total area of object}} \times \text{area of strip}$$

$$= \frac{M}{1/2\,ab}(y\,dx) = \left(\frac{2M}{ab}\right)y\,dx$$

Therefore, the x coordinate of the center of mass is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right)y\,dx = \frac{2}{ab}\int_0^a xy\,dx$$

To evaluate this integral, we must express y in terms of x. From similar triangles in Figure 9.24, we see that

$$\frac{y}{x} = \frac{b}{a} \quad \text{or} \quad y = \frac{b}{a}x$$

With this substitution, x_{CM} becomes

$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right)dx = \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$= \boxed{\frac{2}{3}a}$$

By a similar calculation, we get for the y coordinate of the center of mass

$$y_{CM} = \boxed{\frac{1}{3}b}$$

These values fit our original estimates.

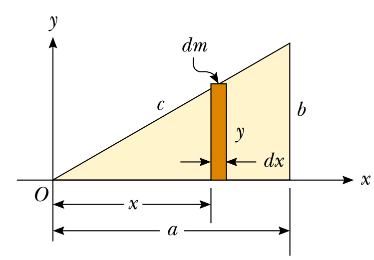


Figure 9.24

9.7 MOTION OF A SYSTEM OF PARTICLES

6.8 We can begin to understand the physical significance and utility of the center of mass concept by taking the time derivative of the position vector given by Equation 9.30. From Section 4.1 we know that the time derivative of a position vector is by definition a velocity. Assuming M remains constant for a system of particles, that is, no particles enter or leave the system, we get the following expression for the **velocity of the center of mass** of the system:

Velocity of the center of mass

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{r}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} \tag{9.34}$$

where $\mathbf{v}_i$ is the velocity of the ith particle. Rearranging Equation 9.34 gives

Total momentum of a system of particles

$$M\mathbf{v}_{\mathrm{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\mathrm{tot}} \tag{9.35}$$

Therefore, we conclude that the **total linear momentum of the system** equals the total mass multiplied by the velocity of the center of mass. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\mathbf{v}_{\mathrm{CM}}$.

If we now differentiate Equation 9.34 with respect to time, we get the **acceleration of the center of mass** of the system:

Acceleration of the center of mass

$$\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i \tag{9.36}$$

Rearranging this expression and using Newton's second law, we obtain

$$M\mathbf{a}_{\mathrm{CM}} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i \tag{9.37}$$

where $\mathbf{F}_i$ is the net force on particle i.

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However, by Newton's third law, the internal force exerted by particle 1 on particle 2, for example, is equal in magnitude and opposite in direction to the internal force exerted by particle 2 on particle 1. Thus, when we sum over all internal forces in Equation 9.37, they cancel in pairs and the net force on the system is caused *only* by external forces. Thus, we can write Equation 9.37 in the form

Newton's second law for a system of particles

$$\sum \mathbf{F}_{\mathrm{ext}} = M\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{p}_{\mathrm{tot}}}{dt} \tag{9.38}$$

That is, the resultant external force on a system of particles equals the total mass of the system multiplied by the acceleration of the center of mass. If we compare this with Newton's second law for a single particle, we see that

The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the resultant external force on the system.

Finally, we see that if the resultant external force is zero, then from Equation 9.38 it follows that

$$\frac{d\mathbf{p}_{\mathrm{tot}}}{dt} = M\mathbf{a}_{\mathrm{CM}} = 0$$

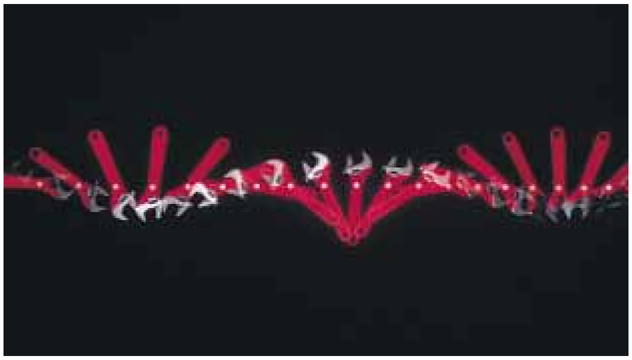


Figure 9.25 Multiflash photograph showing an overhead view of a wrench moving on a horizontal surface. The center of mass of the wrench moves in a straight line as the wrench rotates about this point, shown by the white dots.

so that

$$\mathbf{p}_{\text{tot}} = M\mathbf{v}_{\text{CM}} = \text{constant} \qquad (\text{when } \Sigma\mathbf{F}_{\text{ext}} = 0) \qquad \textbf{(9.39)}$$

That is, the total linear momentum of a system of particles is conserved if no net external force is acting on the system. It follows that for an isolated system of particles, both the total momentum and the velocity of the center of mass are constant in time, as shown in Figure 9.25. This is a generalization to a many-particle system of the law of conservation of momentum discussed in Section 9.1 for a two-particle system.

Suppose an isolated system consisting of two or more members is at rest. The center of mass of such a system remains at rest unless acted upon by an external force. For example, consider a system made up of a swimmer standing on a raft, with the system initially at rest. When the swimmer dives horizontally off the raft, the center of mass of the system remains at rest (if we neglect friction between raft and water). Furthermore, the linear momentum of the diver is equal in magnitude to that of the raft but opposite in direction.

As another example, suppose an unstable atom initially at rest suddenly breaks up into two fragments of masses M_A and M_B, with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. Because the total momentum of the system before the breakup is zero, the total momentum of the system after the breakup must also be zero. Therefore, $M_A\mathbf{v}_A + M_B\mathbf{v}_B = 0$. If the velocity of one of the fragments is known, the recoil velocity of the other fragment can be calculated.

EXAMPLE 9.15 The Sliding Bear

Suppose you tranquilize a polar bear on a smooth glacier as part of a research effort. How might you estimate the bear's mass using a measuring tape, a rope, and knowledge of your own mass?

Solution Tie one end of the rope around the bear, and then lay out the tape measure on the ice with one end at the bear's original position, as shown in Figure 9.26. Grab hold of the free end of the rope and position yourself as shown, noting your location. Take off your spiked shoes and pull on the rope hand over hand. Both you and the bear will slide over the ice until you meet. From the tape, observe how far you have slid, x_p, and how far the bear has slid, x_b. The point where you meet the bear is the constant location of the center of mass of the system (bear plus you), and so you can determine the mass of the bear from $m_b x_b = m_p x_p$. (Unfortunately, you cannot get back to your spiked shoes and so are in big trouble if the bear wakes up!)

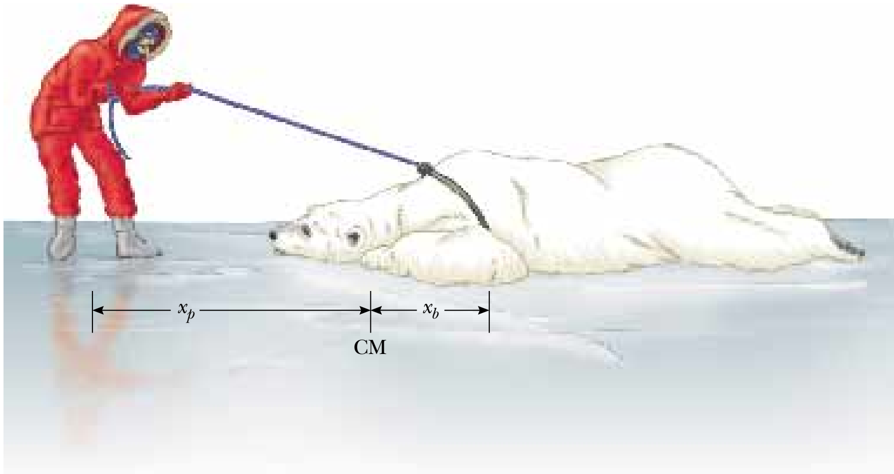
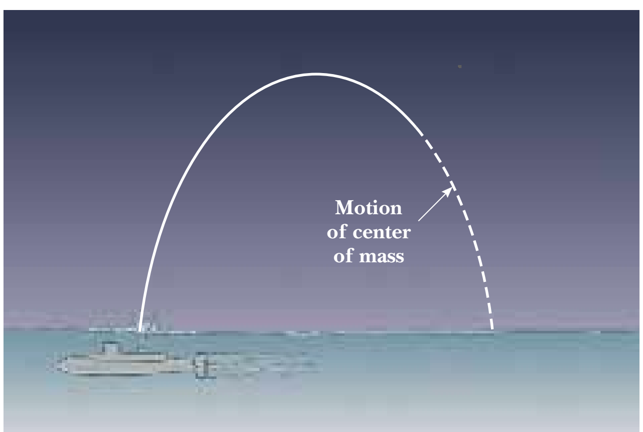


Figure 9.26 The center of mass of an isolated system remains at rest unless acted on by an external force. How can you determine the mass of the polar bear?

CONCEPTUAL EXAMPLE 9.16 Exploding Projectile

A projectile fired into the air suddenly explodes into several fragments (Fig. 9.27). What can be said about the motion of the center of mass of the system made up of all the fragments after the explosion?



Solution Neglecting air resistance, the only external force on the projectile is the gravitational force. Thus, if the projectile did not explode, it would continue to move along the parabolic path indicated by the broken line in Figure 9.27. Because the forces caused by the explosion are internal, they do not affect the motion of the center of mass. Thus, after the explosion the center of mass of the system (the fragments) follows the same parabolic path the projectile would have followed if there had been no explosion.

Figure 9.27 When a projectile explodes into several fragments, the center of mass of the system made up of all the fragments follows the same parabolic path the projectile would have taken had there been no explosion.

EXAMPLE 9.17 The Exploding Rocket

A rocket is fired vertically upward. At the instant it reaches an altitude of 1 000 m and a speed of 300 m/s, it explodes into three equal fragments. One fragment continues to move upward with a speed of 450 m/s following the explosion. The second fragment has a speed of 240 m/s and is moving east right after the explosion. What is the velocity of the third fragment right after the explosion?

Solution Let us call the total mass of the rocket M; hence, the mass of each fragment is $M/3$. Because the forces of the explosion are internal to the system and cannot affect its total momentum, the total momentum $\mathbf{p}_i$ of the rocket just before the explosion must equal the total momentum $\mathbf{p}_f$ of the fragments right after the explosion.

Before the explosion:

$$\mathbf{p}_i = M\mathbf{v}_i = M(300\mathbf{j})\text{ m/s}$$

After the explosion:

$$\mathbf{p}_f = \frac{M}{3}(240\mathbf{i})\text{ m/s} + \frac{M}{3}(450\mathbf{j})\text{ m/s} + \frac{M}{3}\mathbf{v}_f$$

where $\mathbf{v}_f$ is the unknown velocity of the third fragment. Equating these two expressions (because $\mathbf{p}_i = \mathbf{p}_f$) gives

$$\frac{M}{3}\mathbf{v}_f + M(80\mathbf{i})\text{ m/s} + M(150\mathbf{j})\text{ m/s} = M(300\mathbf{j})\text{ m/s}$$

$$\mathbf{v}_f = (-240\mathbf{i} + 450\mathbf{j})\text{ m/s}$$

What does the sum of the momentum vectors for all the fragments look like?

Exercise Find the position of the center of mass of the system of fragments relative to the ground 3.00 s after the explosion. Assume the rocket engine is nonoperative after the explosion.

Answer The x coordinate does not change; $y_{CM} = 1.86$ km.

Optional Section

9.8 ROCKET PROPULSION

When ordinary vehicles, such as automobiles and locomotives, are propelled, the driving force for the motion is friction. In the case of the automobile, the driving force is the force exerted by the road on the car. A locomotive "pushes" against the tracks; hence, the driving force is the force exerted by the tracks on the locomotive. However, a rocket moving in space has no road or tracks to push against. Therefore, the source of the propulsion of a rocket must be something other than friction. Figure 9.28 is a dramatic photograph of a spacecraft at liftoff. **The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.**

Rocket propulsion can be understood by first considering the mechanical system consisting of a machine gun mounted on a cart on wheels. As the gun is fired,

Figure 9.28 Liftoff of the space shuttle *Columbia*. Enormous thrust is generated by the shuttle's liquid-fuel engines, aided by the two solid-fuel boosters. Many physical principles from mechanics, thermodynamics, and electricity and magnetism are involved in such a launch.

The force from a nitrogen-propelled, hand-controlled device allows an astronaut to move about freely in space without restrictive tethers.

each bullet receives a momentum $m\mathbf{v}$ in some direction, where $\mathbf{v}$ is measured with respect to a stationary Earth frame. The momentum of the system made up of cart, gun, and bullets must be conserved. Hence, for each bullet fired, the gun and cart must receive a compensating momentum in the opposite direction. That is, the reaction force exerted by the bullet on the gun accelerates the cart and gun, and the cart moves in the direction opposite that of the bullets. If n is the number of bullets fired each second, then the average force exerted on the gun is $\mathbf{F}_{\text{av}} = nm\mathbf{v}$.

In a similar manner, as a rocket moves in free space, its linear momentum changes when some of its mass is released in the form of ejected gases. **Because the gases are given momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction.** Therefore, the rocket is accelerated as a result of the "push," or thrust, from the exhaust gases. In free space, the center of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process.[5]

Suppose that at some time t, the magnitude of the momentum of a rocket plus its fuel is $(M + \Delta m)v$, where v is the speed of the rocket relative to the Earth (Fig. 9.29a). Over a short time interval Δt, the rocket ejects fuel of mass Δm, and so at the end of this interval the rocket's speed is $v + \Delta v$, where Δv is the change in speed of the rocket (Fig. 9.29b). If the fuel is ejected with a speed v_e relative to the rocket (the subscript "e" stands for *exhaust,* and v_e is usually called the *exhaust speed*), the velocity of the fuel relative to a stationary frame of reference is $v - v_e$. Thus, if we equate the total initial momentum of the system to the total final momentum, we obtain

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

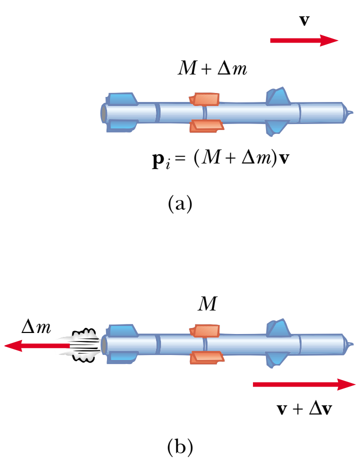


Figure 9.29 Rocket propulsion. (a) The initial mass of the rocket plus all its fuel is $M + \Delta m$ at a time t, and its speed is v. (b) At a time $t + \Delta t$, the rocket's mass has been reduced to M and an amount of fuel Δm has been ejected. The rocket's speed increases by an amount Δv.

where M represents the mass of the rocket and its remaining fuel after an amount of fuel having mass Δm has been ejected. Simplifying this expression gives

$$M\,\Delta v = v_e\,\Delta m$$

We also could have arrived at this result by considering the system in the center-of-mass frame of reference, which is a frame having the same velocity as the center of mass of the system. In this frame, the total momentum of the system is zero; therefore, if the rocket gains a momentum $M\,\Delta v$ by ejecting some fuel, the exhausted fuel obtains a momentum $v_e\,\Delta m$ in the *opposite* direction, so that $M\,\Delta v - v_e\,\Delta m = 0$. If we now take the limit as Δt goes to zero, we get $\Delta v \rightarrow dv$ and $\Delta m \rightarrow dm$. Futhermore, the increase in the exhaust mass dm corresponds to an equal decrease in the rocket mass, so that $dm = -dM$. Note that dM is given a negative sign because it represents a decrease in mass. Using this fact, we obtain

$$M\,dv = v_e\,dm = -v_e\,dM \tag{9.40}$$

Integrating this equation and taking the initial mass of the rocket plus fuel to be M_i and the final mass of the rocket plus its remaining fuel to be M_f, we obtain

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

Expression for rocket propulsion

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \tag{9.41}$$

[5]It is interesting to note that the rocket and machine gun represent cases of the reverse of a perfectly inelastic collision: Momentum is conserved, but the kinetic energy of the system increases (at the expense of chemical potential energy in the fuel).

This is the basic expression of rocket propulsion. First, it tells us that the increase in rocket speed is proportional to the exhaust speed of the ejected gases, v_e. Therefore, the exhaust speed should be very high. Second, the increase in rocket speed is proportional to the natural logarithm of the ratio M_i/M_f. Therefore, this ratio should be as large as possible, which means that the mass of the rocket without its fuel should be as small as possible and the rocket should carry as much fuel as possible.

The **thrust** on the rocket is the force exerted on it by the ejected exhaust gases. We can obtain an expression for the thrust from Equation 9.40:

$$\text{Thrust} = M\frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right| \qquad \textbf{(9.42)}$$

This expression shows us that the thrust increases as the exhaust speed increases and as the rate of change of mass (called the *burn rate*) increases.

EXAMPLE 9.18 A Rocket in Space

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to one-half its mass before ignition?

Solution We can guess that the speed we are looking for must be greater than the original speed because the rocket is accelerating. Applying Equation 9.41, we obtain

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right)$$

$$= 3.0 \times 10^3 \text{ m/s} + (5.0 \times 10^3 \text{ m/s})\ln\left(\frac{M_i}{0.5\ M_i}\right)$$

$$= \boxed{6.5 \times 10^3 \text{ m/s}}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Solution

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right| = (5.0 \times 10^3 \text{ m/s})(50 \text{ kg/s})$$

$$= \boxed{2.5 \times 10^5 \text{ N}}$$

EXAMPLE 9.19 Fighting a Fire

Two firefighters must apply a total force of 600 N to steady a hose that is discharging water at 3 600 L/min. Estimate the speed of the water as it exits the nozzle.

Solution The water is exiting at 3 600 L/min, which is 60 L/s. Knowing that 1 L of water has a mass of 1 kg, we can say that about 60 kg of water leaves the nozzle every second. As the water leaves the hose, it exerts on the hose a thrust that must be counteracted by the 600-N force exerted on the hose by the firefighters. So, applying Equation 9.42 gives

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right|$$

$$600 \text{ N} = |v_e(60 \text{ kg/s})|$$

$$v_e = \boxed{10 \text{ m/s}}$$

Firefighting is dangerous work. If the nozzle should slip from their hands, the movement of the hose due to the thrust it receives from the rapidly exiting water could injure the firefighters.

Firefighters attack a burning house with a hose line.

SUMMARY

The **linear momentum p** of a particle of mass m moving with a velocity $\mathbf{v}$ is

$$\mathbf{p} \equiv m\mathbf{v} \tag{9.1}$$

The law of **conservation of linear momentum** indicates that the total momentum of an isolated system is conserved. If two particles form an isolated system, their total momentum is conserved regardless of the nature of the force between them. Therefore, the total momentum of the system at all times equals its initial total momentum, or

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{9.5}$$

The **impulse** imparted to a particle by a force $\mathbf{F}$ is equal to the change in the momentum of the particle:

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt = \Delta\mathbf{p} \tag{9.9}$$

This is known as the **impulse–momentum theorem.**

Impulsive forces are often very strong compared with other forces on the system and usually act for a very short time, as in the case of collisions.

When two particles collide, the total momentum of the system before the collision always equals the total momentum after the collision, regardless of the nature of the collision. An **inelastic collision** is one for which the total kinetic energy is not conserved. A **perfectly inelastic collision** is one in which the colliding bodies stick together after the collision. An **elastic collision** is one in which kinetic energy is constant.

In a two- or three-dimensional collision, the components of momentum in each of the three directions (x, y, and z) are conserved independently.

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \tag{9.30}$$

where $M = \sum_i m_i$ is the total mass of the system and $\mathbf{r}_i$ is the position vector of the ith particle.

The position vector of the center of mass of a rigid body can be obtained from the integral expression

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \int \mathbf{r}\, dm \tag{9.33}$$

The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{\sum_i m_i \mathbf{v}_i}{M} \tag{9.34}$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

Newton's second law applied to a system of particles is

$$\sum \mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}} = \frac{d\mathbf{p}_{\text{tot}}}{dt} \tag{9.38}$$

where $\mathbf{a}_{\text{CM}}$ is the acceleration of the center of mass and the sum is over all external forces. The center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. It follows from Equation 9.38 that the total momentum of the system is conserved if there are no external forces acting on it.

QUESTIONS

1. If the kinetic energy of a particle is zero, what is its linear momentum?

2. If the speed of a particle is doubled, by what factor is its momentum changed? By what factor is its kinetic energy changed?

3. If two particles have equal kinetic energies, are their momenta necessarily equal? Explain.

4. If two particles have equal momenta, are their kinetic energies necessarily equal? Explain.

5. An isolated system is initially at rest. Is it possible for parts of the system to be in motion at some later time? If so, explain how this might occur.

6. If two objects collide and one is initially at rest, is it possible for both to be at rest after the collision? Is it possible for one to be at rest after the collision? Explain.

7. Explain how linear momentum is conserved when a ball bounces from a floor.

8. Is it possible to have a collision in which all of the kinetic energy is lost? If so, cite an example.

9. In a perfectly elastic collision between two particles, does the kinetic energy of each particle change as a result of the collision?

10. When a ball rolls down an incline, its linear momentum increases. Does this imply that momentum is not conserved? Explain.

11. Consider a perfectly inelastic collision between a car and a large truck. Which vehicle loses more kinetic energy as a result of the collision?

12. Can the center of mass of a body lie outside the body? If so, give examples.

13. Three balls are thrown into the air simultaneously. What is the acceleration of their center of mass while they are in motion?

14. A meter stick is balanced in a horizontal position with the index fingers of the right and left hands. If the two fingers are slowly brought together, the stick remains balanced and the two fingers always meet at the 50-cm mark regardless of their original positions (try it!). Explain.

15. A sharpshooter fires a rifle while standing with the butt of the gun against his shoulder. If the forward momentum of a bullet is the same as the backward momentum of the gun, why is it not as dangerous to be hit by the gun as by the bullet?

16. A piece of mud is thrown against a brick wall and sticks to the wall. What happens to the momentum of the mud? Is momentum conserved? Explain.

17. Early in this century, Robert Goddard proposed sending a rocket to the Moon. Critics took the position that in a vacuum, such as exists between the Earth and the Moon, the gases emitted by the rocket would have nothing to push against to propel the rocket. According to *Scientific American* (January 1975), Goddard placed a gun in a vacuum and fired a blank cartridge from it. (A blank cartridge fires only the wadding and hot gases of the burning gunpowder.) What happened when the gun was fired?

18. A pole-vaulter falls from a height of 6.0 m onto a foam rubber pad. Can you calculate his speed just before he reaches the pad? Can you estimate the force exerted on him due to the collision? Explain.

19. Explain how you would use a balloon to demonstrate the mechanism responsible for rocket propulsion.

20. Does the center of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.

21. A ball is dropped from a tall building. Identify the system for which linear momentum is conserved.

22. A bomb, initially at rest, explodes into several pieces. (a) Is linear momentum conserved? (b) Is kinetic energy conserved? Explain.

23. NASA often uses the gravity of a planet to "slingshot" a probe on its way to a more distant planet. This is actually a collision where the two objects do not touch. How can the probe have its speed increased in this manner?

24. The Moon revolves around the Earth. Is the Moon's linear momentum conserved? Is its kinetic energy conserved? Assume that the Moon's orbit is circular.

25. A raw egg dropped to the floor breaks apart upon impact. However, a raw egg dropped onto a thick foam rubber cushion from a height of about 1 m rebounds without breaking. Why is this possible? (If you try this experiment, be sure to catch the egg after the first bounce.)

26. On the subject of the following positions, state your own view and argue to support it: (a) The best theory of motion is that force causes acceleration. (b) The true measure of a force's effectiveness is the work it does, and the best theory of motion is that work on an object changes its energy. (c) The true measure of a force's effect is impulse, and the best theory of motion is that impulse injected into an object changes its momentum.

Problems

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 9.1 Linear Momentum and Its Conservation

1. A 3.00-kg particle has a velocity of $(3.00\mathbf{i} - 4.00\mathbf{j})$ m/s. (a) Find its x and y components of momentum. (b) Find the magnitude and direction of its momentum.
2. A 0.100-kg ball is thrown straight up into the air with an initial speed of 15.0 m/s. Find the momentum of the ball (a) at its maximum height and (b) halfway up to its maximum height.
3. A 40.0-kg child standing on a frozen pond throws a 0.500-kg stone to the east with a speed of 5.00 m/s. Neglecting friction between child and ice, find the recoil velocity of the child.
4. A pitcher claims he can throw a baseball with as much momentum as a 3.00-g bullet moving with a speed of 1 500 m/s. A baseball has a mass of 0.145 kg. What must be its speed if the pitcher's claim is valid?
5. How fast can you set the Earth moving? In particular, when you jump straight up as high as you can, you give the Earth a maximum recoil speed of what order of magnitude? Model the Earth as a perfectly solid object. In your solution, state the physical quantities you take as data and the values you measure or estimate for them.
6. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them, and the blocks are pushed together with the spring between them (Fig. P9.6). A cord initially holding the blocks together is burned; after this, the block of mass $3M$ moves to the right with a speed of 2.00 m/s. (a) What is the speed of the block of mass M? (b) Find the original elastic energy in the spring if M = 0.350 kg.

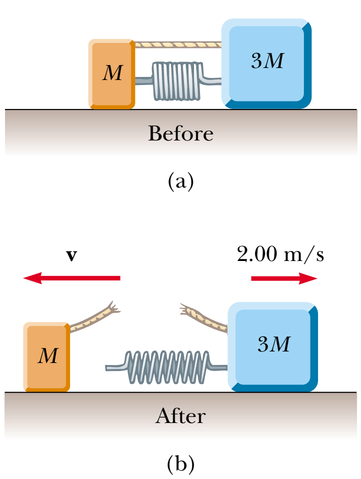


Figure P9.6

7. (a) A particle of mass m moves with momentum p. Show that the kinetic energy of the particle is given by $K = p^2/2m$. (b) Express the magnitude of the particle's momentum in terms of its kinetic energy and mass.

Section 9.2 Impulse and Momentum

8. A car is stopped for a traffic signal. When the light turns green, the car accelerates, increasing its speed from zero to 5.20 m/s in 0.832 s. What linear impulse and average force does a 70.0-kg passenger in the car experience?
9. An estimated force–time curve for a baseball struck by a bat is shown in Figure P9.9. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

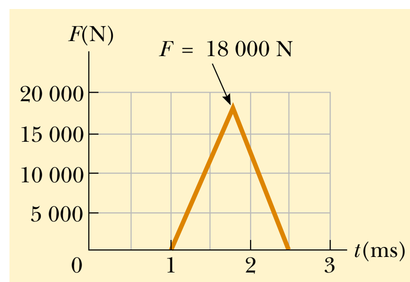


Figure P9.9

10. A tennis player receives a shot with the ball (0.060 0 kg) traveling horizontally at 50.0 m/s and returns the shot with the ball traveling horizontally at 40.0 m/s in the opposite direction. (a) What is the impulse delivered to the ball by the racket? (b) What work does the racket do on the ball?
11. WEB A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle (Fig. P9.11). If the ball is in contact with the wall for 0.200 s, what is the average force exerted on the ball by the wall?
12. In a slow-pitch softball game, a 0.200-kg softball crossed the plate at 15.0 m/s at an angle of 45.0° below the horizontal. The ball was hit at 40.0 m/s, 30.0° above the horizontal. (a) Determine the impulse delivered to the ball. (b) If the force on the ball increased linearly for 4.00 ms, held constant for 20.0 ms, and then decreased to zero linearly in another 4.00 ms, what was the maximum force on the ball?

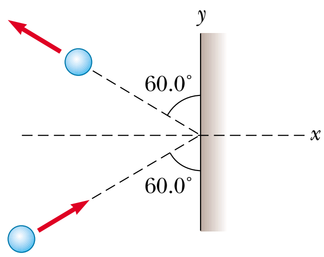


Figure P9.11

13. A garden hose is held in the manner shown in Figure P9.13. The hose is initially full of motionless water. What additional force is necessary to hold the nozzle stationary after the water is turned on if the discharge rate is 0.600 kg/s with a speed of 25.0 m/s?

Figure P9.13

14. A professional diver performs a dive from a platform 10 m above the water surface. Estimate the order of magnitude of the average impact force she experiences in her collision with the water. State the quantities you take as data and their values.

Section 9.3 Collisions

Section 9.4 Elastic and Inelastic Collisions in One Dimension

15. High-speed stroboscopic photographs show that the head of a golf club of mass 200 g is traveling at 55.0 m/s just before it strikes a 46.0-g golf ball at rest on a tee. After the collision, the club head travels (in the same direction) at 40.0 m/s. Find the speed of the golf ball just after impact.
16. A 75.0-kg ice skater, moving at 10.0 m/s, crashes into a stationary skater of equal mass. After the collision, the two skaters move as a unit at 5.00 m/s. Suppose the average force a skater can experience without breaking a bone is 4 500 N. If the impact time is 0.100 s, does a bone break?
17. A 10.0-g bullet is fired into a stationary block of wood (m = 5.00 kg). The relative motion of the bullet stops inside the block. The speed of the bullet-plus-wood combination immediately after the collision is measured as 0.600 m/s. What was the original speed of the bullet?
18. As shown in Figure P9.18, a bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed of $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

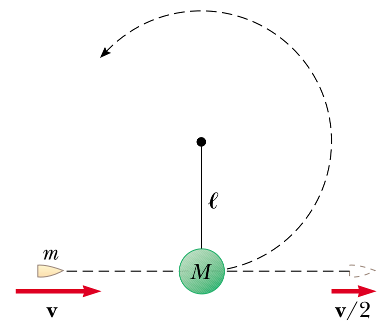


Figure P9.18

19. A 45.0-kg girl is standing on a plank that has a mass of 150 kg. The plank, originally at rest, is free to slide on a frozen lake, which is a flat, frictionless supporting surface. The girl begins to walk along the plank at a constant speed of 1.50 m/s relative to the plank. (a) What is her speed relative to the ice surface? (b) What is the speed of the plank relative to the ice surface?
20. Gayle runs at a speed of 4.00 m/s and dives on a sled, which is initially at rest on the top of a frictionless snow-covered hill. After she has descended a vertical distance of 5.00 m, her brother, who is initially at rest, hops on her back and together they continue down the hill. What is their speed at the bottom of the hill if the total vertical drop is 15.0 m? Gayle's mass is 50.0 kg, the sled has a mass of 5.00 kg and her brother has a mass of 30.0 kg.
21. A 1 200-kg car traveling initially with a speed of 25.0 m/s in an easterly direction crashes into the rear end of a 9 000-kg truck moving in the same direction at 20.0 m/s (Fig. P9.21). The velocity of the car right after the collision is 18.0 m/s to the east. (a) What is the velocity of the truck right after the collision? (b) How much mechanical energy is lost in the collision? Account for this loss in energy.
22. A railroad car of mass 2.50×10^4 kg is moving with a speed of 4.00 m/s. It collides and couples with three other coupled railroad cars, each of the same mass as the single car and moving in the same direction with an initial speed of 2.00 m/s. (a) What is the speed of the four cars after the collision? (b) How much energy is lost in the collision?

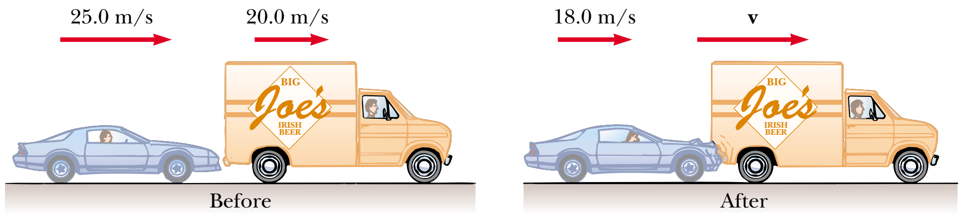


Figure P9.21

23. Four railroad cars, each of mass 2.50×10^4 kg, are coupled together and coasting along horizontal tracks at a speed of v_i toward the south. A very strong but foolish movie actor, riding on the second car, uncouples the front car and gives it a big push, increasing its speed to 4.00 m/s southward. The remaining three cars continue moving toward the south, now at 2.00 m/s. (a) Find the initial speed of the cars. (b) How much work did the actor do? (c) State the relationship between the process described here and the process in Problem 22.

24. A 7.00-kg bowling ball collides head-on with a 2.00-kg bowling pin. The pin flies forward with a speed of 3.00 m/s. If the ball continues forward with a speed of 1.80 m/s, what was the initial speed of the ball? Ignore rotation of the ball.

WEB **25.** A neutron in a reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. (a) What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (b) If the initial kinetic energy of the neutron is 1.60×10^{-13} J, find its final kinetic energy and the kinetic energy of the carbon nucleus after the collision. (The mass of the carbon nucleus is about 12.0 times greater than the mass of the neutron.)

26. Consider a frictionless track *ABC* as shown in Figure P9.26. A block of mass $m_1 = 5.00$ kg is released from *A*. It makes a head-on elastic collision at *B* with a block of mass $m_2 = 10.0$ kg that is initially at rest. Calculate the maximum height to which m_1 rises after the collision.

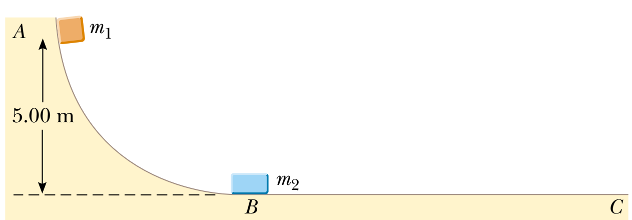


Figure P9.26

WEB **27.** A 12.0-g bullet is fired into a 100-g wooden block initially at rest on a horizontal surface. After impact, the block slides 7.50 m before coming to rest. If the coefficient of friction between the block and the surface is 0.650, what was the speed of the bullet immediately before impact?

28. A 7.00-g bullet, when fired from a gun into a 1.00-kg block of wood held in a vise, would penetrate the block to a depth of 8.00 cm. This block of wood is placed on a frictionless horizontal surface, and a 7.00-g bullet is fired from the gun into the block. To what depth will the bullet penetrate the block in this case?

Section 9.5 Two-Dimensional Collisions

29. A 90.0-kg fullback running east with a speed of 5.00 m/s is tackled by a 95.0-kg opponent running north with a speed of 3.00 m/s. If the collision is perfectly inelastic, (a) calculate the speed and direction of the players just after the tackle and (b) determine the energy lost as a result of the collision. Account for the missing energy.

30. The mass of the blue puck in Figure P9.30 is 20.0% greater than the mass of the green one. Before colliding, the pucks approach each other with equal and opposite momenta, and the green puck has an initial speed of 10.0 m/s. Find the speeds of the pucks after the collision if half the kinetic energy is lost during the collision.

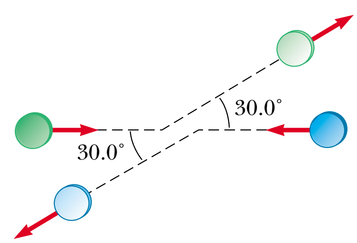


Figure P9.30

31. Two automobiles of equal mass approach an intersection. One vehicle is traveling with velocity 13.0 m/s toward the east and the other is traveling north with a speed of v_{2i}. Neither driver sees the other. The vehicles collide in the intersection and stick together, leaving parallel skid marks at an angle of 55.0° north of east. The speed limit for both roads is 35 mi/h, and the driver of the northward-moving vehicle claims he was within the speed limit when the collision occurred. Is he telling the truth?

32. A proton, moving with a velocity of $v_i\mathbf{i}$, collides elastically with another proton that is initially at rest. If the two protons have equal speeds after the collision, find (a) the speed of each proton after the collision in terms of v_i and (b) the direction of the velocity vectors after the collision.

33. A billiard ball moving at 5.00 m/s strikes a stationary ball of the same mass. After the collision, the first ball moves at 4.33 m/s and at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision (and ignoring friction and rotational motion), find the struck ball's velocity.

34. A 0.300-kg puck, initially at rest on a horizontal, frictionless surface, is struck by a 0.200-kg puck moving initially along the x axis with a speed of 2.00 m/s. After the collision, the 0.200-kg puck has a speed of 1.00 m/s at an angle of $\theta = 53.0°$ to the positive x axis (see Fig. 9.14). (a) Determine the velocity of the 0.300-kg puck after the collision. (b) Find the fraction of kinetic energy lost in the collision.

35. A 3.00-kg mass with an initial velocity of $5.00\mathbf{i}$ m/s collides with and sticks to a 2.00-kg mass with an initial velocity of $-3.00\mathbf{j}$ m/s. Find the final velocity of the composite mass.

36. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed of 5.00 m/s. After the collision, the orange disk moves along a direction that makes an angle of 37.0° with its initial direction of motion, and the velocity of the yellow disk is perpendicular to that of the orange disk (after the collision). Determine the final speed of each disk.

37. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed v_i. After the collision, the orange disk moves along a direction that makes an angle θ with its initial direction of motion, and the velocity of the yellow disk is perpendicular to that of the orange disk (after the collision). Determine the final speed of each disk.

38. During the battle of Gettysburg, the gunfire was so intense that several bullets collided in midair and fused together. Assume a 5.00-g Union musket ball was moving to the right at a speed of 250 m/s, 20.0° above the horizontal, and that a 3.00-g Confederate ball was moving to the left at a speed of 280 m/s, 15.0° above the horizontal. Immediately after they fuse together, what is their velocity?

WEB 39. An unstable nucleus of mass 17.0×10^{-27} kg initially at rest disintegrates into three particles. One of the particles, of mass 5.00×10^{-27} kg, moves along the y axis with a velocity of 6.00×10^6 m/s. Another particle, of mass 8.40×10^{-27} kg, moves along the x axis with a speed of 4.00×10^6 m/s. Find (a) the velocity of the third particle and (b) the total kinetic energy increase in the process.

Section 9.6 The Center of Mass

40. Four objects are situated along the y axis as follows: A 2.00-kg object is at +3.00 m, a 3.00-kg object is at +2.50 m, a 2.50-kg object is at the origin, and a 4.00-kg object is at −0.500 m. Where is the center of mass of these objects?

41. A uniform piece of sheet steel is shaped as shown in Figure P9.41. Compute the x and y coordinates of the center of mass of the piece.

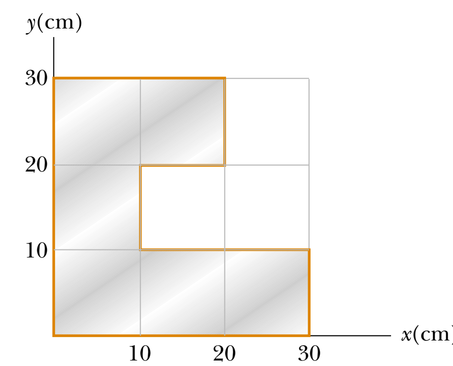


Figure P9.41

42. The mass of the Earth is 5.98×10^{24} kg, and the mass of the Moon is 7.36×10^{22} kg. The distance of separation, measured between their centers, is 3.84×10^8 m. Locate the center of mass of the Earth–Moon system as measured from the center of the Earth.

43. A water molecule consists of an oxygen atom with two hydrogen atoms bound to it (Fig. P9.43). The angle between the two bonds is 106°. If the bonds are 0.100 nm long, where is the center of mass of the molecule?

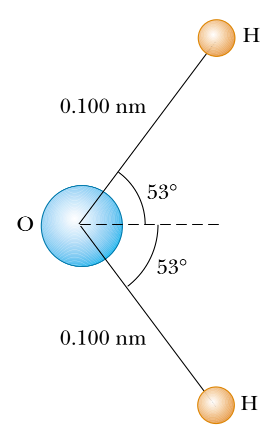


Figure P9.43

44. A 0.400-kg mass m_1 has position $\mathbf{r}_1 = 12.0\mathbf{j}$ cm. A 0.800-kg mass m_2 has position $\mathbf{r}_2 = -12.0\mathbf{i}$ cm. Another 0.800-kg mass m_3 has position $\mathbf{r}_3 = (12.0\mathbf{i} - 12.0\mathbf{j})$ cm. Make a drawing of the masses. Start from the origin and, to the scale 1 cm = 1 kg·cm, construct the vector $m_1\mathbf{r}_1$, then the vector $m_1\mathbf{r}_1 + m_2\mathbf{r}_2$, then the vector $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$, and at last $\mathbf{r}_{CM} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)/(m_1 + m_2 + m_3)$. Observe that the head of the vector $\mathbf{r}_{CM}$ indicates the position of the center of mass.

45. A rod of length 30.0 cm has linear density (mass-per-length) given by

$$\lambda = 50.0 \text{ g/m} + 20.0x \text{ g/m}^2$$

where x is the distance from one end, measured in meters. (a) What is the mass of the rod? (b) How far from the $x = 0$ end is its center of mass?

Section 9.7 Motion of a System of Particles

46. Consider a system of two particles in the xy plane: $m_1 = 2.00$ kg is at $\mathbf{r}_1 = (1.00\mathbf{i} + 2.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} + 0.500\mathbf{j})$ m/s; $m_2 = 3.00$ kg is at $\mathbf{r}_2 = (-4.00\mathbf{i} - 3.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} - 2.00\mathbf{j})$ m/s. (a) Plot these particles on a grid or graph paper. Draw their position vectors and show their velocities. (b) Find the position of the center of mass of the system and mark it on the grid. (c) Determine the velocity of the center of mass and also show it on the diagram. (d) What is the total linear momentum of the system?

47. Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat at rest in still water, 2.70 m away from Juliet who is in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?

48. Two masses, 0.600 kg and 0.300 kg, begin uniform motion at the same speed, 0.800 m/s, from the origin at $t = 0$ and travel in the directions shown in Figure P9.48. (a) Find the velocity of the center of mass in unit–vector notation. (b) Find the magnitude and direction of the velocity of the center of mass. (c) Write the position vector of the center of mass as a function of time.

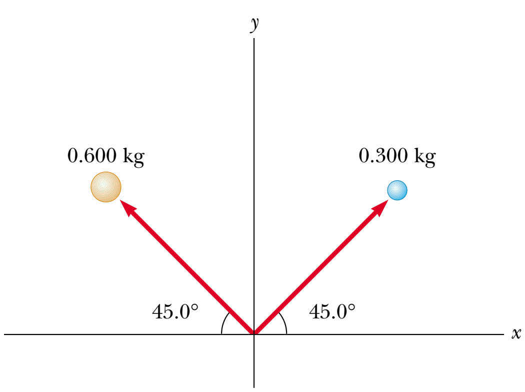


Figure P9.48

49. A 2.00-kg particle has a velocity of $(2.00\mathbf{i} - 3.00\mathbf{j})$ m/s, and a 3.00-kg particle has a velocity of $(1.00\mathbf{i} + 6.00\mathbf{j})$ m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system.

50. A ball of mass 0.200 kg has a velocity of $1.50\mathbf{i}$ m/s; a ball of mass 0.300 kg has a velocity of $-0.400\mathbf{i}$ m/s. They meet in a head-on elastic collision. (a) Find their velocities after the collision. (b) Find the velocity of their center of mass before and after the collision.

(Optional)

Section 9.8 Rocket Propulsion

WEB **51.** The first stage of a Saturn V space vehicle consumes fuel and oxidizer at the rate of 1.50×10^4 kg/s, with an exhaust speed of 2.60×10^3 m/s. (a) Calculate the thrust produced by these engines. (b) Find the initial acceleration of the vehicle on the launch pad if its initial mass is 3.00×10^6 kg. [*Hint:* You must include the force of gravity to solve part (b).]

52. A large rocket with an exhaust speed of $v_e = 3\,000$ m/s develops a thrust of 24.0 million newtons. (a) How much mass is being blasted out of the rocket exhaust per second? (b) What is the maximum speed the rocket can attain if it starts from rest in a force-free environment with $v_e = 3.00$ km/s and if 90.0% of its initial mass is fuel and oxidizer?

53. A rocket for use in deep space is to have the capability of boosting a total load (payload plus rocket frame and engine) of 3.00 metric tons to a speed of 10 000 m/s. (a) It has an engine and fuel designed to produce an exhaust speed of 2 000 m/s. How much fuel plus oxidizer is required? (b) If a different fuel and engine design could give an exhaust speed of 5 000 m/s, what amount of fuel and oxidizer would be required for the same task?

54. A rocket car has a mass of 2 000 kg unfueled and a mass of 5 000 kg when completely fueled. The exhaust velocity is 2 500 m/s. (a) Calculate the amount of fuel used to accelerate the completely fueled car from rest to 225 m/s (about 500 mi/h). (b) If the burn rate is constant at 30.0 kg/s, calculate the time it takes the car to reach this speed. Neglect friction and air resistance.

ADDITIONAL PROBLEMS

55. Review Problem. A 60.0-kg person running at an initial speed of 4.00 m/s jumps onto a 120-kg cart initially at rest (Fig. P9.55). The person slides on the cart's top surface and finally comes to rest relative to the cart. The coefficient of kinetic friction between the person and the cart is 0.400. Friction between the cart and ground can be neglected. (a) Find the final velocity of the person and cart relative to the ground. (b) Find the frictional force acting on the person while he is sliding across the top surface of the cart. (c) How long does the frictional force act on the person? (d) Find the change in momentum of the person and the change in momentum of the cart. (e) Determine the displacement of the person relative to the ground while he is sliding on the cart. (f) Determine the displacement of the cart relative to the ground while the person is sliding. (g) Find the change in kinetic energy of the person. (h) Find the change in kinetic energy of the cart. (i) Explain why the answers to parts (g) and (h) differ. (What kind of collision is this, and what accounts for the loss of mechanical energy?)

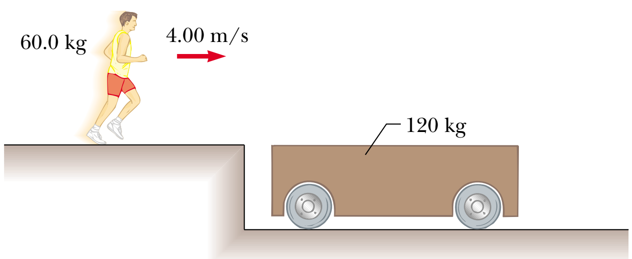


Figure P9.55

56. A golf ball (m = 46.0 g) is struck a blow that makes an angle of 45.0° with the horizontal. The ball lands 200 m away on a flat fairway. If the golf club and ball are in contact for 7.00 ms, what is the average force of impact? (Neglect air resistance.)

57. An 8.00-g bullet is fired into a 2.50-kg block that is initially at rest at the edge of a frictionless table of height 1.00 m (Fig. P9.57). The bullet remains in the block, and after impact the block lands 2.00 m from the bottom of the table. Determine the initial speed of the bullet.

58. A bullet of mass m is fired into a block of mass M that is initially at rest at the edge of a frictionless table of height h (see Fig. P9.57). The bullet remains in the block, and after impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet.

59. An 80.0-kg astronaut is working on the engines of his ship, which is drifting through space with a constant velocity. The astronaut, wishing to get a better view of the Universe, pushes against the ship and much later finds himself 30.0 m behind the ship and at rest with respect to it. Without a thruster, the only way to return to the ship is to throw his 0.500-kg wrench directly away from the ship. If he throws the wrench with a speed of 20.0 m/s relative to the ship, how long does it take the astronaut to reach the ship?

60. A small block of mass m_1 = 0.500 kg is released from rest at the top of a curve-shaped frictionless wedge of mass m_2 = 3.00 kg, which sits on a frictionless horizontal surface, as shown in Figure P9.60a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right, as in Figure P9.60b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height h of the wedge?

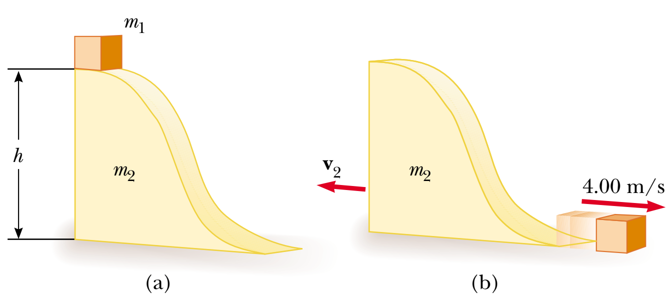


Figure P9.60

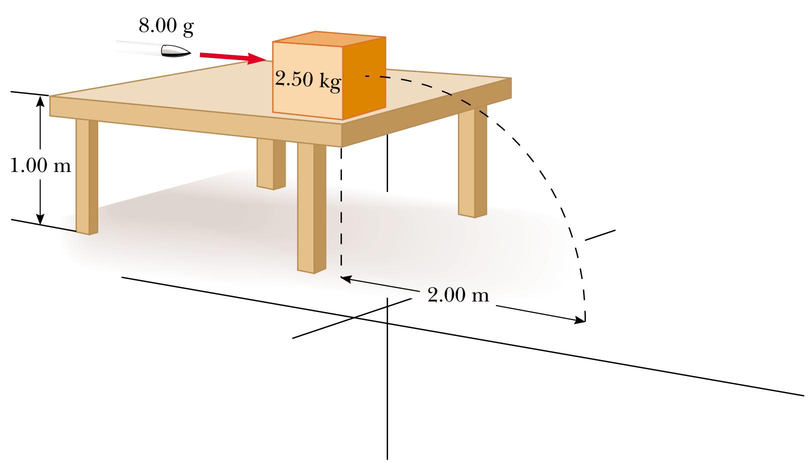


Figure P9.57 Problems 57 and 58.

61. Tarzan, whose mass is 80.0 kg, swings from a 3.00-m vine that is horizontal when he starts. At the bottom of his arc, he picks up 60.0-kg Jane in a perfectly inelastic collision. What is the height of the highest tree limb they can reach on their upward swing?

62. A jet aircraft is traveling at 500 mi/h (223 m/s) in horizontal flight. The engine takes in air at a rate of 80.0 kg/s and burns fuel at a rate of 3.00 kg/s. If the exhaust gases are ejected at 600 m/s relative to the aircraft, find the thrust of the jet engine and the delivered horsepower.

63. A 75.0-kg firefighter slides down a pole while a constant frictional force of 300 N retards her motion. A horizontal 20.0-kg platform is supported by a spring at the bottom of the pole to cushion the fall. The firefighter starts from rest 4.00 m above the platform, and the spring constant is 4 000 N/m. Find (a) the firefighter's speed just before she collides with the platform and (b) the maximum distance the spring is compressed. (Assume the frictional force acts during the entire motion.)

64. A cannon is rigidly attached to a carriage, which can move along horizontal rails but is connected to a post by a large spring, initially unstretched and with force constant $k = 2.00 \times 10^4$ N/m, as shown in Figure P9.64. The cannon fires a 200-kg projectile at a velocity of 125 m/s directed 45.0° above the horizontal. (a) If the mass of the cannon and its carriage is 5 000 kg, find the recoil speed of the cannon. (b) Determine the maximum extension of the spring. (c) Find the maximum force the spring exerts on the carriage. (d) Consider the system consisting of the cannon, carriage, and shell. Is the momentum of this system conserved during the firing? Why or why not?

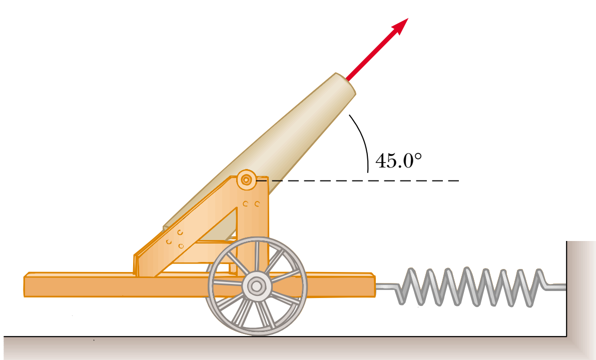


Figure P9.64

65. A chain of length L and total mass M is released from rest with its lower end just touching the top of a table, as shown in Figure P9.65a. Find the force exerted by the table on the chain after the chain has fallen through a distance x, as shown in Figure P9.65b. (Assume each link comes to rest the instant it reaches the table.)

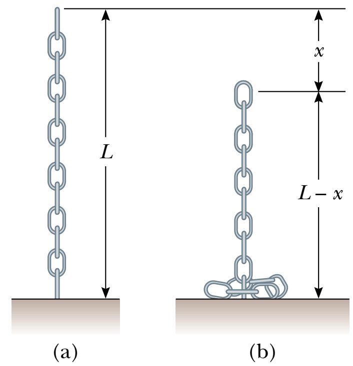


Figure P9.65

66. Two gliders are set in motion on an air track. A spring of force constant k is attached to the near side of one glider. The first glider of mass m_1 has a velocity of $\mathbf{v}_1$, and the second glider of mass m_2 has a velocity of $\mathbf{v}_2$, as shown in Figure P9.66 ($v_1 > v_2$). When m_1 collides with the spring attached to m_2 and compresses the spring to its maximum compression x_m, the velocity of the gliders is $\mathbf{v}$. In terms of $\mathbf{v}_1$, $\mathbf{v}_2$, m_1, m_2, and k, find (a) the velocity $\mathbf{v}$ at maximum compression, (b) the maximum compression x_m, and (c) the velocities of each glider after m_1 has lost contact with the spring.

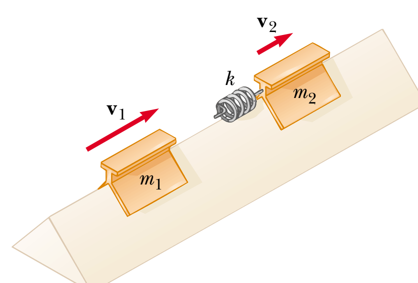


Figure P9.66

67. Sand from a stationary hopper falls onto a moving conveyor belt at the rate of 5.00 kg/s, as shown in Figure P9.67. The conveyor belt is supported by frictionless rollers and moves at a constant speed of 0.750 m/s under the action of a constant horizontal external force $\mathbf{F}_{ext}$ supplied by the motor that drives the belt. Find (a) the sand's rate of change of momentum in the horizontal direction, (b) the force of friction exerted by the belt on the sand, (c) the external force $\mathbf{F}_{ext}$, (d) the work done by $\mathbf{F}_{ext}$ in 1 s, and (e) the kinetic energy acquired by the falling sand each second due to the change in its horizontal motion. (f) Why are the answers to parts (d) and (e) different?

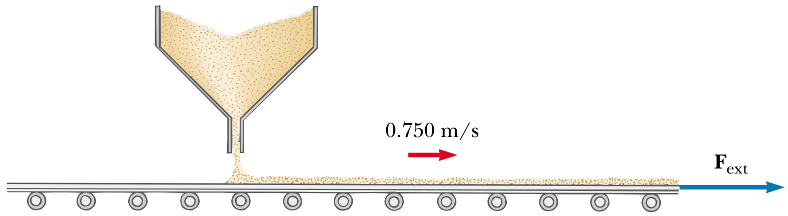


Figure P9.67

68. A rocket has total mass $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest, turns on its engine at time $t = 0$, and puts out exhaust with a relative speed of $v_e = 1\,500$ m/s at the constant rate $k = 2.50$ kg/s. Although the fuel will last for an actual burn time of 330 kg/(2.5 kg/s) = 132 s, define a "projected depletion time" as $T_p = M_i/k = 360$ kg/(2.5 kg/s) = 144 s. (This would be the burn time if the rocket could use its payload, fuel tanks, and even the walls of the combustion chamber as fuel.) (a) Show that during the burn the velocity of the rocket is given as a function of time by

$$v(t) = -v_e \ln(1 - t/T_p)$$

(b) Make a graph of the velocity of the rocket as a function of time for times running from 0 to 132 s. (c) Show that the acceleration of the rocket is

$$a(t) = v_e/(T_p - t)$$

(d) Graph the acceleration as a function of time. (e) Show that the displacement of the rocket from its initial position at $t = 0$ is

$$x(t) = v_e(T_p - t)\ln(1 - t/T_p) + v_e t$$

(f) Graph the displacement during the burn.

69. A 40.0-kg child stands at one end of a 70.0-kg boat that is 4.00 m in length (Fig. P9.69). The boat is initially 3.00 m from the pier. The child notices a turtle on a rock near the far end of the boat and proceeds to walk to that end to catch the turtle. Neglecting friction between the boat and the water, (a) describe the subsequent motion of the system (child plus boat). (b) Where is the child *relative to the pier* when he reaches the far end of the boat? (c) Will he catch the turtle? (Assume he can reach out 1.00 m from the end of the boat.)

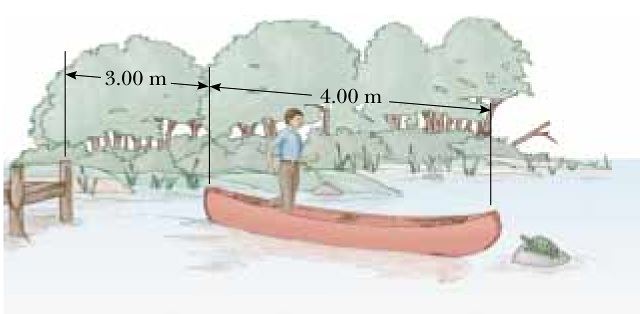


Figure P9.69

70. A student performs a ballistic pendulum experiment, using an apparatus similar to that shown in Figure 9.11b. She obtains the following average data: $h = 8.68$ cm, $m_1 = 68.8$ g, and $m_2 = 263$ g. The symbols refer to the quantities in Figure 9.11a. (a) Determine the initial speed v_{1i} of the projectile. (b) In the second part of her experiment she is to obtain v_{1i} by firing the same projectile horizontally (with the pendulum removed from the path) and measuring its horizontal displacement x and vertical displacement y (Fig. P9.70). Show that the initial speed of the projectile is related to x and y through the relationship

$$v_{1i} = \frac{x}{\sqrt{2y/g}}$$

What numerical value does she obtain for v_{1i} on the basis of her measured values of $x = 257$ cm and $y = 85.3$ cm? What factors might account for the difference in this value compared with that obtained in part (a)?

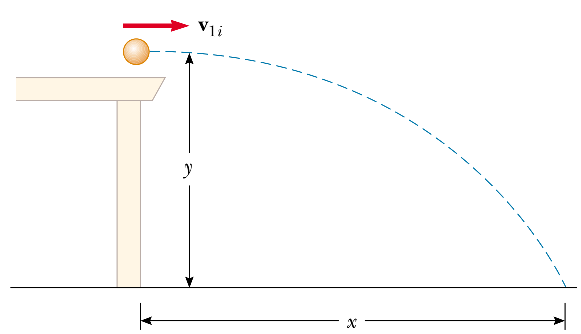


Figure P9.70

71. A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block, as shown in Figure P9.71. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring of force constant 900 N/m. If the block moves 5.00 cm to the right after impact, find (a) the speed at which the bullet emerges from the block and (b) the energy lost in the collision.

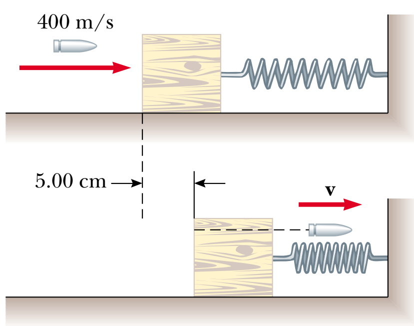


Figure P9.71

72. Two masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i. Mass m is traveling to the left, while mass $3m$ is traveling to the right. They undergo a head-on elastic collision and each rebounds along the same line as it approached. Find the final speeds of the masses.

73. Two masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i. Mass m is traveling to the left, while mass $3m$ is traveling to the right. They undergo an elastic glancing collision such that mass m is moving downward after the collision at right angles from its initial direction. (a) Find the final speeds of the two masses. (b) What is the angle θ at which the mass $3m$ is scattered?

74. Review Problem. There are (one can say) three coequal theories of motion: Newton's second law, stating that the total force on an object causes its acceleration; the work–kinetic energy theorem, stating that the total work on an object causes its change in kinetic energy; and the impulse–momentum theorem, stating that the total impulse on an object causes its change in momentum. In this problem, you compare predictions of the three theories in one particular case. A 3.00-kg object has a velocity of $7.00\mathbf{j}$ m/s. Then, a total force $12.0\mathbf{i}$ N acts on the object for 5.00 s. (a) Calculate the object's final velocity, using the impulse–momentum theorem. (b) Calculate its acceleration from $\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_i)/t$. (c) Calculate its acceleration from $\mathbf{a} = \Sigma\mathbf{F}/m$. (d) Find the object's vector displacement from $\mathbf{r} = \mathbf{v}_i t + \frac{1}{2}\mathbf{a}\,t^2$. (e) Find the work done on the object from $W = \mathbf{F}\cdot\mathbf{r}$. (f) Find the final kinetic energy from $\frac{1}{2}mv_f^2 = \frac{1}{2}m\mathbf{v}_f\cdot\mathbf{v}_f$. (g) Find the final kinetic energy from $\frac{1}{2}mv_i^2 + W$.

75. A rocket has a total mass of $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest. Its engine is turned on at time $t = 0$, and it puts out exhaust with a relative speed of $v_e = 1\,500$ m/s at the constant rate 2.50 kg/s. The burn lasts until the fuel runs out at time 330 kg/(2.5 kg/s) = 132 s. Set up and carry out a computer analysis of the motion according to Euler's method. Find (a) the final velocity of the rocket and (b) the distance it travels during the burn.

Answers to Quick Quizzes

9.1 (d). Two identical objects ($m_1 = m_2$) traveling in the same direction at the same speed ($v_1 = v_2$) have the same kinetic energies and the same momenta. However, this is not true if the two objects are moving at the same speed but in different directions. In the latter case, $K_1 = K_2$, but the differing velocity directions indicate that $\mathbf{p}_1 \neq \mathbf{p}_2$ because momentum is a vector quantity.

It also is possible for particular combinations of masses and velocities to satisfy $K_1 = K_2$ but not $p_1 = p_2$. For example, a 1-kg object moving at 2 m/s has the same kinetic energy as a 4-kg object moving at 1 m/s, but the two clearly do not have the same momenta.

9.2 (b), (c), (a). The slower the ball, the easier it is to catch. If the momentum of the medicine ball is the same as the momentum of the baseball, the speed of the medicine ball must be 1/10 the speed of the baseball because the medicine ball has 10 times the mass. If the kinetic energies are the same, the speed of the medicine ball must be $1/\sqrt{10}$ the speed of the baseball because of the squared speed term in the formula for K. The medicine ball is hardest to catch when it has the same speed as the baseball.

9.3 (c) and (e). Object 2 has a greater acceleration because of its smaller mass. Therefore, it takes less time to travel the distance d. Thus, even though the force applied to objects 1 and 2 is the same, the change in momentum is less for object 2 because Δt is smaller. Therefore, because the initial momenta were the same (both zero), $p_1 > p_2$. The work $W = Fd$ done on both objects is the same because both F and d are the same in the two cases. Therefore, $K_1 = K_2$.

9.4 Because the passenger is brought from the car's initial speed to a full stop, the change in momentum (the impulse) is the same regardless of whether the passenger is stopped by dashboard, seatbelt, or airbag. However, the dashboard stops the passenger very quickly in a front-end collision. The seatbelt takes somewhat more time. Used along with the seatbelt, the airbag can extend the passenger's stopping time further, notably for his head, which would otherwise snap forward. Therefore, the dashboard applies the greatest force, the seatbelt an intermediate force, and the airbag the least force. Airbags are designed to work in conjunction with seatbelts. Make sure you wear your seatbelt at all times while in a moving vehicle.

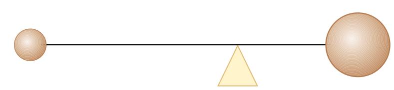
9.5 If we define the ball as our system, momentum is not conserved. The ball's speed—and hence its momentum—continually increase. This is consistent with the fact that the gravitational force is external to this chosen system. However, if we define our system as the ball and the Earth, momentum is conserved, for the Earth also has momentum because the ball exerts a gravitational force on it. As the ball falls, the Earth moves up to meet it (although the Earth's speed is on the order of 10^{25} times less than that of the ball!). This upward movement changes the Earth's momentum. The change in the Earth's momentum is numerically equal to the change in the ball's momentum but is in the opposite direction. Therefore, the total momentum of the Earth–ball system is conserved. Because the Earth's mass is so great, its upward motion is negligibly small.

9.6 (c). The greatest impulse (greatest change in momentum) is imparted to the Frisbee when the skater reverses its momentum vector by catching it and throwing it back. Since this is when the skater imparts the greatest impulse to the Frisbee, then this also is when the Frisbee imparts the greatest impulse to her.

9.7 Both are equally bad. Imagine watching the collision from a safer location alongside the road. As the "crush zones" of the two cars are compressed, you will see that the actual point of contact is stationary. You would see the same thing if your car were to collide with a solid wall.

9.8 No, such movement can never occur if we assume the collisions are elastic. The momentum of the system before the collision is mv, where m is the mass of ball 1 and v is its speed just before the collision. After the collision, we would have two balls, each of mass m and moving with a speed of $v/2$. Thus, the total momentum of the system after the collision would be $m(v/2) + m(v/2) = mv$. Thus, momentum is conserved. However, the kinetic energy just before the collision is $K_i = \frac{1}{2}mv^2$, and that after the collision is $K_f = \frac{1}{2}m(v/2)^2 + \frac{1}{2}m(v/2)^2 = \frac{1}{4}mv^2$. Thus, kinetic energy is *not* conserved. Both momentum and kinetic energy are conserved only when one ball moves out when one ball is released, two balls move out when two are released, and so on.

9.9 No they will not! The piece with the handle will have less mass than the piece made up of the end of the bat. To see why this is so, take the origin of coordinates as the center of mass before the bat was cut. Replace each cut piece by a small sphere located at the center of mass for each piece. The sphere representing the handle piece is farther from the origin, but the product of lesser mass and greater distance balances the product of greater mass and lesser distance for the end piece:



PUZZLER

Did you know that the CD inside this player spins at different speeds, depending on which song is playing? Why would such a strange characteristic be incorporated into the design of every CD player? *(George Semple)*

chapter

10

Rotation of a Rigid Object About a Fixed Axis

Chapter Outline

When an extended object, such as a wheel, rotates about its axis, the motion cannot be analyzed by treating the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. For this reason, it is convenient to consider an extended object as a large number of particles, each of which has its own linear velocity and linear acceleration.

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A **rigid object** is one that is nondeformable—that is, it is an object in which the separations between all pairs of particles remain constant. All real bodies are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.

Rigid object

In this chapter, we treat the rotation of a rigid object about a fixed axis, which is commonly referred to as *pure rotational motion.*

10.1 ANGULAR DISPLACEMENT, VELOCITY, AND ACCELERATION

Figure 10.1 illustrates a planar (flat), rigid object of arbitrary shape confined to the xy plane and rotating about a fixed axis through O. The axis is perpendicular to the plane of the figure, and O is the origin of an xy coordinate system. Let us look at the motion of only one of the millions of "particles" making up this object. A particle at P is at a fixed distance r from the origin and rotates about it in a circle of radius r. (In fact, *every* particle on the object undergoes circular motion about O.) It is convenient to represent the position of P with its polar coordinates (r, θ), where r is the distance from the origin to P and θ is measured *counterclockwise* from some preferred direction—in this case, the positive x axis. In this representation, the only coordinate that changes in time is the angle θ; r remains constant. (In cartesian coordinates, both x and y vary in time.) As the particle moves along the circle from the positive x axis ($\theta = 0$) to P, it moves through an arc of length s, which is related to the angular position θ through the relationship

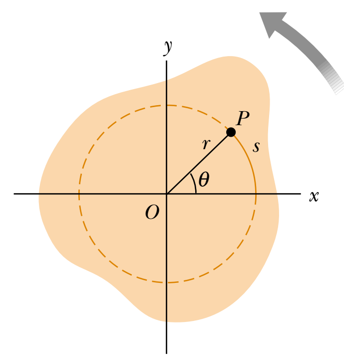


Figure 10.1 A rigid object rotating about a fixed axis through O perpendicular to the plane of the figure. (In other words, the axis of rotation is the z axis.) A particle at P rotates in a circle of radius r centered at O.

$$s = r\theta \qquad \textbf{(10.1a)}$$

$$\theta = \frac{s}{r} \qquad \textbf{(10.1b)}$$

It is important to note the units of θ in Equation 10.1b. Because θ is the ratio of an arc length and the radius of the circle, it is a pure number. However, we commonly give θ the artificial unit **radian** (rad), where

one radian is the angle subtended by an arc length equal to the radius of the arc.

Radian

Because the circumference of a circle is $2\pi r$, it follows from Equation 10.1b that 360° corresponds to an angle of $2\pi r/r$ rad $= 2\pi$ rad (one revolution). Hence, 1 rad $= 360°/2\pi \approx 57.3°$. To convert an angle in degrees to an angle in radians, we use the fact that 2π rad $= 360°$:

$$\theta\ (\text{rad}) = \frac{\pi}{180°}\,\theta\ (\text{deg})$$

For example, 60° equals $\pi/3$ rad, and 45° equals $\pi/4$ rad.

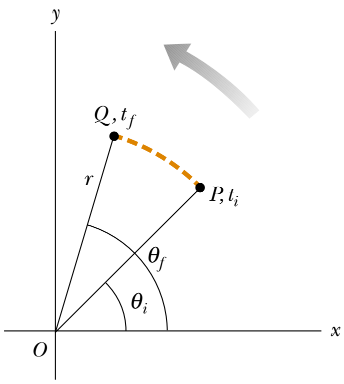
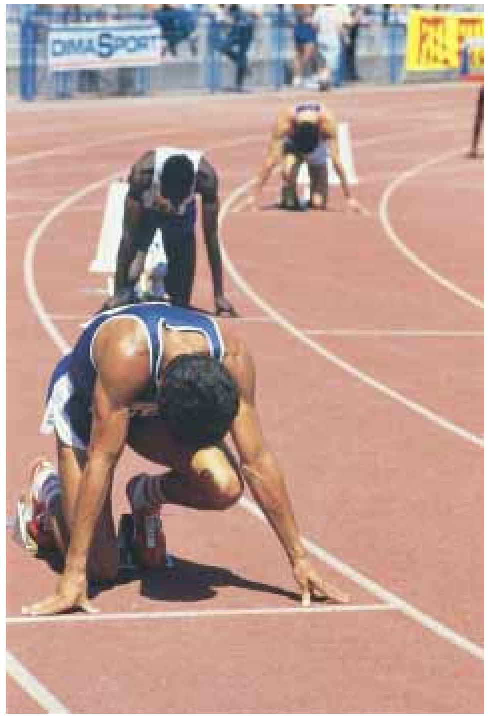


Figure 10.2 A particle on a rotating rigid object moves from P to Q along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radius vector sweeps out an angle $\Delta\theta = \theta_f - \theta_i$.



In a short track event, such as a 200-m or 400-m sprint, the runners begin from staggered positions on the track. Why don't they all begin from the same line?

As the particle in question on our rigid object travels from position P to position Q in a time Δt as shown in Figure 10.2, the radius vector sweeps out an angle $\Delta\theta = \theta_f - \theta_i$. This quantity $\Delta\theta$ is defined as the **angular displacement** of the particle:

$$\Delta\theta = \theta_f - \theta_i \tag{10.2}$$

We define the **average angular speed** $\overline{\omega}$ (omega) as the ratio of this angular displacement to the time interval Δt:

Average angular speed

$$\overline{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \tag{10.3}$$

In analogy to linear speed, the **instantaneous angular speed** ω is defined as the limit of the ratio $\Delta\theta/\Delta t$ as Δt approaches zero:

Instantaneous angular speed

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{10.4}$$

Angular speed has units of radians per second (rad/s), or rather second^{-1} (s^{-1}) because radians are not dimensional. We take ω to be positive when θ is increasing (counterclockwise motion) and negative when θ is decreasing (clockwise motion).

If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration** $\overline{\alpha}$ (alpha) of a rotating object is defined as the ratio of the change in the angular speed to the time interval Δt:

Average angular acceleration

$$\overline{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \tag{10.5}$$

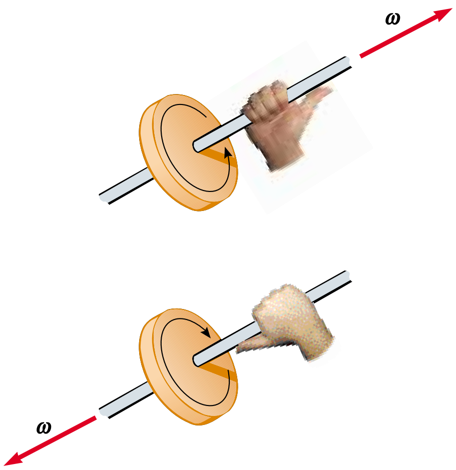


Figure 10.3 The right-hand rule for determining the direction of the angular velocity vector.

In analogy to linear acceleration, the **instantaneous angular acceleration** is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero:

Instantaneous angular acceleration

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad (10.6)$$

Angular acceleration has units of radians per second squared (rad/s^2), or just second^{-2} (s^{-2}). Note that α is positive when the rate of counterclockwise rotation is increasing or when the rate of clockwise rotation is decreasing.

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and the same angular acceleration. That is, the quantities θ, ω, and α characterize the rotational motion of the entire rigid object. Using these quantities, we can greatly simplify the analysis of rigid-body rotation.

Angular position (θ), angular speed (ω), and angular acceleration (α) are analogous to linear position (x), linear speed (v), and linear acceleration (a). The variables θ, ω, and α differ dimensionally from the variables x, v, and a only by a factor having the unit of length.

We have not specified any direction for ω and α. Strictly speaking, these variables are the magnitudes of the angular velocity and the angular acceleration vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, respectively, and they should always be positive. Because we are considering rotation about a fixed axis, however, we can indicate the directions of the vectors by assigning a positive or negative sign to ω and α, as discussed earlier with regard to Equations 10.4 and 10.6. For rotation about a fixed axis, the only direction that uniquely specifies the rotational motion is the direction along the axis of rotation. Therefore, the directions of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are along this axis. If an object rotates in the xy plane as in Figure 10.1, the direction of $\boldsymbol{\omega}$ is out of the plane of the diagram when the rotation is counterclockwise and into the plane of the diagram when the rotation is clockwise. To illustrate this convention, it is convenient to use the *right-hand rule* demonstrated in Figure 10.3. When the four fingers of the right hand are wrapped in the direction of rotation, the extended right thumb points in the direction of $\boldsymbol{\omega}$. The direction of $\boldsymbol{\alpha}$ follows from its definition $d\boldsymbol{\omega}/dt$. It is the same as the direction of $\boldsymbol{\omega}$ if the angular speed is increasing in time, and it is antiparallel to $\boldsymbol{\omega}$ if the angular speed is decreasing in time.

Quick Quiz 10.1

Describe a situation in which $\omega < 0$ and $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are antiparallel.

10.2 ROTATIONAL KINEMATICS: ROTATIONAL MOTION WITH CONSTANT ANGULAR ACCELERATION

7.2 In our study of linear motion, we found that the simplest form of accelerated motion to analyze is motion under constant linear acceleration. Likewise, for rotational motion about a fixed axis, the simplest accelerated motion to analyze is motion under constant angular acceleration. Therefore, we next develop kinematic relationships for this type of motion. If we write Equation 10.6 in the form $d\omega = \alpha\, dt$, and let $t_i = 0$ and $t_f = t$, we can integrate this expression directly:

$$\omega_f = \omega_i + \alpha t \qquad \text{(for constant } \alpha\text{)} \tag{10.7}$$

Substituting Equation 10.7 into Equation 10.4 and integrating once more we obtain

Rotational kinematic equations

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \qquad \text{(for constant } \alpha\text{)} \tag{10.8}$$

If we eliminate t from Equations 10.7 and 10.8, we obtain

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \qquad \text{(for constant } \alpha\text{)} \tag{10.9}$$

Notice that these kinematic expressions for rotational motion under constant angular acceleration are of the same form as those for linear motion under constant linear acceleration with the substitutions $x \rightarrow \theta$, $v \rightarrow \omega$, and $a \rightarrow \alpha$. Table 10.1 compares the kinematic equations for rotational and linear motion.

EXAMPLE 10.1 Rotating Wheel

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, (a) through what angle does the wheel rotate in 2.00 s?

Solution We can use Figure 10.2 to represent the wheel, and so we do not need a new drawing. This is a straightforward application of an equation from Table 10.1:

$$\theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2 = (2.00 \text{ rad/s})(2.00 \text{ s}) + \tfrac{1}{2}(3.50 \text{ rad/s}^2)(2.00 \text{ s})^2$$

$$= 11.0 \text{ rad} = (11.0 \text{ rad})(57.3°/\text{rad}) = \boxed{630°}$$

$$= \frac{630°}{360°/\text{rev}} = \boxed{1.75 \text{ rev}}$$

(b) What is the angular speed at $t = 2.00$ s?

Solution Because the angular acceleration and the angular speed are both positive, we can be sure our answer must be greater than 2.00 rad/s.

$$\omega_f = \omega_i + \alpha t = 2.00 \text{ rad/s} + (3.50 \text{ rad/s}^2)(2.00 \text{ s})$$

$$= \boxed{9.00 \text{ rad/s}}$$

We could also obtain this result using Equation 10.9 and the results of part (a). Try it! You also may want to see if you can formulate the linear motion analog to this problem.

Exercise Find the angle through which the wheel rotates between $t = 2.00$ s and $t = 3.00$ s.

Answer 10.8 rad.

TABLE 10.1 Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$

10.3 ANGULAR AND LINEAR QUANTITIES

In this section we derive some useful relationships between the angular speed and acceleration of a rotating rigid object and the linear speed and acceleration of an arbitrary point in the object. To do so, we must keep in mind that when a rigid object rotates about a fixed axis, as in Figure 10.4, every particle of the object moves in a circle whose center is the axis of rotation.

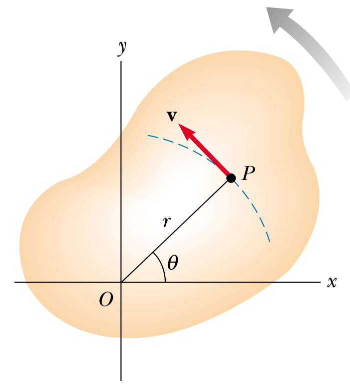


Figure 10.4 As a rigid object rotates about the fixed axis through O, the point P has a linear velocity $\mathbf{v}$ that is always tangent to the circular path of radius r.

We can relate the angular speed of the rotating object to the tangential speed of a point P on the object. Because point P moves in a circle, the linear velocity vector $\mathbf{v}$ is always tangent to the circular path and hence is called *tangential velocity*. The magnitude of the tangential velocity of the point P is by definition the tangential speed $v = ds/dt$, where s is the distance traveled by this point measured along the circular path. Recalling that $s = r\theta$ (Eq. 10.1a) and noting that r is constant, we obtain

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Because $d\theta/dt = \omega$ (see Eq. 10.4), we can say

Relationship between linear and angular speed

$$v = r\omega \tag{10.10}$$

That is, the tangential speed of a point on a rotating rigid object equals the perpendicular distance of that point from the axis of rotation multiplied by the angular speed. Therefore, although every point on the rigid object has the same *angular* speed, not every point has the same *linear* speed because r is not the same for all points on the object. Equation 10.10 shows that the linear speed of a point on the rotating object increases as one moves outward from the center of rotation, as we would intuitively expect. The outer end of a swinging baseball bat moves much faster than the handle.

QuickLab

Spin a tennis ball or basketball and watch it gradually slow down and stop. Estimate α and a_t as accurately as you can.

We can relate the angular acceleration of the rotating rigid object to the tangential acceleration of the point P by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

Relationship between linear and angular acceleration

$$a_t = r\alpha \tag{10.11}$$

That is, the tangential component of the linear acceleration of a point on a rotating rigid object equals the point's distance from the axis of rotation multiplied by the angular acceleration.

In Section 4.4 we found that a point rotating in a circular path undergoes a centripetal, or radial, acceleration $\mathbf{a}_r$ of magnitude v^2/r directed toward the center of rotation (Fig. 10.5). Because $v = r\omega$ for a point P on a rotating object, we can express the radial acceleration of that point as

$$a_r = \frac{v^2}{r} = r\omega^2 \tag{10.12}$$

The total linear acceleration vector of the point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$. ($\mathbf{a}_t$ describes the change in how fast the point is moving, and $\mathbf{a}_r$ represents the change in its direction of travel.) Because $\mathbf{a}$ is a vector having a radial and a tangential component, the magnitude of $\mathbf{a}$ for the point P on the rotating rigid object is

$$a = \sqrt{{a_t}^2 + {a_r}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \tag{10.13}$$

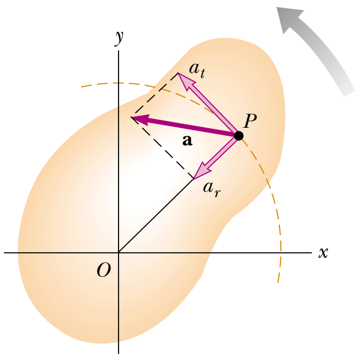


Figure 10.5 As a rigid object rotates about a fixed axis through O, the point P experiences a tangential component of linear acceleration a_t and a radial component of linear acceleration a_r. The total linear acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

Quick Quiz 10.2

When a wheel of radius R rotates about a fixed axis, do all points on the wheel have (a) the same angular speed and (b) the same linear speed? If the angular speed is constant and equal to ω, describe the linear speeds and linear accelerations of the points located at (c) $r = 0$, (d) $r = R/2$, and (e) $r = R$, all measured from the center of the wheel.

EXAMPLE 10.2 CD Player

On a compact disc, audio information is stored in a series of pits and flat areas on the surface of the disc. The information is stored digitally, and the alternations between pits and flat areas on the surface represent binary ones and zeroes to be read by the compact disc player and converted back to sound waves. The pits and flat areas are detected by a system consisting of a laser and lenses. The length of a certain number of ones and zeroes is the same everywhere on the disc, whether the information is near the center of the disc or near its outer edge. In order that this length of ones and zeroes always passes by the laser–lens system in the same time period, the linear speed of the disc surface at the location of the lens must be constant. This requires, according to Equation 10.10, that the angular speed vary as the laser–lens system moves radially along the disc. In a typical compact disc player, the disc spins counterclockwise (Fig. 10.6), and the constant speed of the surface at the point of the laser–lens system is 1.3 m/s. (a) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track (r = 23 mm) and the outermost final track (r = 58 mm).

Solution Using Equation 10.10, we can find the angular speed; this will give us the required linear speed at the position of the inner track,

$$\omega_i = \frac{v}{r_i} = \frac{1.3 \text{ m/s}}{2.3 \times 10^{-2} \text{ m}} = 56.5 \text{ rad/s}$$

$$= (56.5 \text{ rad/s})\left(\frac{1}{2\pi} \text{ rev/rad}\right)(60 \text{ s/min})$$

$$= 5.4 \times 10^2 \text{ rev/min}$$

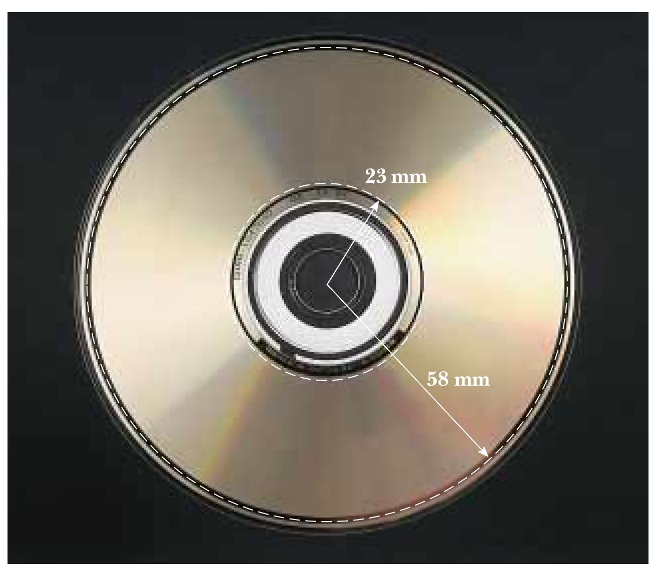


Figure 10.6 A compact disc.

For the outer track,

$$\omega_f = \frac{v}{r_f} = \frac{1.3 \text{ m/s}}{5.8 \times 10^{-2} \text{ m}} = 22.4 \text{ rad/s}$$

$$= 2.1 \times 10^2 \text{ rev/min}$$

The player adjusts the angular speed ω of the disc within this range so that information moves past the objective lens at a constant rate. These angular velocity values are positive because the direction of rotation is counterclockwise.

(b) The maximum playing time of a standard music CD is 74 minutes and 33 seconds. How many revolutions does the disc make during that time?

Solution We know that the angular speed is always decreasing, and we assume that it is decreasing steadily, with α constant. The time interval t is (74 min)(60 s/min) + 33 s = 4 473 s. We are looking for the angular position θ_f, where we set the initial angular position $\theta_i = 0$. We can use Equation 10.3, replacing the average angular speed $\overline{\omega}$ with its mathematical equivalent $(\omega_i + \omega_f)/2$:

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)t$$
$$= 0 + \tfrac{1}{2}(540 \text{ rev/min} + 210 \text{ rev/min})$$
$$(1 \text{ min}/60 \text{ s})(4\,473 \text{ s})$$
$$= 2.8 \times 10^4 \text{ rev}$$

(c) What total length of track moves past the objective lens during this time?

Solution Because we know the (constant) linear velocity and the time interval, this is a straightforward calculation:

$$x_f = v_i t = (1.3 \text{ m/s})(4\,473 \text{ s}) = 5.8 \times 10^3 \text{ m}$$

More than 3.6 miles of track spins past the objective lens!

(d) What is the angular acceleration of the CD over the 4 473-s time interval? Assume that α is constant.

Solution We have several choices for approaching this problem. Let us use the most direct approach by utilizing Equation 10.5, which is based on the definition of the term we are seeking. We should obtain a negative number for the angular acceleration because the disc spins more and more slowly in the positive direction as time goes on. Our answer should also be fairly small because it takes such a long time—more than an hour—for the change in angular speed to be accomplished:

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{22.4 \text{ rad/s} - 56.5 \text{ rad/s}}{4\,473 \text{ s}}$$

$$= -7.6 \times 10^{-3} \text{ rad/s}^2$$

The disc experiences a very gradual decrease in its rotation rate, as expected.

10.4 ROTATIONAL ENERGY

7.3

Let us now look at the kinetic energy of a rotating rigid object, considering the object as a collection of particles and assuming it rotates about a fixed z axis with an angular speed ω (Fig. 10.7). Each particle has kinetic energy determined by its mass and linear speed. If the mass of the ith particle is m_i and its linear speed is v_i, its kinetic energy is

$$K_i = \tfrac{1}{2} m_i v_i^2$$

To proceed further, we must recall that although every particle in the rigid object has the same angular speed ω, the individual linear speeds depend on the distance r_i from the axis of rotation according to the expression $v_i = r_i\omega$ (see Eq. 10.10). The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i K_i = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\sum_i m_i r_i^2 \omega^2$$

We can write this expression in the form

$$K_R = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \qquad \textbf{(10.14)}$$

where we have factored ω^2 from the sum because it is common to every particle.

web

If you want to learn more about the physics of CD players, visit the Special Interest Group on CD Applications and Technology at **www.sigcat.org**

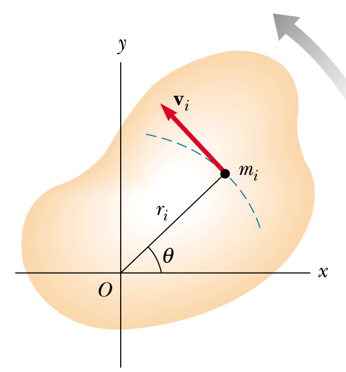


Figure 10.7 A rigid object rotating about a z axis with angular speed ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_i v_i^2$. The total kinetic energy of the object is called its rotational kinetic energy.

We simplify this expression by defining the quantity in parentheses as the **moment of inertia** $\boldsymbol{I}$**:**

Moment of inertia

$$I \equiv \sum_i m_i r_i^{\,2} \tag{10.15}$$

From the definition of moment of inertia, we see that it has dimensions of ML^2 ($kg \cdot m^2$ in SI units).[1] With this notation, Equation 10.14 becomes

Rotational kinetic energy

$$K_R = \tfrac{1}{2} I \omega^2 \tag{10.16}$$

Although we commonly refer to the quantity $\frac{1}{2}I\omega^2$ as **rotational kinetic energy,** it is not a new form of energy. It is ordinary kinetic energy because it is derived from a sum over individual kinetic energies of the particles contained in the rigid object. However, the mathematical form of the kinetic energy given by Equation 10.16 is a convenient one when we are dealing with rotational motion, provided we know how to calculate I.

It is important that you recognize the analogy between kinetic energy associated with linear motion $\frac{1}{2}mv^2$ and rotational kinetic energy $\frac{1}{2}I\omega^2$. The quantities I and ω in rotational motion are analogous to m and v in linear motion, respectively. (In fact, I takes the place of m every time we compare a linear-motion equation with its rotational counterpart.) The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion. Note, however, that mass is an intrinsic property of an object, whereas I depends on the physical arrangement of that mass. Can you think of a situation in which an object's moment of inertia changes even though its mass does not?

EXAMPLE 10.3 The Oxygen Molecule

Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature the average separation between the two atoms is $d = 1.21 \times 10^{-10}$ m (the atoms are treated as point masses). (a) Calculate the moment of inertia of the molecule about the z axis.

Solution This is a straightforward application of the definition of I. Because each atom is a distance $d/2$ from the z axis, the moment of inertia about the axis is

$$I = \sum_i m_i r_i^{\,2} = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \tfrac{1}{2}md^2$$

$$= \tfrac{1}{2}(2.66 \times 10^{-26}\ \text{kg})(1.21 \times 10^{-10}\ \text{m})^2$$

$$= \boxed{1.95 \times 10^{-46}\ \text{kg} \cdot \text{m}^2}$$

This is a very small number, consistent with the minuscule masses and distances involved.

(b) If the angular speed of the molecule about the z axis is 4.60×10^{12} rad/s, what is its rotational kinetic energy?

Solution We apply the result we just calculated for the moment of inertia in the formula for K_R:

$$K_R = \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}(1.95 \times 10^{-46}\ \text{kg} \cdot \text{m}^2)(4.60 \times 10^{12}\ \text{rad/s})^2$$

$$= \boxed{2.06 \times 10^{-21}\ \text{J}}$$

[1] Civil engineers use moment of inertia to characterize the elastic properties (rigidity) of such structures as loaded beams. Hence, it is often useful even in a nonrotational context.

EXAMPLE 10.4 Four Rotating Masses

Four tiny spheres are fastened to the corners of a frame of negligible mass lying in the *xy* plane (Fig. 10.8). We shall assume that the spheres' radii are small compared with the dimensions of the frame. (a) If the system rotates about the *y* axis with an angular speed ω, find the moment of inertia and the rotational kinetic energy about this axis.

Solution First, note that the two spheres of mass *m*, which lie on the *y* axis, do not contribute to I_y (that is, $r_i = 0$ for these spheres about this axis). Applying Equation 10.15, we obtain

$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

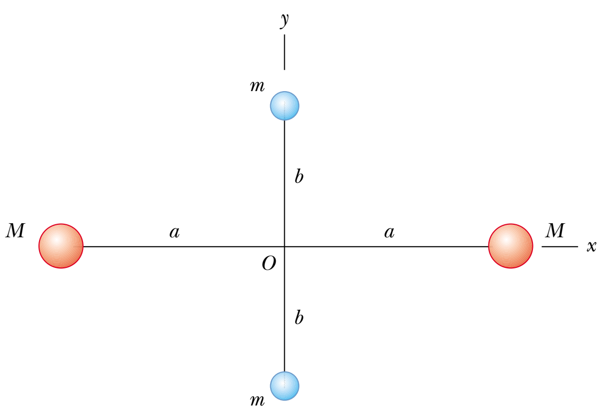


Figure 10.8 The four spheres are at a fixed separation as shown. The moment of inertia of the system depends on the axis about which it is evaluated.

Therefore, the rotational kinetic energy about the *y* axis is

$$K_R = \tfrac{1}{2}I_y\omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = \boxed{Ma^2\omega^2}$$

The fact that the two spheres of mass *m* do not enter into this result makes sense because they have no motion about the axis of rotation; hence, they have no rotational kinetic energy. By similar logic, we expect the moment of inertia about the *x* axis to be $I_x = 2mb^2$ with a rotational kinetic energy about that axis of $K_R = mb^2\omega^2$.

(b) Suppose the system rotates in the *xy* plane about an axis through *O* (the *z* axis). Calculate the moment of inertia and rotational kinetic energy about this axis.

Solution Because r_i in Equation 10.15 is the *perpendicular* distance to the axis of rotation, we obtain

$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

$$K_R = \tfrac{1}{2}I_z\omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = \boxed{(Ma^2 + mb^2)\omega^2}$$

Comparing the results for parts (a) and (b), we conclude that the moment of inertia and therefore the rotational kinetic energy associated with a given angular speed depend on the axis of rotation. In part (b), we expect the result to include all four spheres and distances because all four spheres are rotating in the *xy* plane. Furthermore, the fact that the rotational kinetic energy in part (a) is smaller than that in part (b) indicates that it would take less effort (work) to set the system into rotation about the *y* axis than about the *z* axis.

10.5 CALCULATION OF MOMENTS OF INERTIA

7.5 We can evaluate the moment of inertia of an extended rigid object by imagining the object divided into many small volume elements, each of which has mass Δm. We use the definition $I = \sum_i r_i^2\,\Delta m_i$ and take the limit of this sum as $\Delta m \rightarrow 0$. In this limit, the sum becomes an integral over the whole object:

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2\,\Delta m_i = \int r^2\,dm \qquad (10.17)$$

It is usually easier to calculate moments of inertia in terms of the volume of the elements rather than their mass, and we can easily make that change by using Equation 1.1, $\rho = m/V$, where ρ is the density of the object and V is its volume. We want this expression in its differential form $\rho = dm/dV$ because the volumes we are dealing with are very small. Solving for $dm = \rho\,dV$ and substituting the result into Equation 10.17 gives

$$I = \int \rho r^2 \, dV$$

If the object is homogeneous, then ρ is constant and the integral can be evaluated for a known geometry. If ρ is not constant, then its variation with position must be known to complete the integration.

The density given by $\rho = m/V$ sometimes is referred to as *volume density* for the obvious reason that it relates to volume. Often we use other ways of expressing density. For instance, when dealing with a sheet of uniform thickness t, we can define a *surface density* $\sigma = \rho t$, which signifies *mass per unit area.* Finally, when mass is distributed along a uniform rod of cross-sectional area A, we sometimes use *linear density* $\lambda = M/L = \rho A$, which is the *mass per unit length.*

EXAMPLE 10.5 Uniform Hoop

Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center (Fig. 10.9).

Solution All mass elements dm are the same distance $r = R$ from the axis, and so, applying Equation 10.17, we obtain for the moment of inertia about the z axis through O:

$$I_z = \int r^2 \, dm = R^2 \int dm = MR^2$$

Note that this moment of inertia is the same as that of a single particle of mass M located a distance R from the axis of rotation.

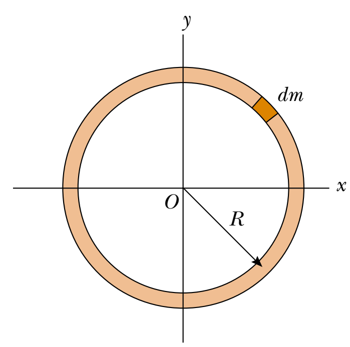


Figure 10.9 The mass elements dm of a uniform hoop are all the same distance from O.

Quick Quiz 10.3

(a) Based on what you have learned from Example 10.5, what do you expect to find for the moment of inertia of two particles, each of mass $M/2$, located anywhere on a circle of radius R around the axis of rotation? (b) How about the moment of inertia of four particles, each of mass $M/4$, again located a distance R from the rotation axis?

EXAMPLE 10.6 Uniform Rigid Rod

Calculate the moment of inertia of a uniform rigid rod of length L and mass M (Fig. 10.10) about an axis perpendicular to the rod (the y axis) and passing through its center of mass.

Solution The shaded length element dx has a mass dm equal to the mass per unit length λ multiplied by dx:

$$dm = \lambda \, dx = \frac{M}{L} \, dx$$

Substituting this expression for dm into Equation 10.17, with $r = x$, we obtain

$$I_y = \int r^2 \, dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} \, dx = \frac{M}{L} \int_{-L/2}^{L/2} x^2 \, dx$$

$$= \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2} = \frac{1}{12} ML^2$$

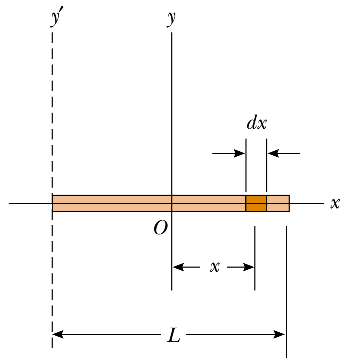


Figure 10.10 A uniform rigid rod of length L. The moment of inertia about the y axis is less than that about the y' axis. The latter axis is examined in Example 10.8.

EXAMPLE 10.7 Uniform Solid Cylinder

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis (the z axis in Fig. 10.11).

Solution It is convenient to divide the cylinder into many cylindrical shells, each of which has radius r, thickness dr, and length L, as shown in Figure 10.11. The volume dV of each shell is its cross-sectional area multiplied by its length: $dV = dA \cdot L = (2\pi r\ dr)L$. If the mass per unit volume is ρ, then the mass of this differential volume element is $dm = \rho dV = \rho 2\pi r L\ dr$. Substituting this expression for dm into Equation 10.17, we obtain

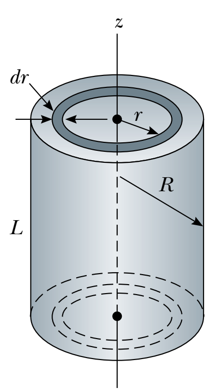


Figure 10.11 Calculating I about the z axis for a uniform solid cylinder.

$$I_z = \int r^2\ dm = 2\pi\rho L \int_0^R r^3\ dr = \tfrac{1}{2}\pi\rho L R^4$$

Because the total volume of the cylinder is $\pi R^2 L$, we see that $\rho = M/V = M/\pi R^2 L$. Substituting this value for ρ into the above result gives

$$(1) \qquad I_z = \tfrac{1}{2}MR^2$$

Note that this result does not depend on L, the length of the cylinder. In other words, it applies equally well to a long cylinder and a flat disc. Also note that this is exactly half the value we would expect were all the mass concentrated at the outer edge of the cylinder or disc. (See Example 10.5.)

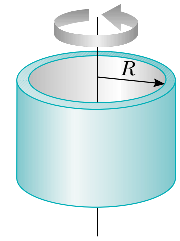
Table 10.2 gives the moments of inertia for a number of bodies about specific axes. The moments of inertia of rigid bodies with simple geometry (high symmetry) are relatively easy to calculate provided the rotation axis coincides with an axis of symmetry. The calculation of moments of inertia about an arbitrary axis can be cumbersome, however, even for a highly symmetric object. Fortunately, use of an important theorem, called the **parallel-axis theorem,** often simplifies the calculation. Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance D away from this axis is

$$I = I_{CM} + MD^2 \qquad \textbf{(10.18)}$$

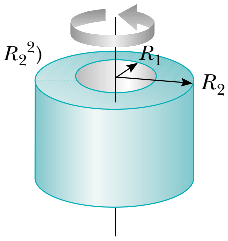
Parallel-axis theorem

TABLE 10.2 Moments of Inertia of Homogeneous Rigid Bodies with Different Geometries

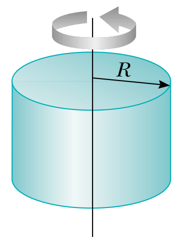
Hoop or cylindrical shell
$I_{CM} = MR^2$



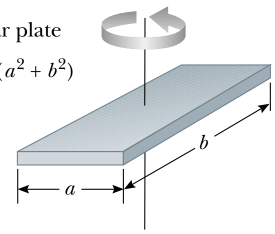
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



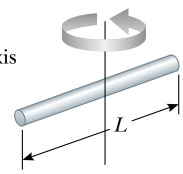
Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$



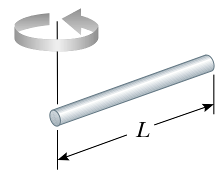
Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



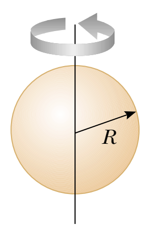
Long thin rod with rotation axis through center
$I_{CM} = \frac{1}{12}ML^2$



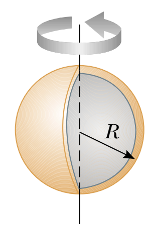
Long thin rod with rotation axis through end
$I = \frac{1}{3}ML^2$



Solid sphere
$I_{CM} = \frac{2}{5}MR^2$



Thin spherical shell
$I_{CM} = \frac{2}{3}MR^2$



Proof of the Parallel-Axis Theorem *(Optional)*. Suppose that an object rotates in the xy plane about the z axis, as shown in Figure 10.12, and that the coordinates of the center of mass are x_{CM}, y_{CM}. Let the mass element dm have coordinates x, y. Because this element is a distance $r = \sqrt{x^2 + y^2}$ from the z axis, the moment of inertia about the z axis is

$$I = \int r^2\,dm = \int (x^2 + y^2)\,dm$$

However, we can relate the coordinates x, y of the mass element dm to the coordinates of this same element located in a coordinate system having the object's center of mass as its origin. If the coordinates of the center of mass are x_{CM}, y_{CM} in the original coordinate system centered on O, then from Figure 10.12a we see that the relationships between the unprimed and primed coordinates are $x = x' + x_{CM}$

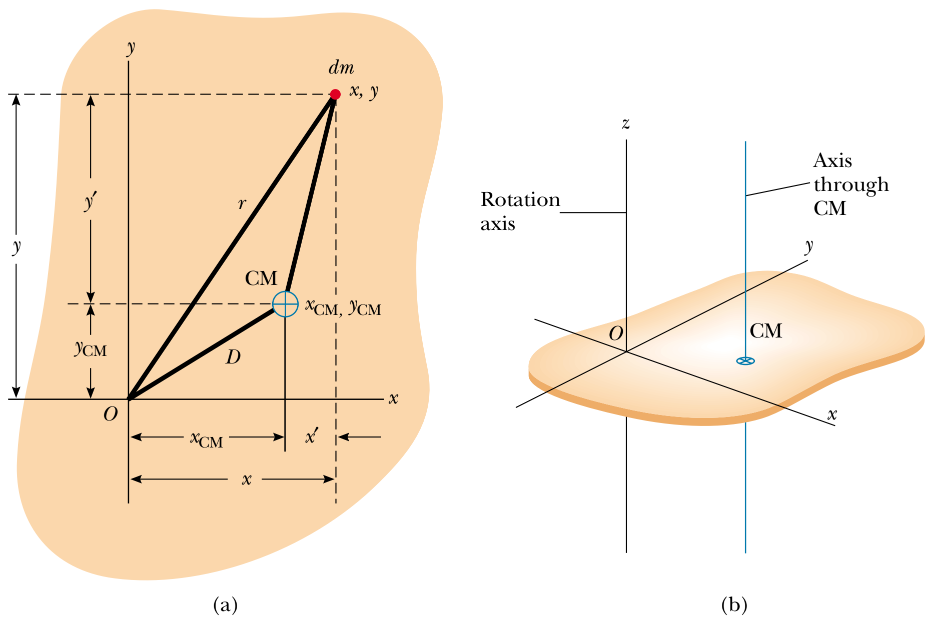


Figure 10.12 (a) The parallel-axis theorem: If the moment of inertia about an axis perpendicular to the figure through the center of mass is I_{CM}, then the moment of inertia about the z axis is $I_z = I_{CM} + MD^2$. (b) Perspective drawing showing the z axis (the axis of rotation) and the parallel axis through the CM.

and $y = y' + y_{CM}$. Therefore,

$$I = \int [(x' + x_{CM})^2 + (y' + y_{CM})^2]\, dm$$

$$= \int [(x')^2 + (y')^2]\, dm + 2x_{CM} \int x'\, dm + 2y_{CM} \int y'\, dm + ({x_{CM}}^2 + {y_{CM}}^2) \int dm$$

The first integral is, by definition, the moment of inertia about an axis that is parallel to the z axis and passes through the center of mass. The second two integrals are zero because, by definition of the center of mass, $\int x'\, dm = \int y'\, dm = 0$. The last integral is simply MD^2 because $\int dm = M$ and $D^2 = {x_{CM}}^2 + {y_{CM}}^2$. Therefore, we conclude that

$$I = I_{CM} + MD^2$$

EXAMPLE 10.8 Applying the Parallel-Axis Theorem

Consider once again the uniform rigid rod of mass M and length L shown in Figure 10.10. Find the moment of inertia of the rod about an axis perpendicular to the rod through one end (the y' axis in Fig. 10.10).

Solution Intuitively, we expect the moment of inertia to be greater than $I_{CM} = \frac{1}{12}ML^2$ because it should be more difficult to change the rotational motion of a rod spinning about an axis at one end than one that is spinning about its center. Because the distance between the center-of-mass axis and the y' axis is $D = L/2$, the parallel-axis theorem gives

$$I = I_{CM} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

So, it is four times more difficult to change the rotation of a rod spinning about its end than it is to change the motion of one spinning about its center.

Exercise Calculate the moment of inertia of the rod about a perpendicular axis through the point $x = L/4$.

Answer $I = \frac{7}{48}ML^2$.

10.6 TORQUE

7.6

Why are a door's doorknob and hinges placed near opposite edges of the door? This question actually has an answer based on common sense ideas. The harder we push against the door and the farther we are from the hinges, the more likely we are to open or close the door. When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** $\boldsymbol{\tau}$ (tau).

Consider the wrench pivoted on the axis through O in Figure 10.13. The applied force **F** acts at an angle ϕ to the horizontal. We define the magnitude of the torque associated with the force **F** by the expression

Definition of torque

$$\tau \equiv rF\sin\phi = Fd \tag{10.19}$$

where r is the distance between the pivot point and the point of application of **F** and d is the perpendicular distance from the pivot point to the line of action of **F**. (The *line of action* of a force is an imaginary line extending out both ends of the vector representing the force. The dashed line extending from the tail of **F** in Figure 10.13 is part of the line of action of **F**.) From the right triangle in Figure 10.13 that has the wrench as its hypotenuse, we see that $d = r\sin\phi$. This quantity d is called the **moment arm** (or *lever arm*) of **F**.

Moment arm

It is very important that you recognize that *torque is defined only when a reference axis is specified.* Torque is the product of a force and the moment arm of that force, and moment arm is defined only in terms of an axis of rotation.

In Figure 10.13, the only component of **F** that tends to cause rotation is $F\sin\phi$, the component perpendicular to r. The horizontal component $F\cos\phi$, because it passes through O, has no tendency to produce rotation. From the definition of torque, we see that the rotating tendency increases as **F** increases and as d increases. This explains the observation that it is easier to close a door if we push at the doorknob rather than at a point close to the hinge. We also want to apply our push as close to perpendicular to the door as we can. Pushing sideways on the doorknob will not cause the door to rotate.

If two or more forces are acting on a rigid object, as shown in Figure 10.14, each tends to produce rotation about the pivot at O. In this example, $\mathbf{F}_2$ tends to

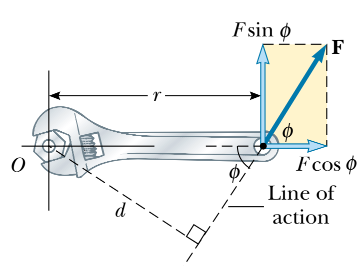


Figure 10.13 The force **F** has a greater rotating tendency about O as F increases and as the moment arm d increases. It is the component $F\sin\phi$ that tends to rotate the wrench about O.

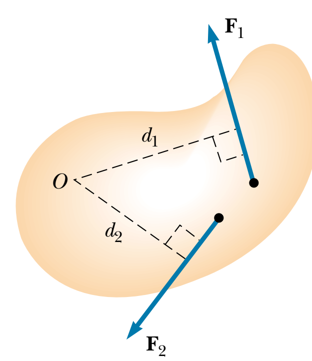


Figure 10.14 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about O, and $\mathbf{F}_2$ tends to rotate it clockwise.

rotate the object clockwise, and $\mathbf{F}_1$ tends to rotate it counterclockwise. We use the convention that the sign of the torque resulting from a force is positive if the turning tendency of the force is counterclockwise and is negative if the turning tendency is clockwise. For example, in Figure 10.14, the torque resulting from $\mathbf{F}_1$, which has a moment arm d_1, is positive and equal to $+F_1 d_1$; the torque from $\mathbf{F}_2$ is negative and equal to $-F_2 d_2$. Hence, the net torque about O is

$$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

Torque should not be confused with force. Forces can cause a change in linear motion, as described by Newton's second law. Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces, in the combination that we call *torque*. Torque has units of force times length—newton · meters in SI units—and should be reported in these units. Do not confuse torque and work, which have the same units but are very different concepts.

Example 10.9 The Net Torque on a Cylinder

A one-piece cylinder is shaped as shown in Figure 10.15, with a core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the drawing. A rope wrapped around the drum, which has radius R_1, exerts a force $\mathbf{F}_1$ to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force $\mathbf{F}_2$ downward on the cylinder. (a) What is the net torque acting on the cylinder about the rotation axis (which is the z axis in Figure 10.15)?

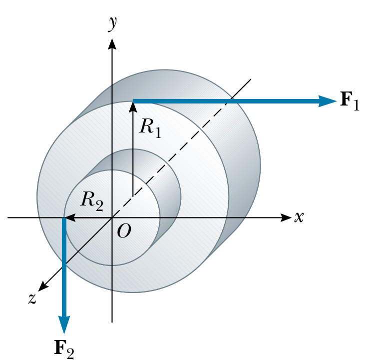


Figure 10.15 A solid cylinder pivoted about the z axis through O. The moment arm of $\mathbf{F}_1$ is R_1, and the moment arm of $\mathbf{F}_2$ is R_2.

Solution The torque due to $\mathbf{F}_1$ is $-R_1F_1$ (the sign is negative because the torque tends to produce clockwise rotation). The torque due to $\mathbf{F}_2$ is $+R_2F_2$ (the sign is positive because the torque tends to produce counterclockwise rotation). Therefore, the net torque about the rotation axis is

$$\sum \tau = \tau_1 + \tau_2 = \boxed{-R_1F_1 + R_2F_2}$$

We can make a quick check by noting that if the two forces are of equal magnitude, the net torque is negative because $R_1 > R_2$. Starting from rest with both forces acting on it, the cylinder would rotate clockwise because $\mathbf{F}_1$ would be more effective at turning it than would $\mathbf{F}_2$.

(b) Suppose $F_1 = 5.0$ N, $R_1 = 1.0$ m, $F_2 = 15.0$ N, and $R_2 = 0.50$ m. What is the net torque about the rotation axis, and which way does the cylinder rotate from rest?

$$\sum \tau = -(5.0\text{ N})(1.0\text{ m}) + (15.0\text{ N})(0.50\text{ m}) = \boxed{2.5\text{ N}\cdot\text{m}}$$

Because the net torque is positive, if the cylinder starts from rest, it will commence rotating counterclockwise with increasing angular velocity. (If the cylinder's initial rotation is clockwise, it will slow to a stop and then rotate counterclockwise with increasing angular speed.)

10.7 RELATIONSHIP BETWEEN TORQUE AND ANGULAR ACCELERATION

7.6 In this section we show that the angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis. Before discussing the more complex case of rigid-body rotation, however, it is instructive first to discuss the case of a particle rotating about some fixed point under the influence of an external force.

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.16. (As we learned in Chapter 6, the radial force must be present to keep the particle moving in its circular path.) The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

Figure 10.16 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ in the radial direction also must be present to maintain the circular motion.

$$F_t = ma_t$$

The torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t r = (ma_t) r$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$ (see Eq. 10.11), the torque can be expressed as

$$\tau = (mr\alpha) r = (mr^2)\alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the rotating particle about the z axis passing through the origin, so that

Relationship between torque and angular acceleration

$$\tau = I\alpha \quad \textbf{(10.20)}$$

That is, **the torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. It is important to note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis, as shown in Figure 10.17. The object can be regarded as an infinite number of mass elements dm of infinitesimal size. If we impose a cartesian coordinate system on the object, then each mass element rotates in a circle about the origin, and each has a tangential acceleration $\mathbf{a}_t$ produced by an external tangential force $d\mathbf{F}_t$. For any given element, we know from Newton's second law that

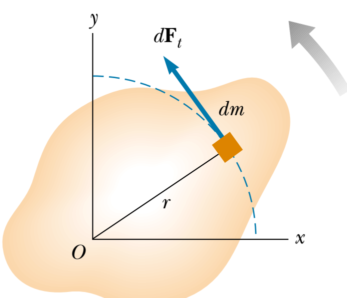


Figure 10.17 A rigid object rotating about an axis through O. Each mass element dm rotates about O with the same angular acceleration α, and the net torque on the object is proportional to α.

$$dF_t = (dm) a_t$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ acts about the origin and is given by

$$d\tau = r\, dF_t = (r\, dm) a_t$$

Because $a_t = r\alpha$, the expression for $d\tau$ becomes

$$d\tau = (r\, dm) r\alpha = (r^2\, dm)\alpha$$

It is important to recognize that although each mass element of the rigid object may have a different linear acceleration $\mathbf{a}_t$, they all have the *same* angular acceleration α. With this in mind, we can integrate the above expression to obtain the net torque about O due to the external forces:

$$\sum \tau = \int (r^2\, dm)\alpha = \alpha \int r^2\, dm$$

where α can be taken outside the integral because it is common to all mass elements. From Equation 10.17, we know that $\int r^2\, dm$ is the moment of inertia of the object about the rotation axis through O, and so the expression for $\Sigma\tau$ becomes

Torque is proportional to angular acceleration

$$\sum \tau = I\alpha \quad \textbf{(10.21)}$$

Note that this is the same relationship we found for a particle rotating in a circle (see Eq. 10.20). So, again we see that the net torque about the rotation axis is proportional to the angular acceleration of the object, with the proportionality factor being I, a quantity that depends upon the axis of rotation and upon the size and shape of the object. In view of the complex nature of the system, it is interesting to note that the relationship $\Sigma\tau = I\alpha$ is strikingly simple and in complete agreement with experimental observations. The simplicity of the result lies in the manner in which the motion is described.

Although each point on a rigid object rotating about a fixed axis may not experience the same force, linear acceleration, or linear speed, each point experiences the same angular acceleration and angular speed at any instant. Therefore, at any instant the rotating rigid object as a whole is characterized by specific values for angular acceleration, net torque, and angular speed.

Every point has the same ω and α

Finally, note that the result $\Sigma\tau = I\alpha$ also applies when the forces acting on the mass elements have radial components as well as tangential components. This is because the line of action of all radial components must pass through the axis of rotation, and hence all radial components produce zero torque about that axis.

QuickLab

Tip over a child's tall tower of blocks. Try this several times. Does the tower "break" at the same place each time? What affects where the tower comes apart as it tips? If the tower were made of toy bricks that snap together, what would happen? (Refer to Conceptual Example 10.11.)

EXAMPLE 10.10 Rotating Rod

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane, as shown in Figure 10.18. The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?

Solution We cannot use our kinematic equations to find α or a because the torque exerted on the rod varies with its position, and so neither acceleration is constant. We have enough information to find the torque, however, which we can then use in the torque–angular acceleration relationship (Eq. 10.21) to find α and then a.

The only force contributing to torque about an axis through the pivot is the gravitational force $M\mathbf{g}$ exerted on the rod. (The force exerted by the pivot on the rod has zero torque about the pivot because its moment arm is zero.) To compute the torque on the rod, we can assume that the gravitational force acts at the center of mass of the rod, as shown in Figure 10.18. The torque due to this force about an axis through the pivot is

$$\tau = Mg\left(\frac{L}{2}\right)$$

With $\Sigma\tau = I\alpha$, and $I = \frac{1}{3}ML^2$ for this axis of rotation (see Table 10.2), we obtain

$$\alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{1/3\,ML^2} = \boxed{\frac{3g}{2L}}$$

All points on the rod have this angular acceleration.

To find the linear acceleration of the right end of the rod, we use the relationship $a_t = r\alpha$ (Eq. 10.11), with $r = L$:

$$a_t = L\alpha = \boxed{\tfrac{3}{2}g}$$

This result—that $a_t > g$ for the free end of the rod—is rather interesting. It means that if we place a coin at the tip of the rod, hold the rod in the horizontal position, and then release the rod, the tip of the rod falls faster than the coin does!

Other points on the rod have a linear acceleration that is less than $\frac{3}{2}g$. For example, the middle of the rod has an acceleration of $\frac{3}{4}g$.

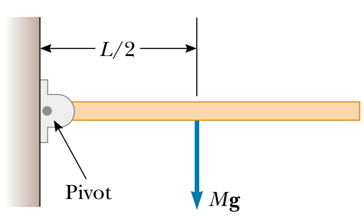


Figure 10.18 The uniform rod is pivoted at the left end.

CONCEPTUAL EXAMPLE 10.11 Falling Smokestacks and Tumbling Blocks

When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground, as shown in Figure 10.19. The same thing happens with a tall tower of children's toy blocks. Why does this happen?

Solution As the smokestack rotates around its base, each higher portion of the smokestack falls with an increasing tangential acceleration. (The tangential acceleration of a given point on the smokestack is proportional to the distance of that portion from the base.) As the acceleration increases, higher portions of the smokestack experience an acceleration greater than that which could result from gravity alone; this is similar to the situation described in Example 10.10. This can happen only if these portions are being pulled downward by a force in addition to the gravitational force. The force that causes this to occur is the shear force from lower portions of the smokestack. Eventually the shear force that provides this acceleration is greater than the smokestack can withstand, and the smokestack breaks.

Figure 10.19 A falling smokestack.

EXAMPLE 10.12 Angular Acceleration of a Wheel

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle, as shown in Figure 10.20. A light cord wrapped around the wheel supports an object of mass m. Calculate the angular acceleration of the wheel, the linear acceleration of the object, and the tension in the cord.

Solution The torque acting on the wheel about its axis of rotation is $\tau = TR$, where T is the force exerted by the cord on the rim of the wheel. (The gravitational force exerted by the Earth on the wheel and the normal force exerted by the axle on the wheel both pass through the axis of rotation and thus produce no torque.) Because $\sum \tau = I\alpha$, we obtain

$$\sum \tau = I\alpha = TR$$

$$(1) \qquad \alpha = \frac{TR}{I}$$

Now let us apply Newton's second law to the motion of the object, taking the downward direction to be positive:

$$\sum F_y = mg - T = ma$$

$$(2) \qquad a = \frac{mg - T}{m}$$

Equations (1) and (2) have three unknowns, α, a, and T. Because the object and wheel are connected by a string that does not slip, the linear acceleration of the suspended object is equal to the linear acceleration of a point on the rim of the

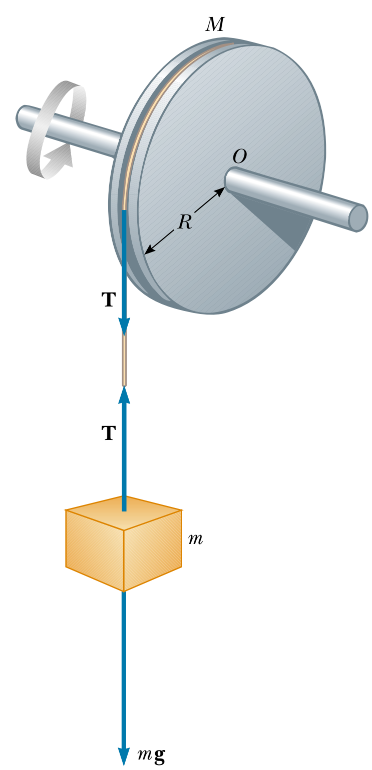


Figure 10.20 The tension in the cord produces a torque about the axle passing through O.

wheel. Therefore, the angular acceleration of the wheel and this linear acceleration are related by $a = R\alpha$. Using this fact together with Equations (1) and (2), we obtain

$$(3) \qquad a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

$$(4) \qquad T = \frac{mg}{1 + \dfrac{mR^2}{I}}$$

Substituting Equation (4) into Equation (2), and solving for a and α, we find that

$$a = \frac{g}{1 + I/mR^2}$$

$$\alpha = \frac{a}{R} = \frac{g}{R + I/mR}$$

Exercise The wheel in Figure 10.20 is a solid disk of $M = 2.00$ kg, $R = 30.0$ cm, and $I = 0.090\,0$ kg·m^2. The suspended object has a mass of $m = 0.500$ kg. Find the tension in the cord and the angular acceleration of the wheel.

Answer 3.27 N; 10.9 rad/s^2.

EXAMPLE 10.13 Atwood's Machine Revisited

Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over two identical, frictionless pulleys, each having a moment of inertia I and radius R, as shown in Figure 10.21a. Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord. (Assume no slipping between cord and pulleys.)

Solution We shall define the downward direction as positive for m_1 and upward as the positive direction for m_2. This allows us to represent the acceleration of both masses by a single variable a and also enables us to relate a positive a to a positive (counterclockwise) angular acceleration α. Let us write Newton's second law of motion for each block, using the free-body diagrams for the two blocks as shown in Figure 10.21b:

$$(1) \qquad m_1 g - T_1 = m_1 a$$

$$(2) \qquad T_3 - m_2 g = m_2 a$$

Next, we must include the effect of the pulleys on the motion. Free-body diagrams for the pulleys are shown in Figure 10.21c. The net torque about the axle for the pulley on the left is $(T_1 - T_2)R$, while the net torque for the pulley on the right is $(T_2 - T_3)R$. Using the relation $\Sigma\tau = I\alpha$ for each pulley and noting that each pulley has the same angular acceleration α, we obtain

$$(3) \qquad (T_1 - T_2)R = I\alpha$$

$$(4) \qquad (T_2 - T_3)R = I\alpha$$

We now have four equations with four unknowns: a, T_1, T_2, and T_3. These can be solved simultaneously. Adding Equations (3) and (4) gives

$$(5) \qquad (T_1 - T_3)R = 2I\alpha$$

Adding Equations (1) and (2) gives

$$T_3 - T_1 + m_1 g - m_2 g = (m_1 + m_2)a$$

$$(6) \qquad T_1 - T_3 = (m_1 - m_2)g - (m_1 + m_2)a$$

Substituting Equation (6) into Equation (5), we have

$$[(m_1 - m_2)g - (m_1 + m_2)a]R = 2I\alpha$$

Because $\alpha = a/R$, this expression can be simplified to

$$(m_1 - m_2)g - (m_1 + m_2)a = 2I\frac{a}{R^2}$$

$$(7) \qquad a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2\dfrac{I}{R^2}}$$

This value of a can then be substituted into Equations (1)

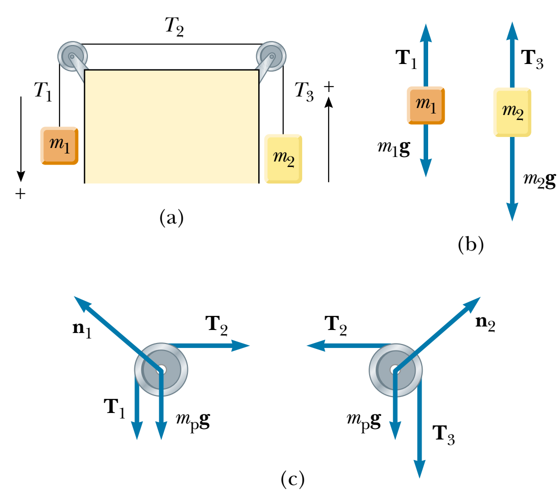


Figure 10.21 (a) Another look at Atwood's machine. (b) Free-body diagrams for the blocks. (c) Free-body diagrams for the pulleys, where $m_p\mathbf{g}$ represents the force of gravity acting on each pulley.

and (2) to give T_1 and T_3. Finally, T_2 can be found from Equation (3) or Equation (4). Note that if $m_1 > m_2$, the acceleration is positive; this means that the left block accelerates downward, the right block accelerates upward, and both pulleys accelerate counterclockwise. If $m_1 < m_2$, then all the values are negative and the motions are reversed. If $m_1 = m_2$, then no acceleration occurs at all. You should compare these results with those found in Example 5.9 on page 129.

10.8 WORK, POWER, AND ENERGY IN ROTATIONAL MOTION

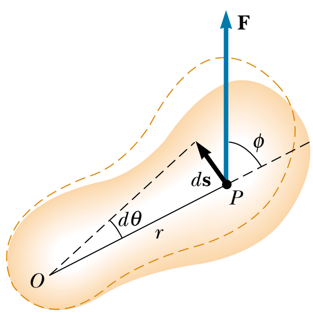


Figure 10.22 A rigid object rotates about an axis through O under the action of an external force **F** applied at P.

In this section, we consider the relationship between the torque acting on a rigid object and its resulting rotational motion in order to generate expressions for the power and a rotational analog to the work–kinetic energy theorem. Consider the rigid object pivoted at O in Figure 10.22. Suppose a single external force **F** is applied at P, where **F** lies in the plane of the page. The work done by **F** as the object rotates through an infinitesimal distance $ds = r\,d\theta$ in a time dt is

$$dW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)r\,d\theta$$

where $F\sin\phi$ is the tangential component of **F**, or, in other words, the component of the force along the displacement. Note that *the radial component of* **F** *does no work because it is perpendicular to the displacement.*

Because the magnitude of the torque due to **F** about O is defined as $rF\sin\phi$ by Equation 10.19, we can write the work done for the infinitesimal rotation as

$$dW = \tau\,d\theta \tag{10.22}$$

The rate at which work is being done by **F** as the object rotates about the fixed axis is

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt}$$

Because dW/dt is the instantaneous power $\mathcal{P}$ (see Section 7.5) delivered by the force, and because $d\theta/dt = \omega$, this expression reduces to

Power delivered to a rigid object

$$\mathcal{P} = \frac{dW}{dt} = \tau\omega \tag{10.23}$$

This expression is analogous to $\mathcal{P} = Fv$ in the case of linear motion, and the expression $dW = \tau\,d\theta$ is analogous to $dW = F_x\,dx$.

Work and Energy in Rotational Motion

In studying linear motion, we found the energy concept—and, in particular, the work–kinetic energy theorem—extremely useful in describing the motion of a system. The energy concept can be equally useful in describing rotational motion. From what we learned of linear motion, we expect that when a symmetric object rotates about a fixed axis, the work done by external forces equals the change in the rotational energy.

To show that this is in fact the case, let us begin with $\sum\tau = I\alpha$. Using the chain rule from the calculus, we can express the resultant torque as

$$\sum\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\,\omega$$

TABLE 10.3 Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Resultant torque $\Sigma\tau = I\alpha$	Resultant force $\Sigma F = ma$
If α = constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f{}^2 = \omega_i{}^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If a = constant $\begin{cases} v_f = v_i + at \\ x_f - x_i = v_i t + \frac{1}{2}at^2 \\ v_f{}^2 = v_i{}^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$	Work $W = \int_{x_i}^{x_f} F_x\, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Resultant torque $\Sigma\tau = dL/dt$	Resultant force $\Sigma F = dp/dt$

Rearranging this expression and noting that $\Sigma\tau\, d\theta = dW$, we obtain

$$\sum \tau\, d\theta = dW = I\omega\, d\omega$$

Integrating this expression, we get for the total work done by the net external force acting on a rotating system

$$\sum W = \int_{\theta_i}^{\theta_f} \sum \tau\, d\theta = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = \tfrac{1}{2}I\omega_f{}^2 - \tfrac{1}{2}I\omega_i{}^2 \qquad \textbf{(10.24)}$$

Work–kinetic energy theorem for rotational motion

where the angular speed changes from ω_i to ω_f as the angular position changes from θ_i to θ_f. That is,

> the net work done by external forces in rotating a symmetric rigid object about a fixed axis equals the change in the object's rotational energy.

Table 10.3 lists the various equations we have discussed pertaining to rotational motion, together with the analogous expressions for linear motion. The last two equations in Table 10.3, involving angular momentum L, are discussed in Chapter 11 and are included here only for the sake of completeness.

Quick Quiz 10.4

For a hoop lying in the xy plane, which of the following requires that more work be done by an external agent to accelerate the hoop from rest to an angular speed ω: (a) rotation about the z axis through the center of the hoop, or (b) rotation about an axis parallel to z passing through a point P on the hoop rim?

EXAMPLE 10.14 Rotating Rod Revisited

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end (Fig 10.23). The rod is released from rest in the horizontal position. (a) What is its angular speed when it reaches its lowest position?

Solution The question can be answered by considering the mechanical energy of the system. When the rod is horizontal, it has no rotational energy. The potential energy relative to the lowest position of the center of mass of the rod (O') is $MgL/2$. When the rod reaches its lowest position, the energy is entirely rotational energy, $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the pivot. Because $I = \frac{1}{3}ML^2$ (see Table 10.2) and because mechanical energy is constant, we have $E_i = E_f$ or

$$\tfrac{1}{2}MgL = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(\tfrac{1}{3}ML^2)\omega^2$$

$$\omega = \boxed{\sqrt{\frac{3g}{L}}}$$

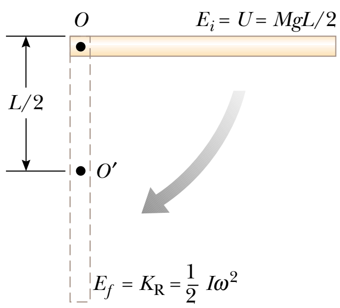


Figure 10.23 A uniform rigid rod pivoted at O rotates in a vertical plane under the action of gravity.

(b) Determine the linear speed of the center of mass and the linear speed of the lowest point on the rod when it is in the vertical position.

Solution These two values can be determined from the relationship between linear and angular speeds. We know ω from part (a), and so the linear speed of the center of mass is

$$v_{\text{CM}} = r\omega = \frac{L}{2}\omega = \boxed{\tfrac{1}{2}\sqrt{3gL}}$$

Because r for the lowest point on the rod is twice what it is for the center of mass, the lowest point has a linear speed equal to

$$2v_{\text{CM}} = \boxed{\sqrt{3gL}}$$

EXAMPLE 10.15 Connected Cylinders

Consider two cylinders having masses m_1 and m_2, where $m_1 \neq m_2$, connected by a string passing over a pulley, as shown in Figure 10.24. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h, and the angular speed of the pulley at this time.

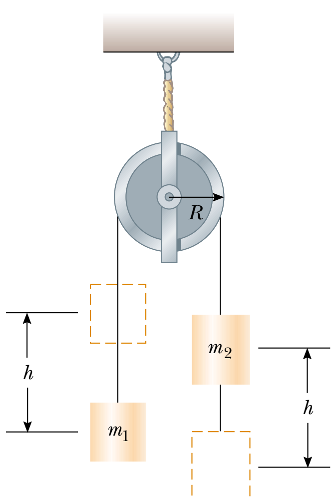


Figure 10.24

Solution We are now able to account for the effect of a massive pulley. Because the string does not slip, the pulley rotates. We neglect friction in the axle about which the pulley rotates for the following reason: Because the axle's radius is small relative to that of the pulley, the frictional torque is much smaller than the torque applied by the two cylinders, provided that their masses are quite different. Mechanical energy is constant; hence, the increase in the system's kinetic energy (the system being the two cylinders, the pulley, and the Earth) equals the decrease in its potential energy. Because $K_i = 0$ (the system is initially at rest), we have

$$\Delta K = K_f - K_i = (\tfrac{1}{2}m_1 v_f{}^2 + \tfrac{1}{2}m_2 v_f{}^2 + \tfrac{1}{2}I\omega_f{}^2) - 0$$

where v_f is the same for both blocks. Because $v_f = R\omega_f$, this expression becomes

$$\Delta K = \tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f{}^2$$

From Figure 10.24, we see that the system loses potential energy as cylinder 2 descends and gains potential energy as cylinder 1 rises. That is, $\Delta U_2 = -m_2gh$ and $\Delta U_1 = m_1gh$. Applying the principle of conservation of energy in the form $\Delta K + \Delta U_1 + \Delta U_2 = 0$ gives

$$\frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2 + m_1gh - m_2gh = 0$$

$$v_f = \left[\frac{2(m_2 - m_1)gh}{\left(m_1 + m_2 + \dfrac{I}{R^2}\right)}\right]^{1/2}$$

Because $v_f = R\omega_f$, the angular speed of the pulley at this instant is

$$\omega_f = \frac{v_f}{R} = \frac{1}{R}\left[\frac{2(m_2 - m_1)gh}{\left(m_1 + m_2 + \dfrac{I}{R^2}\right)}\right]^{1/2}$$

Exercise Repeat the calculation of v_f, using $\Sigma\tau = I\alpha$ applied to the pulley and Newton's second law applied to the two cylinders. Use the procedures presented in Examples 10.12 and 10.13.

SUMMARY

If a particle rotates in a circle of radius r through an angle θ (measured in radians), the arc length it moves through is $s = r\theta$.

The **angular displacement** of a particle rotating in a circle or of a rigid object rotating about a fixed axis is

$$\Delta\theta = \theta_f - \theta_i \qquad \textbf{(10.2)}$$

The **instantaneous angular speed** of a particle rotating in a circle or of a rigid object rotating about a fixed axis is

$$\omega = \frac{d\theta}{dt} \qquad \textbf{(10.4)}$$

The **instantaneous angular acceleration** of a rotating object is

$$\alpha = \frac{d\omega}{dt} \qquad \textbf{(10.6)}$$

When a rigid object rotates about a fixed axis, every part of the object has the same angular speed and the same angular acceleration.

If a particle or object rotates about a fixed axis under constant angular acceleration, one can apply equations of kinematics that are analogous to those for linear motion under constant linear acceleration:

$$\omega_f = \omega_i + \alpha t \qquad \textbf{(10.7)}$$

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \qquad \textbf{(10.8)}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \qquad \textbf{(10.9)}$$

A useful technique in solving problems dealing with rotation is to visualize a linear version of the same problem.

When a rigid object rotates about a fixed axis, the angular position, angular speed, and angular acceleration are related to the linear position, linear speed, and linear acceleration through the relationships

$$s = r\theta \qquad \textbf{(10.1a)}$$

$$v = r\omega \qquad \textbf{(10.10)}$$

$$a_t = r\alpha \qquad \textbf{(10.11)}$$

You must be able to easily alternate between the linear and rotational variables that describe a given situation.

The **moment of inertia of a system of particles** is

$$I \equiv \sum_i m_i r_i^2 \quad \textbf{(10.15)}$$

If a rigid object rotates about a fixed axis with angular speed ω, its **rotational energy** can be written

$$K_R = \tfrac{1}{2} I \omega^2 \quad \textbf{(10.16)}$$

where I is the moment of inertia about the axis of rotation.

The **moment of inertia of a rigid object** is

$$I = \int r^2 \, dm \quad \textbf{(10.17)}$$

where r is the distance from the mass element dm to the axis of rotation.

The magnitude of the **torque** associated with a force **F** acting on an object is

$$\tau = Fd \quad \textbf{(10.19)}$$

where d is the moment arm of the force, which is the perpendicular distance from some origin to the line of action of the force. Torque is a measure of the tendency of the force to change the rotation of the object about some axis.

If a rigid object free to rotate about a fixed axis has a **net external torque** acting on it, the object undergoes an angular acceleration α, where

$$\sum \tau = I\alpha \quad \textbf{(10.21)}$$

The rate at which work is done by an external force in rotating a rigid object about a fixed axis, or the **power** delivered, is

$$\mathcal{P} = \tau\omega \quad \textbf{(10.23)}$$

The net work done by external forces in rotating a rigid object about a fixed axis equals the change in the rotational kinetic energy of the object:

$$\sum W = \tfrac{1}{2} I \omega_f^2 - \tfrac{1}{2} I \omega_i^2 \quad \textbf{(10.24)}$$

QUESTIONS

1. What is the angular speed of the second hand of a clock? What is the direction of $\boldsymbol{\omega}$ as you view a clock hanging vertically? What is the magnitude of the angular acceleration vector $\boldsymbol{\alpha}$ of the second hand?
2. A wheel rotates counterclockwise in the xy plane. What is the direction of $\boldsymbol{\omega}$? What is the direction of $\boldsymbol{\alpha}$ if the angular velocity is decreasing in time?
3. Are the kinematic expressions for θ, ω, and α valid when the angular displacement is measured in degrees instead of in radians?
4. A turntable rotates at a constant rate of 45 rev/min. What is its angular speed in radians per second? What is the magnitude of its angular acceleration?
5. Suppose $a = b$ and $M > m$ for the system of particles described in Figure 10.8. About what axis (x, y, or z) does the moment of inertia have the smallest value? the largest value?
6. Suppose the rod in Figure 10.10 has a nonuniform mass distribution. In general, would the moment of inertia about the y axis still equal $ML^2/12$? If not, could the moment of inertia be calculated without knowledge of the manner in which the mass is distributed?
7. Suppose that only two external forces act on a rigid body, and the two forces are equal in magnitude but opposite in direction. Under what condition does the body rotate?
8. Explain how you might use the apparatus described in Example 10.12 to determine the moment of inertia of the wheel. (If the wheel does not have a uniform mass density, the moment of inertia is not necessarily equal to $\frac{1}{2}MR^2$.)

9. Using the results from Example 10.12, how would you calculate the angular speed of the wheel and the linear speed of the suspended mass at $t = 2$ s, if the system is released from rest at $t = 0$? Is the expression $v = R\omega$ valid in this situation?
10. If a small sphere of mass M were placed at the end of the rod in Figure 10.23, would the result for ω be greater than, less than, or equal to the value obtained in Example 10.14?
11. Explain why changing the axis of rotation of an object changes its moment of inertia.
12. Is it possible to change the translational kinetic energy of an object without changing its rotational energy?
13. Two cylinders having the same dimensions are set into rotation about their long axes with the same angular speed. One is hollow, and the other is filled with water. Which cylinder will be easier to stop rotating? Explain your answer.
14. Must an object be rotating to have a nonzero moment of inertia?
15. If you see an object rotating, is there necessarily a net torque acting on it?
16. Can a (momentarily) stationary object have a nonzero angular acceleration?
17. The polar diameter of the Earth is slightly less than the equatorial diameter. How would the moment of inertia of the Earth change if some mass from near the equator were removed and transferred to the polar regions to make the Earth a perfect sphere?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

1. A wheel starts from rest and rotates with constant angular acceleration and reaches an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle (in radians) through which it rotates in this time.
2. What is the angular speed in radians per second of (a) the Earth in its orbit about the Sun and (b) the Moon in its orbit about the Earth?
3. An airliner arrives at the terminal, and its engines are shut off. The rotor of one of the engines has an initial clockwise angular speed of 2 000 rad/s. The engine's rotation slows with an angular acceleration of magnitude 80.0 rad/s^2. (a) Determine the angular speed after 10.0 s. (b) How long does it take for the rotor to come to rest?
4. (a) The positions of the hour and minute hand on a clock face coincide at 12 o'clock. Determine all other times (up to the second) at which the positions of the hands coincide. (b) If the clock also has a second hand, determine all times at which the positions of all three hands coincide, given that they all coincide at 12 o'clock.

WEB 5. An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative acceleration of magnitude 2.00 rad/s^2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn during the time found in part (a)?

6. A centrifuge in a medical laboratory rotates at a rotational speed of 3 600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.
7. The angular position of a swinging door is described by $\theta = 5.00 + 10.0t + 2.00t^2$ rad. Determine the angular position, angular speed, and angular acceleration of the door (a) at $t = 0$ and (b) at $t = 3.00$ s.
8. The tub of a washer goes into its spin cycle, starting from rest and gaining angular speed steadily for 8.00 s, when it is turning at 5.00 rev/s. At this point the person doing the laundry opens the lid, and a safety switch turns off the washer. The tub smoothly slows to rest in 12.0 s. Through how many revolutions does the tub turn while it is in motion?
9. A rotating wheel requires 3.00 s to complete 37.0 revolutions. Its angular speed at the end of the 3.00-s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?
10. (a) Find the angular speed of the Earth's rotation on its axis. As the Earth turns toward the east, we see the sky turning toward the west at this same rate.
(b) *The rainy Pleiads wester*
And seek beyond the sea
The head that I shall dream of
That shall not dream of me.

A. E. Housman (*© Robert E. Symons*)

Cambridge, England, is at longitude 0°, and Saskatoon, Saskatchewan, is at longitude 107° west. How much time elapses after the Pleiades set in Cambridge until these stars fall below the western horizon in Saskatoon?

Section 10.3 Angular and Linear Quantities

11. Make an order-of-magnitude estimate of the number of revolutions through which a typical automobile tire

turns in 1 yr. State the quantities you measure or estimate and their values.

12. The diameters of the main rotor and tail rotor of a single-engine helicopter are 7.60 m and 1.02 m, respectively. The respective rotational speeds are 450 rev/min and 4 138 rev/min. Calculate the speeds of the tips of both rotors. Compare these speeds with the speed of sound, 343 m/s.

Figure P10.12 *(Ross Harrrison Koty/Tony Stone Images)*

13. A racing car travels on a circular track with a radius of 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration.

14. A car is traveling at 36.0 km/h on a straight road. The radius of its tires is 25.0 cm. Find the angular speed of one of the tires, with its axle taken as the axis of rotation.

15. A wheel 2.00 m in diameter lies in a vertical plane and rotates with a constant angular acceleration of 4.00 rad/s^2. The wheel starts at rest at $t = 0$, and the radius vector of point P on the rim makes an angle of 57.3° with the horizontal at this time. At $t = 2.00$ s, find (a) the angular speed of the wheel, (b) the linear speed and acceleration of the point P, and (c) the angular position of the point P.

16. A discus thrower accelerates a discus from rest to a speed of 25.0 m/s by whirling it through 1.25 rev. Assume the discus moves on the arc of a circle 1.00 m in radius. (a) Calculate the final angular speed of the discus. (b) Determine the magnitude of the angular acceleration of the discus, assuming it to be constant. (c) Calculate the acceleration time.

Figure P10.16 *(Bruce Ayers/Tony Stone Images)*

17. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming that no slipping occurs. (b) What is the final rotational speed of a tire in revolutions per second?

18. A 6.00-kg block is released from A on the frictionless track shown in Figure P10.18. Determine the radial and tangential components of acceleration for the block at P.

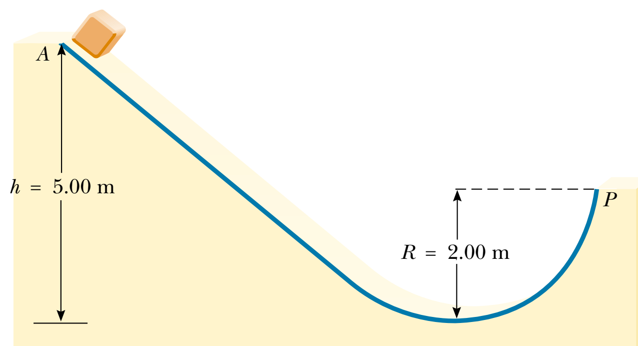


Figure P10.18

WEB 19. A disc 8.00 cm in radius rotates at a constant rate of 1 200 rev/min about its central axis. Determine (a) its angular speed, (b) the linear speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.

20. A car traveling on a flat (unbanked) circular track accelerates uniformly from rest with a tangential acceleration of 1.70 m/s^2. The car makes it one quarter of the way around the circle before it skids off the track. Determine the coefficient of static friction between the car and track from these data.

21. A small object with mass 4.00 kg moves counterclockwise with constant speed 4.50 m/s in a circle of radius 3.00 m centered at the origin. (a) It started at the point with cartesian coordinates (3 m, 0). When its angular displacement is 9.00 rad, what is its position vector, in cartesian unit-vector notation? (b) In what quadrant is the particle located, and what angle does its position vector make with the positive x axis? (c) What is its velocity vector, in unit–vector notation? (d) In what direction is it moving? Make a sketch of the position and velocity vectors. (e) What is its acceleration, expressed in unit–vector notation? (f) What total force acts on the object? (Express your answer in unit vector notation.)

22. A standard cassette tape is placed in a standard cassette player. Each side plays for 30 min. The two tape wheels of the cassette fit onto two spindles in the player. Suppose that a motor drives one spindle at a constant angular speed of ~ 1 rad/s and that the other spindle is free to rotate at any angular speed. Estimate the order of magnitude of the thickness of the tape.

Section 10.4 Rotational Energy

23. Three small particles are connected by rigid rods of negligible mass lying along the y axis (Fig. P10.23). If the system rotates about the x axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the x axis and the total rotational kinetic energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the linear speed of each particle and the total kinetic energy evaluated from $\Sigma\frac{1}{2}m_iv_i^2$.

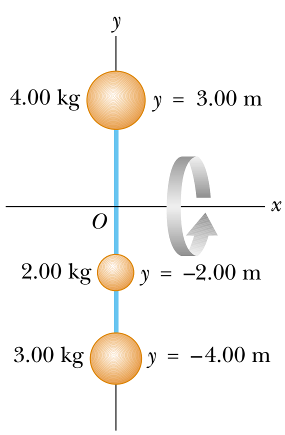


Figure P10.23

24. The center of mass of a pitched baseball (3.80-cm radius) moves at 38.0 m/s. The ball spins about an axis through its center of mass with an angular speed of 125 rad/s. Calculate the ratio of the rotational energy to the translational kinetic energy. Treat the ball as a uniform sphere.

WEB **25.** The four particles in Figure P10.25 are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the xy plane about the z axis with an angular speed of 6.00 rad/s, calculate (a) the moment of inertia of the system about the z axis and (b) the rotational energy of the system.

26. The hour hand and the minute hand of Big Ben, the famous Parliament tower clock in London, are 2.70 m long and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively. Calculate the total rotational kinetic energy of the two hands about the axis of rotation. (You may model the hands as long thin rods.)

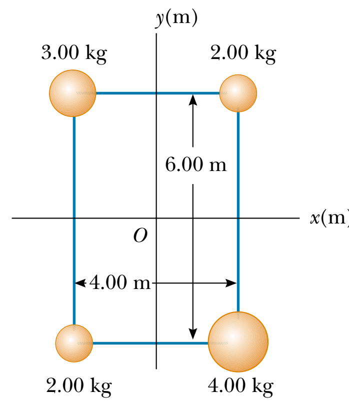


Figure P10.25

Figure P10.26 Problems 26 and 74. *(John Lawrence/Tony Stone Images)*

27. Two masses M and m are connected by a rigid rod of length L and of negligible mass, as shown in Figure P10.27. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the center of mass. Show that this moment of inertia is $I = \mu L^2$, where $\mu = mM/(m + M)$.

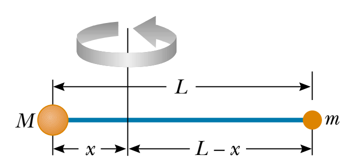


Figure P10.27

Section 10.5 Calculation of Moments of Inertia

28. Three identical thin rods, each of length L and mass m, are welded perpendicular to each other, as shown in Figure P10.28. The entire setup is rotated about an axis

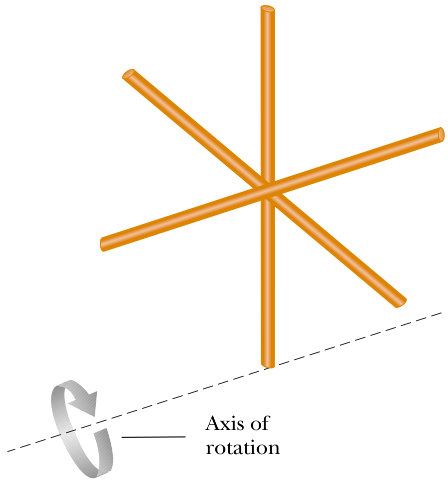


Figure P10.28

that passes through the end of one rod and is parallel to another. Determine the moment of inertia of this arrangement.

29. Figure P10.29 shows a side view of a car tire and its radial dimensions. The rubber tire has two sidewalls of uniform thickness 0.635 cm and a tread wall of uniform thickness 2.50 cm and width 20.0 cm. Suppose its density is uniform, with the value 1.10×10^3 kg/m^3. Find its moment of inertia about an axis through its center perpendicular to the plane of the sidewalls.

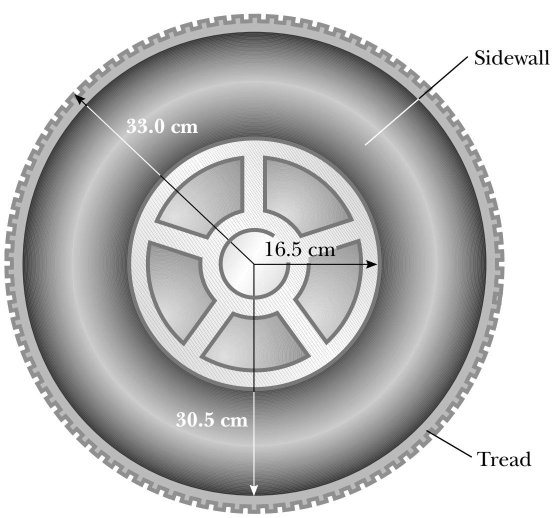


Figure P10.29

30. Use the parallel-axis theorem and Table 10.2 to find the moments of inertia of (a) a solid cylinder about an axis parallel to the center-of-mass axis and passing through the edge of the cylinder and (b) a solid sphere about an axis tangent to its surface.

31. *Attention! About face!* Compute an order-of-magnitude estimate for the moment of inertia of your body as you stand tall and turn around a vertical axis passing through the top of your head and the point halfway between your ankles. In your solution state the quantities you measure or estimate and their values.

Section 10.6 Torque

32. Find the mass m needed to balance the 1 500-kg truck on the incline shown in Figure P10.32. Assume all pulleys are frictionless and massless.

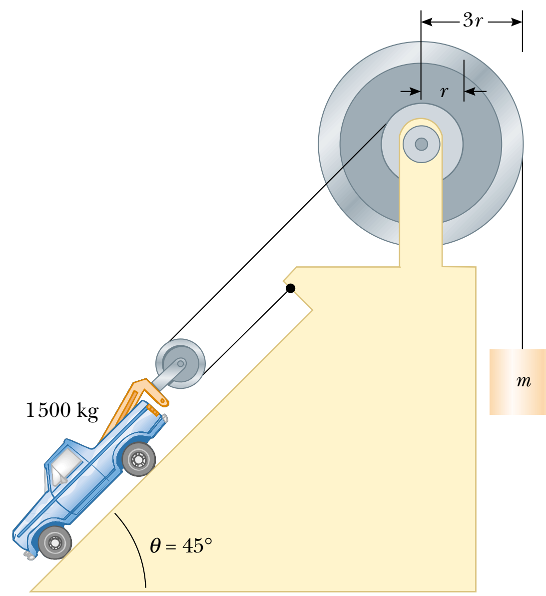


Figure P10.32

WEB **33.** Find the net torque on the wheel in Figure P10.33 about the axle through O if a = 10.0 cm and b = 25.0 cm.

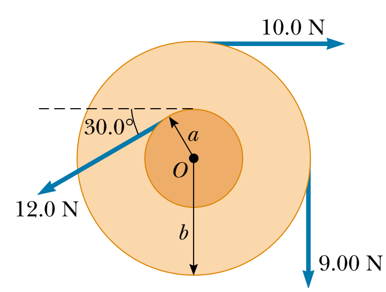


Figure P10.33

34. The fishing pole in Figure P10.34 makes an angle of 20.0° with the horizontal. What is the torque exerted by

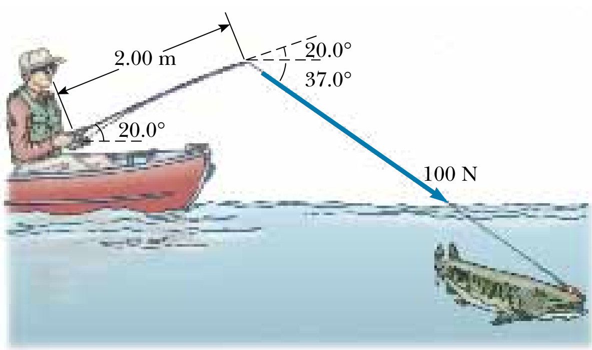


Figure P10.34

the fish about an axis perpendicular to the page and passing through the fisher's hand?

35. The tires of a 1 500-kg car are 0.600 m in diameter, and the coefficients of friction with the road surface are $\mu_s = 0.800$ and $\mu_k = 0.600$. Assuming that the weight is evenly distributed on the four wheels, calculate the maximum torque that can be exerted by the engine on a driving wheel such that the wheel does not spin. If you wish, you may suppose that the car is at rest.

36. Suppose that the car in Problem 35 has a disk brake system. Each wheel is slowed by the frictional force between a single brake pad and the disk-shaped rotor. On this particular car, the brake pad comes into contact with the rotor at an average distance of 22.0 cm from the axis. The coefficients of friction between the brake pad and the disk are $\mu_s = 0.600$ and $\mu_k = 0.500$. Calculate the normal force that must be applied to the rotor such that the car slows as quickly as possible.

Section 10.7 Relationship Between Torque and Angular Acceleration

WEB **37.** A model airplane having a mass of 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.

38. The combination of an applied force and a frictional force produces a constant total torque of 36.0 N·m on a wheel rotating about a fixed axis. The applied force acts for 6.00 s; during this time the angular speed of the wheel increases from 0 to 10.0 rad/s. The applied force is then removed, and the wheel comes to rest in 60.0 s. Find (a) the moment of inertia of the wheel, (b) the magnitude of the frictional torque, and (c) the total number of revolutions of the wheel.

39. A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley in the shape of a disk having radius $R = 0.250$ m and mass $M = 10.0$ kg. These blocks are allowed to move on a fixed block–wedge of angle $\theta = 30.0°$, as shown in Figure P10.39. The coefficient of kinetic friction for both blocks is 0.360. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks and (b) the tensions in the string on both sides of the pulley.

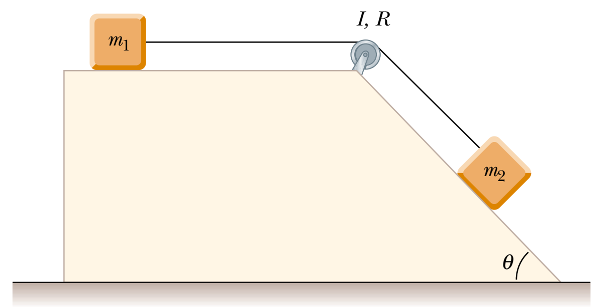


Figure P10.39

40. A potter's wheel—a thick stone disk with a radius of 0.500 m and a mass of 100 kg—is freely rotating at 50.0 rev/min. The potter can stop the wheel in 6.00 s by pressing a wet rag against the rim and exerting a radially inward force of 70.0 N. Find the effective coefficient of kinetic friction between the wheel and the rag.

41. A bicycle wheel has a diameter of 64.0 cm and a mass of 1.80 kg. Assume that the wheel is a hoop with all of its mass concentrated on the outside radius. The bicycle is placed on a stationary stand on rollers, and a resistive force of 120 N is applied tangent to the rim of the tire. (a) What force must be applied by a chain passing over a 9.00-cm-diameter sprocket if the wheel is to attain an acceleration of 4.50 rad/s^2? (b) What force is required if the chain shifts to a 5.60-cm-diameter sprocket?

Section 10.8 Work, Power, and Energy in Rotational Motion

42. A cylindrical rod 24.0 cm long with a mass of 1.20 kg and a radius of 1.50 cm has a ball with a diameter of 8.00 cm and a mass of 2.00 kg attached to one end. The arrangement is originally vertical and stationary, with the ball at the top. The apparatus is free to pivot about the bottom end of the rod. (a) After it falls through 90°, what is its rotational kinetic energy? (b) What is the angular speed of the rod and ball? (c) What is the linear speed of the ball? (d) How does this compare with the speed if the ball had fallen freely through the same distance of 28 cm?

43. A 15.0-kg mass and a 10.0-kg mass are suspended by a pulley that has a radius of 10.0 cm and a mass of 3.00 kg (Fig. P10.43). The cord has a negligible mass and causes the pulley to rotate without slipping. The pulley rotates without friction. The masses start from rest 3.00 m apart. Treating the pulley as a uniform disk, determine the speeds of the two masses as they pass each other.

44. A mass m_1 and a mass m_2 are suspended by a pulley that has a radius R and a mass M (see Fig. P10.43). The cord has a negligible mass and causes the pulley to rotate without slipping. The pulley rotates without friction. The masses start from rest a distance d apart. Treating the pulley as a uniform disk, determine the speeds of the two masses as they pass each other.

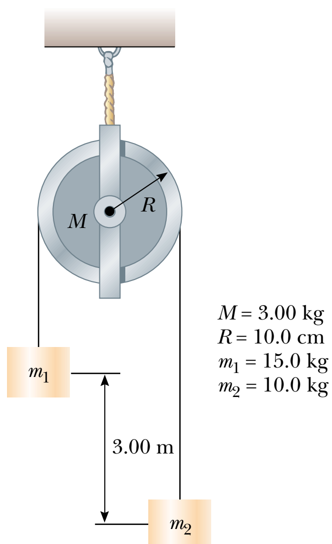


Figure P10.43 Problems 43 and 44.

45. A weight of 50.0 N is attached to the free end of a light string wrapped around a reel with a radius of 0.250 m and a mass of 3.00 kg. The reel is a solid disk, free to rotate in a vertical plane about the horizontal axis passing through its center. The weight is released 6.00 m above the floor. (a) Determine the tension in the string, the acceleration of the mass, and the speed with which the weight hits the floor. (b) Find the speed calculated in part (a), using the principle of conservation of energy.

46. A constant torque of 25.0 N·m is applied to a grindstone whose moment of inertia is 0.130 kg·m^2. Using energy principles, find the angular speed after the grindstone has made 15.0 revolutions. (Neglect friction.)

47. This problem describes one experimental method of determining the moment of inertia of an irregularly shaped object such as the payload for a satellite. Figure P10.47 shows a mass m suspended by a cord wound around a spool of radius r, forming part of a turntable supporting the object. When the mass is released from rest, it descends through a distance h, acquiring a speed v. Show that the moment of inertia I of the equipment (including the turntable) is $mr^2(2gh/v^2 - 1)$.

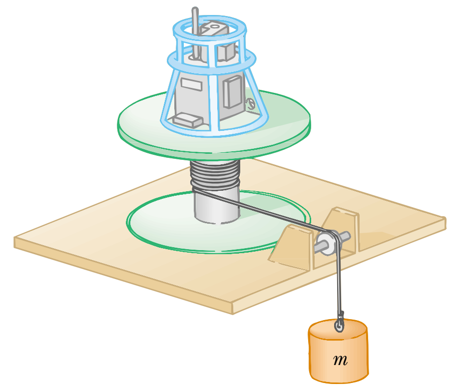


Figure P10.47

48. A bus is designed to draw its power from a rotating flywheel that is brought up to its maximum rate of rotation (3 000 rev/min) by an electric motor. The flywheel is a solid cylinder with a mass of 1 000 kg and a diameter of 1.00 m. If the bus requires an average power of 10.0 kW, how long does the flywheel rotate?

49. (a) A uniform, solid disk of radius R and mass M is free to rotate on a frictionless pivot through a point on its rim (Fig. P10.49). If the disk is released from rest in the position shown by the blue circle, what is the speed of its center of mass when the disk reaches the position indicated by the dashed circle? (b) What is the speed of the lowest point on the disk in the dashed position? (c) Repeat part (a), using a uniform hoop.

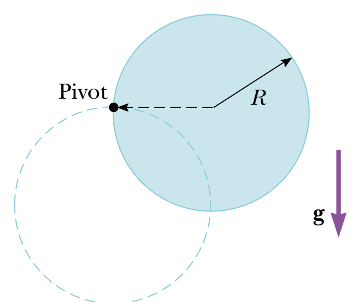


Figure P10.49

50. A horizontal 800-N merry-go-round is a solid disk of radius 1.50 m and is started from rest by a constant horizontal force of 50.0 N applied tangentially to the cylinder. Find the kinetic energy of the solid cylinder after 3.00 s.

ADDITIONAL PROBLEMS

51. Toppling chimneys often break apart in mid-fall (Fig. P10.51) because the mortar between the bricks cannot withstand much shear stress. As the chimney begins to fall, shear forces must act on the topmost sections to accelerate them tangentially so that they can keep up with the rotation of the lower part of the stack. For simplicity, let us model the chimney as a uniform rod of length ℓ pivoted at the lower end. The rod starts at rest in a vertical position (with the frictionless pivot at the bottom) and falls over under the influence of gravity. What fraction of the length of the rod has a tangential acceleration greater than $g \sin \theta$, where θ is the angle the chimney makes with the vertical?

Figure P10.51 A building demolition site in Baltimore, MD. At the left is a chimney, mostly concealed by the building, that has broken apart on its way down. Compare with Figure 10.19. *(Jerry Wachter/Photo Researchers, Inc.)*

52. **Review Problem.** A mixing beater consists of three thin rods: Each is 10.0 cm long, diverges from a central hub, and is separated from the others by 120°. All turn in the same plane. A ball is attached to the end of each rod. Each ball has a cross-sectional area of 4.00 cm^2 and is shaped so that it has a drag coefficient of 0.600. Calculate the power input required to spin the beater at 1 000 rev/min (a) in air and (b) in water.

53. A grinding wheel is in the form of a uniform solid disk having a radius of 7.00 cm and a mass of 2.00 kg. It starts from rest and accelerates uniformly under the action of the constant torque of 0.600 N·m that the motor exerts on the wheel. (a) How long does the wheel take to reach its final rotational speed of 1 200 rev/min? (b) Through how many revolutions does it turn while accelerating?

54. The density of the Earth, at any distance r from its center, is approximately

$$\rho = [14.2 - 11.6\, r/R] \times 10^3 \text{ kg/m}^3$$

where R is the radius of the Earth. Show that this density leads to a moment of inertia $I = 0.330MR^2$ about an axis through the center, where M is the mass of the Earth.

55. A 4.00-m length of light nylon cord is wound around a uniform cylindrical spool of radius 0.500 m and mass 1.00 kg. The spool is mounted on a frictionless axle and is initially at rest. The cord is pulled from the spool with a constant acceleration of magnitude 2.50 m/s^2. (a) How much work has been done on the spool when it reaches an angular speed of 8.00 rad/s? (b) Assuming that there is enough cord on the spool, how long does it take the spool to reach this angular speed? (c) Is there enough cord on the spool?

56. A flywheel in the form of a heavy circular disk of diameter 0.600 m and mass 200 kg is mounted on a frictionless bearing. A motor connected to the flywheel accelerates it from rest to 1 000 rev/min. (a) What is the moment of inertia of the flywheel? (b) How much work is done on it during this acceleration? (c) When the angular speed reaches 1 000 rev/min, the motor is disengaged. A friction brake is used to slow the rotational rate to 500 rev/min. How much energy is dissipated as internal energy in the friction brake?

57. A shaft is turning at 65.0 rad/s at time zero. Thereafter, its angular acceleration is given by

$$\alpha = -10 \text{ rad/s}^2 - 5t \text{ rad/s}^3$$

where t is the elapsed time. (a) Find its angular speed at $t = 3.00$ s. (b) How far does it turn in these 3 s?

58. For any given rotational axis, the *radius of gyration* K of a rigid body is defined by the expression $K^2 = I/M$, where M is the total mass of the body and I is its moment of inertia. Thus, the radius of gyration is equal to the distance between an imaginary point mass M and the axis of rotation such that I for the point mass about that axis is the same as that for the rigid body. Find the radius of gyration of (a) a solid disk of radius R, (b) a uniform rod of length L, and (c) a solid sphere of radius R, all three of which are rotating about a central axis.

59. A long, uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin passing through one end. The rod is released from rest in a vertical position, as shown in Figure P10.59. At the instant the rod is horizontal, find (a) its angular speed, (b) the magnitude of its angular acceleration, (c) the x and y components of the acceleration of its center of mass, and (d) the components of the reaction force at the pivot.

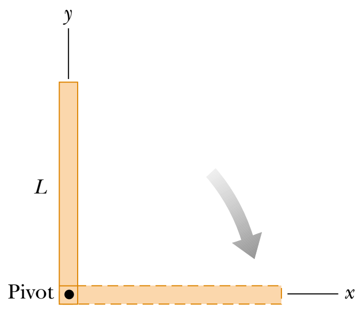


Figure P10.59

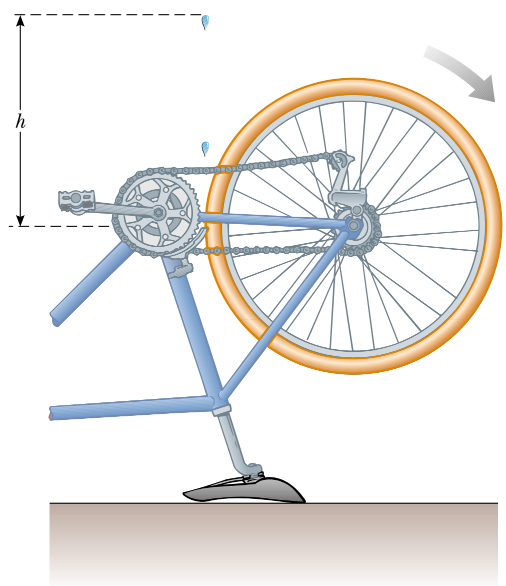


Figure P10.60 Problems 60 and 61.

60. A bicycle is turned upside down while its owner repairs a flat tire. A friend spins the other wheel, of radius 0.381 m, and observes that drops of water fly off tangentially. She measures the height reached by drops moving vertically (Fig. P10.60). A drop that breaks loose from the tire on one turn rises $h = 54.0$ cm above the tangent point. A drop that breaks loose on the next turn rises 51.0 cm above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.

61. A bicycle is turned upside down while its owner repairs a flat tire. A friend spins the other wheel of radius R and observes that drops of water fly off tangentially. She measures the height reached by drops moving vertically (see Fig. P10.60). A drop that breaks loose from the tire on one turn rises a distance h_1 above the tangent point. A drop that breaks loose on the next turn rises a distance $h_2 < h_1$ above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.

62. The top shown in Figure P10.62 has a moment of inertia of 4.00×10^{-4} kg·m^2 and is initially at rest. It is free to rotate about the stationary axis AA'. A string, wrapped around a peg along the axis of the top, is pulled in such a manner that a constant tension of 5.57 N is maintained. If the string does not slip while it is unwound from the peg, what is the angular speed of the top after 80.0 cm of string has been pulled off the peg?

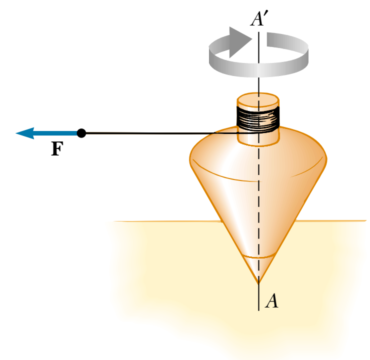


Figure P10.62

63. A cord is wrapped around a pulley of mass m and of radius r. The free end of the cord is connected to a block of mass M. The block starts from rest and then slides down an incline that makes an angle θ with the horizontal. The coefficient of kinetic friction between block and incline is μ. (a) Use energy methods to show that the block's speed as a function of displacement d down the incline is

$$v = [4gdM(m + 2M)^{-1}(\sin\theta - \mu\cos\theta)]^{1/2}$$

(b) Find the magnitude of the acceleration of the block in terms of μ, m, M, g, and θ.

64. (a) What is the rotational energy of the Earth about its spin axis? The radius of the Earth is 6 370 km, and its mass is 5.98×10^{24} kg. Treat the Earth as a sphere of moment of inertia $\frac{2}{5}MR^2$. (b) The rotational energy of the Earth is decreasing steadily because of tidal friction. Estimate the change in one day, given that the rotational period increases by about 10 μs each year.

65. The speed of a moving bullet can be determined by allowing the bullet to pass through two rotating paper disks mounted a distance d apart on the same axle (Fig. P10.65). From the angular displacement $\Delta\theta$ of the two

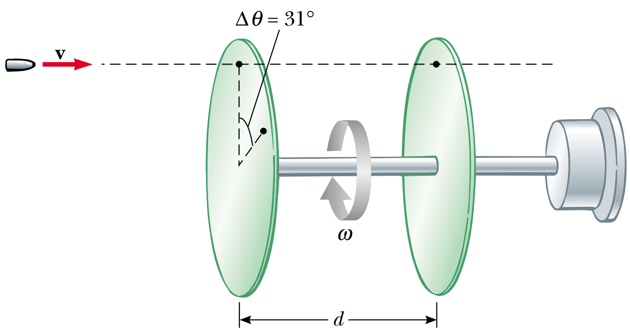


Figure P10.65

bullet holes in the disks and the rotational speed of the disks, we can determine the speed v of the bullet. Find the bullet speed for the following data: $d = 80$ cm, $\omega = 900$ rev/min, and $\Delta\theta = 31.0°$.

66. A wheel is formed from a hoop and n equally spaced spokes extending from the center of the hoop to its rim. The mass of the hoop is M, and the radius of the hoop (and hence the length of each spoke) is R. The mass of each spoke is m. Determine (a) the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel and (b) the moment of inertia of the wheel about an axis through its rim and perpendicular to the plane of the wheel.

67. A uniform, thin, solid door has a height of 2.20 m, a width of 0.870 m, and a mass of 23.0 kg. Find its moment of inertia for rotation on its hinges. Are any of the data unnecessary?

68. A uniform, hollow, cylindrical spool has inside radius $R/2$, outside radius R, and mass M (Fig. P10.68). It is mounted so that it rotates on a massless horizontal axle. A mass m is connected to the end of a string wound around the spool. The mass m falls from rest through a distance y in time t. Show that the torque due to the frictional forces between spool and axle is

$$\tau_f = R[m(g - 2y/t^2) - M(5y/4t^2)]$$

69. An electric motor can accelerate a Ferris wheel of moment of inertia $I = 20\,000$ kg·m^2 from rest to 10.0 rev/min in 12.0 s. When the motor is turned off, friction causes the wheel to slow down from 10.0 to 8.00 rev/min in 10.0 s. Determine (a) the torque generated by the motor to bring the wheel to 10.0 rev/min and (b) the power that would be needed to maintain this rotational speed.

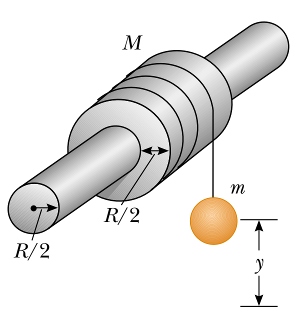


Figure P10.68

70. The pulley shown in Figure P10.70 has radius R and moment of inertia I. One end of the mass m is connected to a spring of force constant k, and the other end is fastened to a cord wrapped around the pulley. The pulley axle and the incline are frictionless. If the pulley is wound counterclockwise so that the spring is stretched a distance d from its unstretched position and is then released from rest, find (a) the angular speed of the pulley when the spring is again unstretched and (b) a numerical value for the angular speed at this point if $I = 1.00$ kg·m^2, $R = 0.300$ m, $k = 50.0$ N/m, $m = 0.500$ kg, $d = 0.200$ m, and $\theta = 37.0°$.

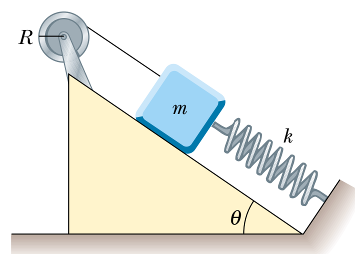


Figure P10.70

71. Two blocks, as shown in Figure P10.71, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia I. The block on the frictionless incline is moving upward with a constant acceleration of 2.00 m/s^2. (a) Determine T_1 and T_2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.

72. A common demonstration, illustrated in Figure P10.72, consists of a ball resting at one end of a uniform board

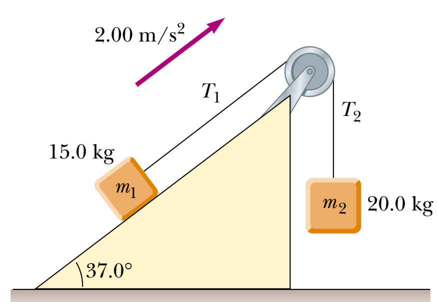


Figure P10.71

of length ℓ, hinged at the other end, and elevated at an angle θ. A light cup is attached to the board at r_c so that it will catch the ball when the support stick is suddenly removed. (a) Show that the ball will lag behind the falling board when θ is less than 35.3° ; and that (b) the ball will fall into the cup when the board is supported at this limiting angle and the cup is placed at

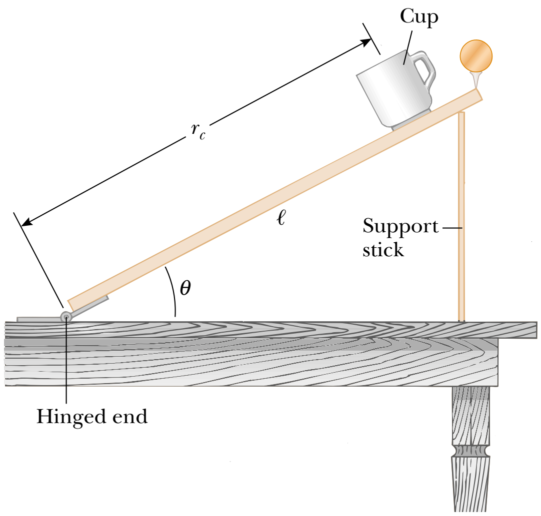


Figure P10.72

$$r_c = \frac{2\,\ell}{3\cos\theta}$$

(c) If a ball is at the end of a 1.00-m stick at this critical angle, show that the cup must be 18.4 cm from the moving end.

73. As a result of friction, the angular speed of a wheel changes with time according to the relationship

$$d\theta/dt = \omega_0 e^{-\sigma t}$$

where ω_0 and σ are constants. The angular speed changes from 3.50 rad/s at $t = 0$ to 2.00 rad/s at $t = 9.30$ s. Use this information to determine σ and ω_0. Then, determine (a) the magnitude of the angular acceleration at $t = 3.00$ s, (b) the number of revolutions the wheel makes in the first 2.50 s, and (c) the number of revolutions it makes before coming to rest.

74. The hour hand and the minute hand of Big Ben, the famous Parliament tower clock in London, are 2.70 m long and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively (see Fig. P10.26). (a) Determine the total torque due to the weight of these hands about the axis of rotation when the time reads (i) 3:00, (ii) 5:15, (iii) 6:00, (iv) 8:20, and (v) 9:45. (You may model the hands as long thin rods.) (b) Determine all times at which the total torque about the axis of rotation is zero. Determine the times to the nearest second, solving a transcendental equation numerically.

Answers to Quick Quizzes

10.1 The fact that ω is negative indicates that we are dealing with an object that is rotating in the clockwise direction. We also know that when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are antiparallel, ω must be decreasing—the object is slowing down. Therefore, the object is spinning more and more slowly (with less and less angular speed) in the clockwise, or negative, direction. This has a linear analogy to a sky diver opening her parachute. The velocity is negative—downward. When the sky diver opens the parachute, a large upward force causes an upward acceleration. As a result, the acceleration and velocity vectors are in opposite directions. Consequently, the parachutist slows down.

10.2 (a) Yes, all points on the wheel have the same angular speed. This is why we use angular quantities to describe rotational motion. (b) No, not all points on the wheel have the same linear speed. (c) $v = 0$, $a = 0$. (d) $v = R\omega/2$, $a = a_r = v^2/(R/2) = R\omega^2/2$ (a_t is zero at all points because ω is constant). (e) $v = R\omega$, $a = R\omega^2$.

10.3 (a) $I = MR^2$. (b) $I = MR^2$. The moment of inertia of a system of masses equidistant from an axis of rotation is always the sum of the masses multiplied by the square of the distance from the axis.

10.4 (b) Rotation about the axis through point P requires more work. The moment of inertia of the hoop about the center axis is $I_{CM} = MR^2$, whereas, by the parallel-axis theorem, the moment of inertia about the axis through point P is $I_P = I_{CM} + MR^2 = MR^2 + MR^2 = 2MR^2$.

PUZZLER

One of the most popular early bicycles was the penny–farthing, introduced in 1870. The bicycle was so named because the size relationship of its two wheels was about the same as the size relationship of the penny and the farthing, two English coins. When the rider looks down at the top of the front wheel, he sees it moving forward faster than he and the handlebars are moving. Yet the center of the wheel does not appear to be moving at all relative to the handlebars. How can different parts of the rolling wheel move at different linear speeds? *(© Steve Lovegrove/Tasmanian Photo Library)*

chapter 11

Rolling Motion and Angular Momentum

Chapter Outline

In the preceding chapter we learned how to treat a rigid body rotating about a fixed axis; in the present chapter, we move on to the more general case in which the axis of rotation is not fixed in space. We begin by describing such motion, which is called *rolling motion.* The central topic of this chapter is, however, angular momentum, a quantity that plays a key role in rotational dynamics. In analogy to the conservation of linear momentum, we find that the angular momentum of a rigid object is always conserved if no external torques act on the object. Like the law of conservation of linear momentum, the law of conservation of angular momentum is a fundamental law of physics, equally valid for relativistic and quantum systems.

11.1 ROLLING MOTION OF A RIGID OBJECT

7.7 In this section we treat the motion of a rigid object rotating about a moving axis. In general, such motion is very complex. However, we can simplify matters by restricting our discussion to a homogeneous rigid object having a high degree of symmetry, such as a cylinder, sphere, or hoop. Furthermore, we assume that the object undergoes rolling motion along a flat surface. We shall see that if an object such as a cylinder rolls without slipping on the surface (we call this *pure rolling motion*), a simple relationship exists between its rotational and translational motions.

Suppose a cylinder is rolling on a straight path. As Figure 11.1 shows, the center of mass moves in a straight line, but a point on the rim moves in a more complex path called a *cycloid.* This means that the axis of rotation remains parallel to its initial orientation in space. Consider a uniform cylinder of radius R rolling without slipping on a horizontal surface (Fig. 11.2). As the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$ (see Eq. 10.1a). Therefore, the linear speed of the center of mass for pure rolling motion is given by

$$v_{\mathrm{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \qquad \textbf{(11.1)}$$

where ω is the angular velocity of the cylinder. Equation 11.1 holds whenever a cylinder or sphere rolls without slipping and is the **condition for pure rolling**

Figure 11.1 One light source at the center of a rolling cylinder and another at one point on the rim illustrate the different paths these two points take. The center moves in a straight line (green line), whereas the point on the rim moves in the path called a *cycloid* (red curve). *(Henry Leap and Jim Lehman)*

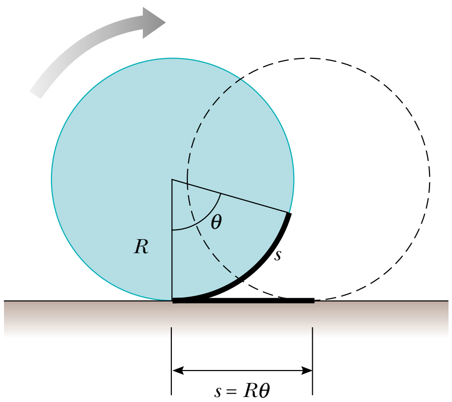


Figure 11.2 In pure rolling motion, as the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$.

motion. The magnitude of the linear acceleration of the center of mass for pure rolling motion is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha \tag{11.2}$$

where α is the angular acceleration of the cylinder.

The linear velocities of the center of mass and of various points on and within the cylinder are illustrated in Figure 11.3. A short time after the moment shown in the drawing, the rim point labeled P will have rotated from the six o'clock position to, say, the seven o'clock position, the point Q will have rotated from the ten o'clock position to the eleven o'clock position, and so on. Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point P. At any instant, the part of the rim that is at point P is at rest relative to the surface because slipping does not occur.

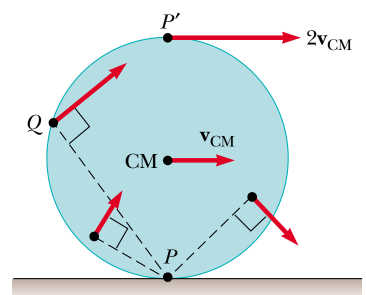


Figure 11.3 All points on a rolling object move in a direction perpendicular to an axis through the instantaneous point of contact P. In other words, all points rotate about P. The center of mass of the object moves with a velocity $\mathbf{v}_{CM}$, and the point P' moves with a velocity $2\mathbf{v}_{CM}$.

All points on the cylinder have the same angular speed. Therefore, because the distance from P' to P is twice the distance from P to the center of mass, P' has a speed $2v_{CM} = 2R\omega$. To see why this is so, let us model the rolling motion of the cylinder in Figure 11.4 as a combination of translational (linear) motion and rotational motion. For the pure translational motion shown in Figure 11.4a, imagine that the cylinder does not rotate, so that each point on it moves to the right with speed v_{CM}. For the pure rotational motion shown in Figure 11.4b, imagine that a rotation axis through the center of mass is stationary, so that each point on the cylinder has the same rotational speed ω. The combination of these two motions represents the rolling motion shown in Figure 11.4c. Note in Figure 11.4c that the top of the cylinder has linear speed $v_{CM} + R\omega = v_{CM} + v_{CM} = 2v_{CM}$, which is greater than the linear speed of any other point on the cylinder. As noted earlier, the center of mass moves with linear speed v_{CM} while the contact point between the surface and cylinder has a linear speed of zero.

7.2 We can express the total kinetic energy of the rolling cylinder as

$$K = \tfrac{1}{2}I_P\omega^2 \tag{11.3}$$

where I_P is the moment of inertia about a rotation axis through P. Applying the parallel-axis theorem, we can substitute $I_P = I_{CM} + MR^2$ into Equation 11.3 to obtain

$$K = \tfrac{1}{2}I_{CM}\omega^2 + \tfrac{1}{2}MR^2\omega^2$$

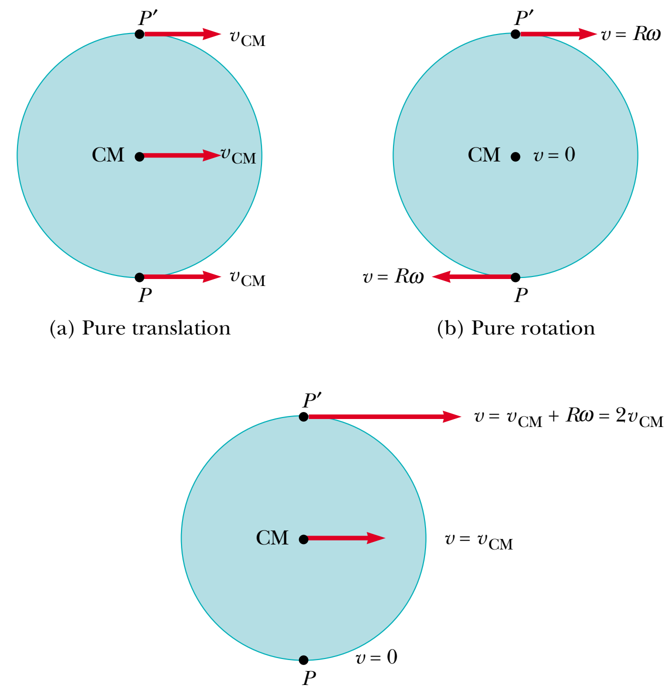


Figure 11.4 The motion of a rolling object can be modeled as a combination of pure translation and pure rotation.

or, because $v_{CM} = R\omega$,

Total kinetic energy of a rolling body

$$K = \tfrac{1}{2}I_{CM}\omega^2 + \tfrac{1}{2}Mv_{CM}^{\,2} \tag{11.4}$$

The term $\frac{1}{2}I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass, and the term $\frac{1}{2}Mv_{CM}^{\,2}$ represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that the **total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**

We can use energy methods to treat a class of problems concerning the rolling motion of a sphere down a rough incline. (The analysis that follows also applies to the rolling motion of a cylinder or hoop.) We assume that the sphere in Figure 11.5 rolls without slipping and is released from rest at the top of the incline. Note that accelerated rolling motion is possible only if a frictional force is present between the sphere and the incline to produce a net torque about the center of mass. Despite the presence of friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant. On the other hand, if the sphere were to slip, mechanical energy would be lost as motion progressed.

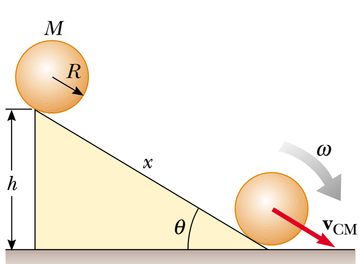


Figure 11.5 A sphere rolling down an incline. Mechanical energy is conserved if no slipping occurs.

Using the fact that $v_{CM} = R\omega$ for pure rolling motion, we can express Equation 11.4 as

$$K = \tfrac{1}{2}I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2}Mv_{CM}^{\,2}$$

$$K = \tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^{\,2} \tag{11.5}$$

By the time the sphere reaches the bottom of the incline, work equal to Mgh has been done on it by the gravitational field, where h is the height of the incline. Because the sphere starts from rest at the top, its kinetic energy at the bottom, given by Equation 11.5, must equal this work done. Therefore, the speed of the center of mass at the bottom can be obtained by equating these two quantities:

$$\frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^2 = Mgh$$

$$v_{CM} = \left(\frac{2gh}{1 + I_{CM}/MR^2}\right)^{1/2} \tag{11.6}$$

Quick Quiz 11.1

Imagine that you slide your textbook across a gymnasium floor with a certain initial speed. It quickly stops moving because of friction between it and the floor. Yet, if you were to start a basketball rolling with the same initial speed, it would probably keep rolling from one end of the gym to the other. Why does a basketball roll so far? Doesn't friction affect its motion?

EXAMPLE 11.1 Sphere Rolling Down an Incline

For the solid sphere shown in Figure 11.5, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.

Solution The sphere starts from the top of the incline with potential energy $U_g = Mgh$ and kinetic energy $K = 0$. As we have seen before, if it fell vertically from that height, it would have a linear speed of $\sqrt{2gh}$ at the moment before it hit the floor. After rolling down the incline, the linear speed of the center of mass must be less than this value because some of the initial potential energy is diverted into rotational kinetic energy rather than all being converted into translational kinetic energy. For a uniform solid sphere, $I_{CM} = \frac{2}{5}MR^2$ (see Table 10.2), and therefore Equation 11.6 gives

$$v_{CM} = \left(\frac{2gh}{1 + \frac{2/5MR^2}{MR^2}}\right)^{1/2} = \left(\frac{10}{7}gh\right)^{1/2}$$

which is less than $\sqrt{2gh}$.

To calculate the linear acceleration of the center of mass, we note that the vertical displacement is related to the displacement x along the incline through the relationship $h = x\sin\theta$. Hence, after squaring both sides, we can express the equation above as

$$v_{CM}^2 = \frac{10}{7}gx\sin\theta$$

Comparing this with the expression from kinematics, $v_{CM}^2 = 2a_{CM}x$ (see Eq. 2.12), we see that the acceleration of the center of mass is

$$a_{CM} = \frac{5}{7}g\sin\theta$$

These results are quite interesting in that both the speed and the acceleration of the center of mass are *independent* of the mass and the radius of the sphere! That is, **all homogeneous solid spheres experience the same speed and acceleration on a given incline.**

If we repeated the calculations for a hollow sphere, a solid cylinder, or a hoop, we would obtain similar results in which only the factor in front of $g\sin\theta$ would differ. The constant factors that appear in the expressions for v_{CM} and a_{CM} depend only on the moment of inertia about the center of mass for the specific body. In all cases, the acceleration of the center of mass is *less* than $g\sin\theta$, the value the acceleration would have if the incline were frictionless and no rolling occurred.

EXAMPLE 11.2 Another Look at the Rolling Sphere

In this example, let us use dynamic methods to verify the results of Example 11.1. The free-body diagram for the sphere is illustrated in Figure 11.6.

Solution Newton's second law applied to the center of mass gives

$$(1) \qquad \Sigma F_x = Mg\sin\theta - f = Ma_{CM}$$

$$\Sigma F_y = n - Mg\cos\theta = 0$$

where x is measured along the slanted surface of the incline.

Now let us write an expression for the torque acting on the sphere. A convenient axis to choose is the one that passes

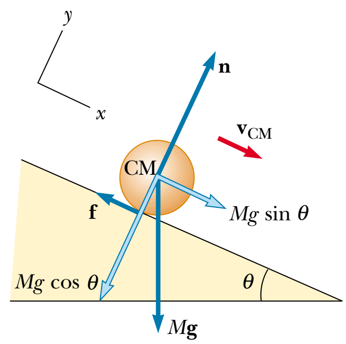


Figure 11.6 Free-body diagram for a solid sphere rolling down an incline.

through the center of the sphere and is perpendicular to the plane of the figure.[1] Because **n** and $M\mathbf{g}$ go through the center of mass, they have zero moment arm about this axis and thus do not contribute to the torque. However, the force of static friction produces a torque about this axis equal to fR in the clockwise direction; therefore, because τ is also in the clockwise direction,

$$\tau_{CM} = fR = I_{CM}\alpha$$

Because $I_{CM} = \frac{2}{5}MR^2$ and $\alpha = a_{CM}/R$, we obtain

$$(2) \qquad f = \frac{I_{CM}\alpha}{R} = \left(\frac{\frac{2}{5}MR^2}{R}\right)\frac{a_{CM}}{R} = \frac{2}{5}Ma_{CM}$$

Substituting Equation (2) into Equation (1) gives

$$a_{CM} = \boxed{\frac{5}{7}g \sin\theta}$$

which agrees with the result of Example 11.1.

Note that $\Sigma\mathbf{F} = m\mathbf{a}$ applies only if $\Sigma\mathbf{F}$ is the net external force exerted on the sphere and **a** is the acceleration of its center of mass. In the case of our sphere rolling down an incline, even though the frictional force does not change the total kinetic energy of the sphere, it does contribute to $\Sigma\mathbf{F}$ and thus decreases the acceleration of the center of mass. As a result, the final translational kinetic energy is less than it would be in the absence of friction. As mentioned in Example 11.1, some of the initial potential energy is converted to rotational kinetic energy.

QuickLab

Hold a basketball and a tennis ball side by side at the top of a ramp and release them at the same time. Which reaches the bottom first? Does the outcome depend on the angle of the ramp? What if the angle were 90° (that is, if the balls were in free fall)?

Quick Quiz 11.2

Which gets to the bottom first: a ball rolling without sliding down incline A or a box sliding down a frictionless incline B having the same dimensions as incline A?

11.2 THE VECTOR PRODUCT AND TORQUE

2.7

Consider a force **F** acting on a rigid body at the vector position **r** (Fig. 11.7). **The origin *O* is assumed to be in an inertial frame, so Newton's first law is valid in this case.** As we saw in Section 10.6, the *magnitude* of the torque due to this force relative to the origin is, by definition, $rF \sin\phi$, where ϕ is the angle between **r** and **F**. The axis about which **F** tends to produce rotation is perpendicular to the plane formed by **r** and **F**. If the force lies in the xy plane, as it does in Figure 11.7, the torque $\boldsymbol{\tau}$ is represented by a vector parallel to the z axis. The force in Figure 11.7 creates a torque that tends to rotate the body counterclockwise about the z axis; thus the direction of $\boldsymbol{\tau}$ is toward increasing z, and $\boldsymbol{\tau}$ is therefore in the positive z direction. If we reversed the direction of **F** in Figure 11.7, then $\boldsymbol{\tau}$ would be in the negative z direction.

The torque $\boldsymbol{\tau}$ involves the two vectors **r** and **F**, and its direction is perpendicular to the plane of **r** and **F**. We can establish a mathematical relationship between $\boldsymbol{\tau}$, **r**, and **F**, using a new mathematical operation called the **vector product,** or **cross product:**

Torque

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \qquad \textbf{(11.7)}$$

[1] Although a coordinate system whose origin is at the center of mass of a rolling object is not an inertial frame, the expression $\tau_{CM} = I\alpha$ still applies in the center-of-mass frame.

We now give a formal definition of the vector product. Given any two vectors **A** and **B**, the **vector product A × B** is defined as a third vector **C**, the magnitude of which is $AB \sin\theta$, where θ is the angle between **A** and **B**. That is, if **C** is given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{11.8}$$

then its magnitude is

$$C \equiv AB \sin\theta \tag{11.9}$$

The quantity $AB \sin\theta$ is equal to the area of the parallelogram formed by **A** and **B**, as shown in Figure 11.8. The *direction* of **C** is perpendicular to the plane formed by **A** and **B**, and the best way to determine this direction is to use the right-hand rule illustrated in Figure 11.8. The four fingers of the right hand are pointed along **A** and then "wrapped" into **B** through the angle θ. The direction of the erect right thumb is the direction of **A × B = C.** Because of the notation, **A × B** is often read "**A** cross **B**"; hence, the term *cross product.*

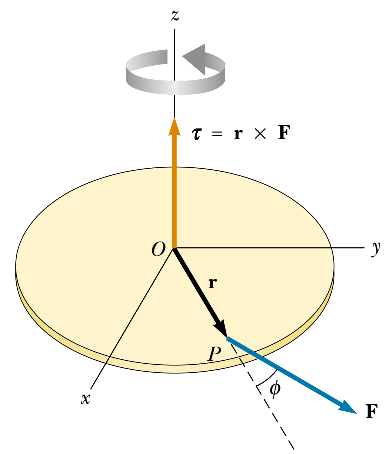


Figure 11.7 The torque vector $\boldsymbol{\tau}$ lies in a direction perpendicular to the plane formed by the position vector **r** and the applied force vector **F**.

Some properties of the vector product that follow from its definition are as follows:

Properties of the vector product

1. Unlike the scalar product, the vector product is *not* commutative. Instead, the order in which the two vectors are multiplied in a cross product is important:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{11.10}$$

   Therefore, if you change the order of the vectors in a cross product, you must change the sign. You could easily verify this relationship with the right-hand rule.
2. If **A** is parallel to **B** ($\theta = 0°$ or $180°$), then $\mathbf{A} \times \mathbf{B} = 0$; therefore, it follows that $\mathbf{A} \times \mathbf{A} = 0$.
3. If **A** is perpendicular to **B**, then $|\mathbf{A} \times \mathbf{B}| = AB$.
4. The vector product obeys the distributive law:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{11.11}$$

5. The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B} \tag{11.12}$$

   where it is important to preserve the multiplicative order of **A** and **B**, in view of Equation 11.10.

It is left as an exercise to show from Equations 11.9 and 11.10 and from the definition of unit vectors that the cross products of the rectangular unit vectors **i**,

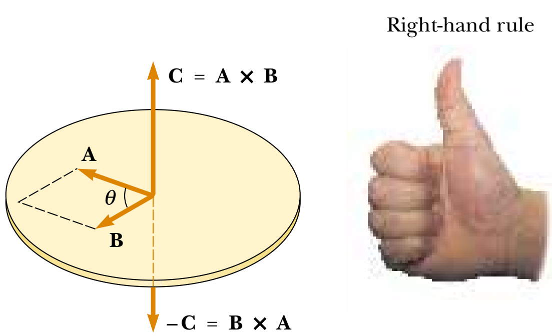


Figure 11.8 The vector product **A × B** is a third vector **C** having a magnitude $AB \sin\theta$ equal to the area of the parallelogram shown. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and this direction is determined by the right-hand rule.

j, and **k** obey the following rules:

Cross products of unit vectors

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \qquad \textbf{(11.13a)}$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k} \qquad \textbf{(11.13b)}$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \qquad \textbf{(11.13c)}$$

$$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j} \qquad \textbf{(11.13d)}$$

Signs are interchangeable in cross products. For example, $\mathbf{A} \times (-\mathbf{B}) = -\mathbf{A} \times \mathbf{B}$ and $\mathbf{i} \times (-\mathbf{j}) = -\mathbf{i} \times \mathbf{j}$.

The cross product of any two vectors **A** and **B** can be expressed in the following determinant form:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

Expanding these determinants gives the result

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \qquad \textbf{(11.14)}$$

EXAMPLE 11.3 The Cross Product

Two vectors lying in the *xy* plane are given by the equations $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. Find $\mathbf{A} \times \mathbf{B}$ and verify that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Solution Using Equations 11.13a through 11.13d, we obtain

$$\mathbf{A} \times \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) \times (-\mathbf{i} + 2\mathbf{j})$$

$$= 2\mathbf{i} \times 2\mathbf{j} + 3\mathbf{j} \times (-\mathbf{i}) = 4\mathbf{k} + 3\mathbf{k} = 7\mathbf{k}$$

(We have omitted the terms containing $\mathbf{i} \times \mathbf{i}$ and $\mathbf{j} \times \mathbf{j}$ because, as Equation 11.13a shows, they are equal to zero.)

We can show that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, since

$$\mathbf{B} \times \mathbf{A} = (-\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j})$$

$$= -\mathbf{i} \times 3\mathbf{j} + 2\mathbf{j} \times 2\mathbf{i} = -3\mathbf{k} - 4\mathbf{k} = -7\mathbf{k}$$

Therefore, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

As an alternative method for finding $\mathbf{A} \times \mathbf{B}$, we could use Equation 11.14, with $A_x = 2$, $A_y = 3$, $A_z = 0$ and $B_x = -1$, $B_y = 2$, $B_z = 0$:

$$\mathbf{A} \times \mathbf{B} = (0)\mathbf{i} - (0)\mathbf{j} + [(2)(2) - (3)(-1)]\mathbf{k} = 7\mathbf{k}$$

Exercise Use the results to this example and Equation 11.9 to find the angle between **A** and **B**.

Answer 60.3°

11.3 ANGULAR MOMENTUM OF A PARTICLE

7.8 Imagine a rigid pole sticking up through the ice on a frozen pond (Fig. 11.9). A skater glides rapidly toward the pole, aiming a little to the side so that she does not hit it. As she approaches a point beside the pole, she reaches out and grabs the pole, an action that whips her rapidly into a circular path around the pole. Just as the idea of linear momentum helps us analyze translational motion, a rotational analog—*angular momentum*—helps us describe this skater and other objects undergoing rotational motion.

To analyze the motion of the skater, we need to know her mass and her velocity, as well as her position relative to the pole. In more general terms, consider a particle of mass m located at the vector position **r** and moving with linear velocity **v** (Fig. 11.10).

The instantaneous angular momentum **L** of the particle relative to the origin O is defined as the cross product of the particle's instantaneous position vector **r** and its instantaneous linear momentum **p**:

Angular momentum of a particle

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{11.15}$$

The SI unit of angular momentum is kg·m^2/s. It is important to note that both the magnitude and the direction of **L** depend on the choice of origin. Following the right-hand rule, note that the direction of **L** is perpendicular to the plane formed by **r** and **p**. In Figure 11.10, **r** and **p** are in the xy plane, and so **L** points in the z direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of **L** is

$$L = mvr \sin \phi \tag{11.16}$$

where ϕ is the angle between **r** and **p**. It follows that L is zero when **r** is parallel to **p** ($\phi = 0$ or $180°$). In other words, when the linear velocity of the particle is along a line that passes through the origin, the particle has zero angular momentum with respect to the origin. On the other hand, if **r** is perpendicular to **p** ($\phi = 90°$), then $L = mvr$. At that instant, the particle moves exactly as if it were on the rim of a wheel rotating about the origin in a plane defined by **r** and **p**.

Figure 11.9 As the skater passes the pole, she grabs hold of it. This causes her to swing around the pole rapidly in a circular path.

Quick Quiz 11.3

Recall the skater described at the beginning of this section. What would be her angular momentum relative to the pole if she were skating directly toward it?

In describing linear motion, we found that the net force on a particle equals the time rate of change of its linear momentum, $\Sigma\mathbf{F} = d\mathbf{p}/dt$ (see Eq. 9.3). We now show that the net torque acting on a particle equals the time rate of change of its angular momentum. Let us start by writing the net torque on the particle in the form

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{11.17}$$

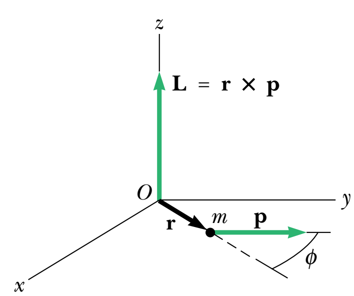


Figure 11.10 The angular momentum **L** of a particle of mass m and linear momentum **p** located at the vector position **r** is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of **L** depends on the origin about which it is measured and is a vector perpendicular to both **r** and **p**.

Now let us differentiate Equation 11.15 with respect to time, using the rule given by Equation 11.12:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

Remember, it is important to adhere to the order of terms because $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The last term on the right in the above equation is zero because $\mathbf{v} = d\mathbf{r}/dt$ is parallel to $\mathbf{p} = m\mathbf{v}$ (property 2 of the vector product). Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{11.18}$$

Comparing Equations 11.17 and 11.18, we see that

The net torque equals time rate of change of angular momentum

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{11.19}$$

which is the rotational analog of Newton's second law, $\Sigma \mathbf{F} = d\mathbf{p}/dt$. Note that torque causes the angular momentum $\mathbf{L}$ to change just as force causes linear momentum $\mathbf{p}$ to change. This rotational result, Equation 11.19, states that

the net torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

It is important to note that Equation 11.19 is valid only if $\Sigma \boldsymbol{\tau}$ and $\mathbf{L}$ are measured about the same origin. (Of course, the same origin must be used in calculating all of the torques.) Furthermore, **the expression is valid for any origin fixed in an inertial frame.**

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

where the vector sum is over all n particles in the system.

Because individual angular momenta can change with time, so can the total angular momentum. In fact, from Equations 11.18 and 11.19, we find that the time rate of change of the total angular momentum equals the vector sum of all torques acting on the system, both those associated with internal forces between particles and those associated with external forces. However, the net torque associated with all internal forces is zero. To understand this, recall that Newton's third law tells us that internal forces between particles of the system are equal in magnitude and opposite in direction. If we assume that these forces lie along the line of separation of each pair of particles, then the torque due to each action–reaction force pair is zero. That is, the moment arm d from O to the line of action of the forces is equal for both particles. In the summation, therefore, we see that the net internal torque vanishes. We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system, so that we have

$$\sum \boldsymbol{\tau}_{\text{ext}} = \sum_i \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt}\sum_i \mathbf{L}_i = \frac{d\mathbf{L}}{dt} \qquad \textbf{(11.20)}$$

That is,

the time rate of change of the total angular momentum of a system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

Note that Equation 11.20 is the rotational analog of Equation 9.38, $\Sigma \mathbf{F}_{\text{ext}} = d\mathbf{p}/dt$, for a system of particles.

EXAMPLE 11.4 Circular Motion

A particle moves in the xy plane in a circular path of radius r, as shown in Figure 11.11. (a) Find the magnitude and direction of its angular momentum relative to O when its linear velocity is $\mathbf{v}$.

Solution You might guess that because the linear momentum of the particle is always changing (in direction, not magnitude), the direction of the angular momentum must also change. In this example, however, this is not the case. The magnitude of $\mathbf{L}$ is given by

$$L = mvr \sin 90° = mvr \qquad \text{(for } \mathbf{r} \text{ perpendicular to } \mathbf{v}\text{)}$$

This value of L is constant because all three factors on the right are constant. The direction of $\mathbf{L}$ also is constant, even though the direction of $\mathbf{p} = m\mathbf{v}$ keeps changing. You can visualize this by sliding the vector $\mathbf{v}$ in Figure 11.11 parallel to itself until its tail meets the tail of $\mathbf{r}$ and by then applying the right-hand rule. (You can use $\mathbf{v}$ to determine the direction of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ because the direction of $\mathbf{p}$ is the same as the direction of $\mathbf{v}$.) Line up your fingers so that they point along $\mathbf{r}$ and wrap your fingers into the vector $\mathbf{v}$. Your thumb points upward and away from the page; this is the direction of $\mathbf{L}$. Hence, we can write the vector expression $\mathbf{L} = (mvr)\mathbf{k}$. If the particle were to move clockwise, $\mathbf{L}$ would point downward and into the page.

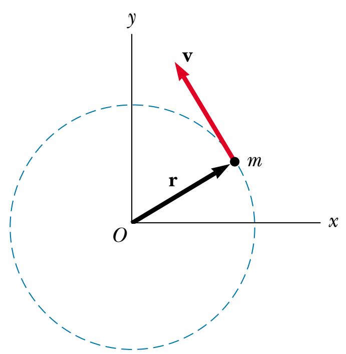


Figure 11.11 A particle moving in a circle of radius r has an angular momentum about O that has magnitude mvr. The vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ points *out* of the diagram.

(b) Find the magnitude and direction of $\mathbf{L}$ in terms of the particle's angular speed ω.

Solution Because $v = r\omega$ for a particle rotating in a circle, we can express L as

$$L = mvr = mr^2\omega = I\omega$$

where I is the moment of inertia of the particle about the z axis through O. Because the rotation is counterclockwise, the direction of $\boldsymbol{\omega}$ is along the z axis (see Section 10.1). The direction of $\mathbf{L}$ is the same as that of $\boldsymbol{\omega}$, and so we can write the angular momentum as $\mathbf{L} = I\boldsymbol{\omega} = I\omega\mathbf{k}$.

Exercise A car of mass 1 500 kg moves with a linear speed of 40 m/s on a circular race track of radius 50 m. What is the magnitude of its angular momentum relative to the center of the track?

Answer $3.0 \times 10^6\ \text{kg}\cdot\text{m}^2/\text{s}$

11.4 ANGULAR MOMENTUM OF A ROTATING RIGID OBJECT

Consider a rigid object rotating about a fixed axis that coincides with the z axis of a coordinate system, as shown in Figure 11.12. Let us determine the angular momentum of this object. Each particle of the object rotates in the xy plane about the z axis with an angular speed ω. The magnitude of the angular momentum of a particle of mass m_i about the origin O is $m_i v_i r_i$. Because $v_i = r_i\omega$, we can express the magnitude of the angular momentum of this particle as

$$L_i = m_i r_i^2 \omega$$

The vector $\mathbf{L}_i$ is directed along the z axis, as is the vector $\boldsymbol{\omega}$.

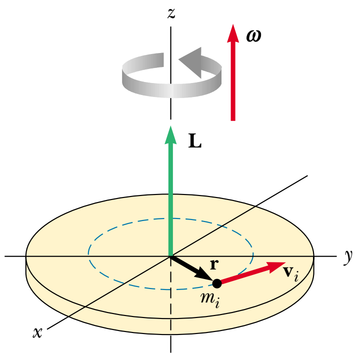


Figure 11.12 When a rigid body rotates about an axis, the angular momentum **L** is in the same direction as the angular velocity **ω**, according to the expression **L** = *I***ω**.

We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of L_i over all particles:

$$L_z = \sum_i m_i {r_i}^2 \omega = \left(\sum_i m_i {r_i}^2\right)\omega$$

$$L_z = I\omega \qquad \textbf{(11.21)}$$

where I is the moment of inertia of the object about the z axis.

Now let us differentiate Equation 11.21 with respect to time, noting that I is constant for a rigid body:

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \qquad \textbf{(11.22)}$$

where α is the angular acceleration relative to the axis of rotation. Because dL_z/dt is equal to the net external torque (see Eq. 11.20), we can express Equation 11.22 as

$$\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I\alpha \qquad \textbf{(11.23)}$$

That is, the net external torque acting on a rigid object rotating about a fixed axis equals the moment of inertia about the rotation axis multiplied by the object's angular acceleration relative to that axis.

Equation 11.23 also is valid for a rigid object rotating about a moving axis provided the moving axis (1) passes through the center of mass and (2) is a symmetry axis.

You should note that if a symmetrical object rotates about a fixed axis passing through its center of mass, you can write Equation 11.21 in vector form as **L** = *I***ω**, where **L** is the total angular momentum of the object measured with respect to the axis of rotation. Furthermore, the expression is valid for any object, regardless of its symmetry, if **L** stands for the component of angular momentum along the axis of rotation.[2]

EXAMPLE 11.5 Bowling Ball

Estimate the magnitude of the angular momentum of a bowling ball spinning at 10 rev/s, as shown in Figure 11.13.

Solution We start by making some estimates of the relevant physical parameters and model the ball as a uniform solid sphere. A typical bowling ball might have a mass of 6 kg and a radius of 12 cm. The moment of inertia of a solid sphere about an axis through its center is, from Table 10.2,

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5}(6\text{ kg})(0.12\text{ m})^2 = 0.035\text{ kg}\cdot\text{m}^2$$

Therefore, the magnitude of the angular momentum is

[2] In general, the expression **L** = *I***ω** is not always valid. If a rigid object rotates about an arbitrary axis, **L** and **ω** may point in different directions. In this case, the moment of inertia cannot be treated as a scalar. Strictly speaking, **L** = *I***ω** applies only to rigid objects of any shape that rotate about one of three mutually perpendicular axes (called *principal axes*) through the center of mass. This is discussed in more advanced texts on mechanics.

$$L = I\omega = (0.035 \text{ kg}\cdot\text{m}^2)(10 \text{ rev/s})(2\pi \text{ rad/rev})$$
$$= 2.2 \text{ kg}\cdot\text{m}^2/\text{s}$$

Because of the roughness of our estimates, we probably want to keep only one significant figure, and so $L \approx 2 \text{ kg}\cdot\text{m}^2/\text{s}$.

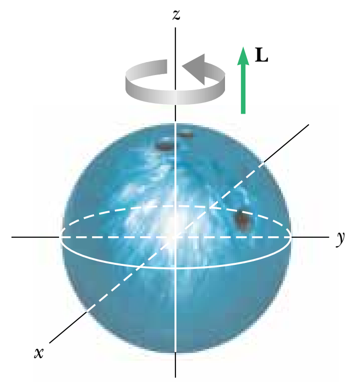


Figure 11.13 A bowling ball that rotates about the z axis in the direction shown has an angular momentum **L** in the positive z direction. If the direction of rotation is reversed, **L** points in the negative z direction.

EXAMPLE 11.6 Rotating Rod

A rigid rod of mass M and length ℓ is pivoted without friction at its center (Fig. 11.14). Two particles of masses m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed ω. (a) Find an expression for the magnitude of the angular momentum of the system.

Solution This is different from the last example in that we now must account for the motion of more than one object. The moment of inertia of the system equals the sum of the moments of inertia of the three components: the rod and the two particles. Referring to Table 10.2 to obtain the expression for the moment of inertia of the rod, and using the expression $I = mr^2$ for each particle, we find that the total moment of inertia about the z axis through O is

$$I = \frac{1}{12}M\ell^2 + m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2$$
$$= \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)$$

Therefore, the magnitude of the angular momentum is

$$L = I\omega = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega$$

(b) Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizontal.

Solution If the masses of the two particles are equal, then the system has no angular acceleration because the net torque on the system is zero when $m_1 = m_2$. If the initial angle θ is exactly $\pi/2$ or $-\pi/2$ (vertical position), then the rod will be in equilibrium. To find the angular acceleration of the system at any angle θ, we first calculate the net torque on the system and then use $\Sigma\tau_{\text{ext}} = I\alpha$ to obtain an expression for α.

The torque due to the force m_1g about the pivot is

$$\tau_1 = m_1g\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_1 \text{ out of page})$$

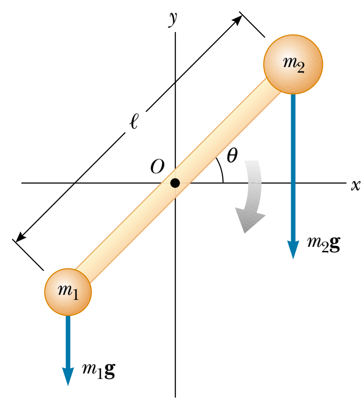


Figure 11.14 Because gravitational forces act on the rotating rod, there is in general a net nonzero torque about O when $m_1 \neq m_2$. This net torque produces an angular acceleration given by $\alpha = \Sigma\tau_{\text{ext}}/I$.

The torque due to the force m_2g about the pivot is

$$\tau_2 = -m_2g\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_2 \text{ into page})$$

Hence, the net torque exerted on the system about O is

$$\sum\tau_{\text{ext}} = \tau_1 + \tau_2 = \tfrac{1}{2}(m_1 - m_2)g\ell\cos\theta$$

The direction of $\Sigma\boldsymbol{\tau}_{\text{ext}}$ is out of the page if $m_1 > m_2$ and is into the page if $m_2 > m_1$.

To find α, we use $\Sigma\tau_{\text{ext}} = I\alpha$, where I was obtained in part (a):

$$\alpha = \frac{\Sigma\tau_{\text{ext}}}{I} = \frac{2(m_1 - m_2)g\cos\theta}{\ell(M/3 + m_1 + m_2)}$$

Note that α is zero when θ is $\pi/2$ or $-\pi/2$ (vertical position) and is a maximum when θ is 0 or π (horizontal position).

Exercise If $m_2 > m_1$, at what value of θ is ω a maximum?

Answer $\theta = -\pi/2$.

EXAMPLE 11.7 Two Connected Masses

A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure 11.15. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

Solution We need to determine the angular momentum of the system, which consists of the two objects and the pulley. Let us calculate the angular momentum about an axis that coincides with the axle of the pulley.

At the instant the sphere and block have a common speed v, the angular momentum of the sphere is m_1vR, and that of the block is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Hence, the total angular momentum of the system is

$$(1) \qquad L = m_1vR + m_2vR + I\frac{v}{R}$$

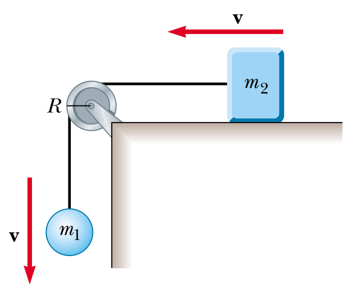


Figure 11.15

Now let us evaluate the total external torque acting on the system about the pulley axle. Because it has a moment arm of zero, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block is balanced by the force of gravity $m_2\mathbf{g}$, and so these forces do not contribute to the torque. The force of gravity $m_1\mathbf{g}$ acting on the sphere produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. (Note that in this situation, the tension is *not* equal to m_1g.) This is the total external torque about the pulley axle; that is, $\Sigma\tau_{ext} = m_1gR$. Using this result, together with Equation (1) and Equation 11.23, we find

$$\sum \tau_{ext} = \frac{dL}{dt}$$

$$m_1gR = \frac{d}{dt}\left[(m_1 + m_2)Rv + I\frac{v}{R}\right]$$

$$(2) \qquad m_1gR = (m_1 + m_2)R\frac{dv}{dt} + \frac{I}{R}\frac{dv}{dt}$$

Because $dv/dt = a$, we can solve this for a to obtain

$$a = \frac{m_1g}{(m_1 + m_2) + I/R^2}$$

You may wonder why we did not include the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration, and we analyzed the system as a whole. Only external torques contribute to the change in the system's angular momentum.

11.5 CONSERVATION OF ANGULAR MOMENTUM

7.9 In Chapter 9 we found that the total linear momentum of a system of particles remains constant when the resultant external force acting on the system is zero. We have an analogous conservation law in rotational motion:

Conservation of angular momentum

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

This follows directly from Equation 11.20, which indicates that if

$$\sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = 0 \qquad \textbf{(11.24)}$$

then

$$\mathbf{L} = \text{constant} \qquad \textbf{(11.25)}$$

For a system of particles, we write this conservation law as $\Sigma\,\mathbf{L}_n = \text{constant}$, where the index n denotes the nth particle in the system.

If the mass of an object undergoes redistribution in some way, then the object's moment of inertia changes; hence, its angular speed must change because $L = I\omega$. In this case we express the conservation of angular momentum in the form

$$\mathbf{L}_i = \mathbf{L}_f = \text{constant} \quad \textbf{(11.26)}$$

If the system is an object rotating about a *fixed* axis, such as the z axis, we can write $L_z = I\omega$, where L_z is the component of $\mathbf{L}$ along the axis of rotation and I is the moment of inertia about this axis. In this case, we can express the conservation of angular momentum as

$$I_i\omega_i = I_f\omega_f = \text{constant} \quad \textbf{(11.27)}$$

This expression is valid both for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system as long as that axis remains parallel to itself. We require only that the net external torque be zero.

Although we do not prove it here, there is an important theorem concerning the angular momentum of an object relative to the object's center of mass:

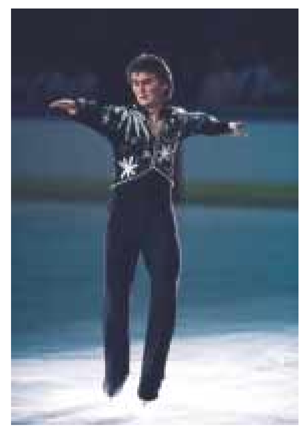
The resultant torque acting on an object about an axis through the center of mass equals the time rate of change of angular momentum regardless of the motion of the center of mass.

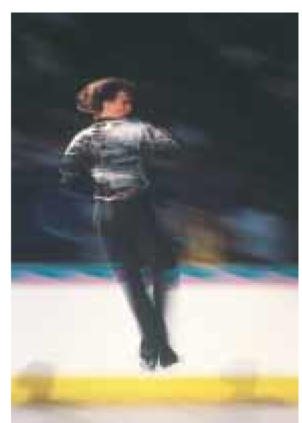
This theorem applies even if the center of mass is accelerating, provided $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

In Equation 11.26 we have a third conservation law to add to our list. We can now state that the energy, linear momentum, and angular momentum of an isolated system all remain constant:

$$\left.\begin{aligned} K_i + U_i &= K_f + U_f \\ \mathbf{p}_i &= \mathbf{p}_f \\ \mathbf{L}_i &= \mathbf{L}_f \end{aligned}\right\} \quad \text{For an isolated system}$$

There are many examples that demonstrate conservation of angular momentum. You may have observed a figure skater spinning in the finale of a program. The angular speed of the skater increases when the skater pulls his hands and feet close to his body, thereby decreasing I. Neglecting friction between skates and ice, no external torques act on the skater. The change in angular speed is due to the fact that, because angular momentum is conserved, the product $I\omega$ remains constant, and a decrease in the moment of inertia of the skater causes an increase in the angular speed. Similarly, when divers or acrobats wish to make several somersaults, they pull their hands and feet close to their bodies to rotate at a higher rate. In these cases, the external force due to gravity acts through the center of mass and hence exerts no torque about this point. Therefore, the angular momentum about the center of mass must be conserved—that is, $I_i\omega_i = I_f\omega_f$. For example, when divers wish to double their angular speed, they must reduce their moment of inertia to one-half its initial value.

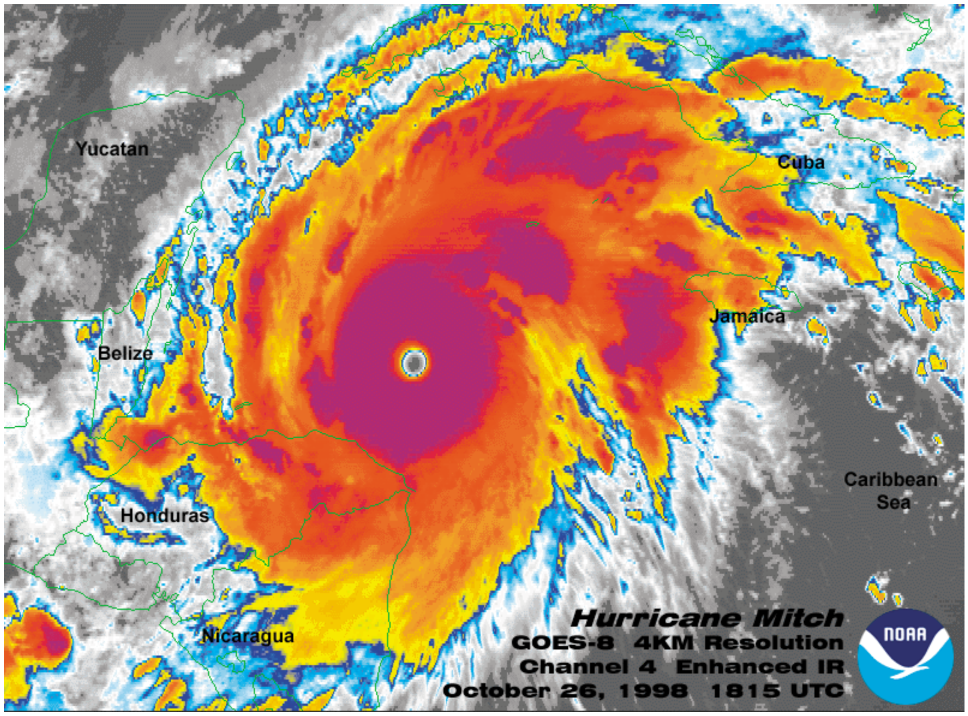




Angular momentum is conserved as figure skater Todd Eldredge pulls his arms toward his body. *(© 1998 David Madison)*

Quick Quiz 11.4

A particle moves in a straight line, and you are told that the net torque acting on it is zero about some unspecified point. Decide whether the following statements are true or false: (a) The net force on the particle must be zero. (b) The particle's velocity must be constant.



A color-enhanced, infrared image of Hurricane Mitch, which devastated large areas of Honduras and Nicaragua in October 1998. The spiral, nonrigid mass of air undergoes rotation and has angular momentum. *(Courtesy of NOAA)*

EXAMPLE 11.8 Formation of a Neutron Star

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0 km. Determine the period of rotation of the neutron star.

Solution The same physics that makes a skater spin faster with his arms pulled in describes the motion of the neutron star. Let us assume that during the collapse of the stellar core, (1) no torque acts on it, (2) it remains spherical, and (3) its mass remains constant. Also, let us use the symbol T for the period, with T_i being the initial period of the star and T_f being the period of the neutron star. The period is the length of time a point on the star's equator takes to make one complete circle around the axis of rotation. The angular speed of a star is given by $\omega = 2\pi/T$. Therefore, because I is proportional to r^2, Equation 11.27 gives

$$T_f = T_i\left(\frac{r_f}{r_i}\right)^2 = (30 \text{ days})\left(\frac{3.0 \text{ km}}{1.0 \times 10^4 \text{ km}}\right)^2$$

$$= 2.7 \times 10^{-6} \text{ days} = \boxed{0.23 \text{ s}}$$

Thus, the neutron star rotates about four times each second; this result is approximately the same as that for a spinning figure skater.

EXAMPLE 11.9 The Merry-Go-Round

A horizontal platform in the shape of a circular disk rotates in a horizontal plane about a frictionless vertical axle (Fig. 11.16). The platform has a mass $M = 100$ kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when he has reached a point $r = 0.50$ m from the center?

Solution The speed change here is similar to the increase in angular speed of the spinning skater when he pulls his arms inward. Let us denote the moment of inertia of the platform as I_p and that of the student as I_s. Treating the student as a point mass, we can write the initial moment of inertia I_i of the system (student plus platform) about the axis of rotation:

$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

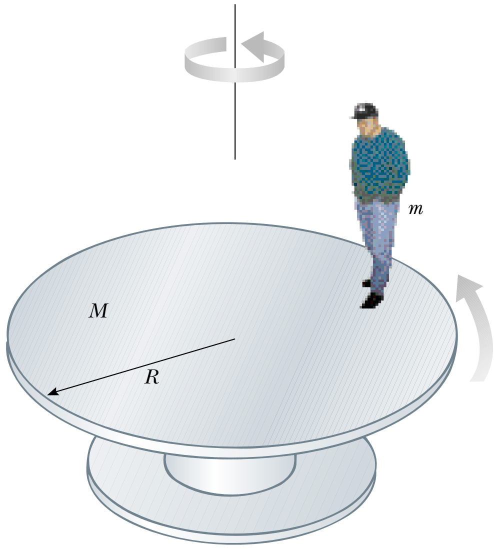


Figure 11.16 As the student walks toward the center of the rotating platform, the angular speed of the system increases because the angular momentum must remain constant.

When the student has walked to the position $r < R$, the moment of inertia of the system reduces to

$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

Note that we still use the greater radius R when calculating I_{pf} because the radius of the platform has not changed. Because no external torques act on the system about the axis of rotation, we can apply the law of conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f$$

$$\left(\tfrac{1}{2}MR^2 + mR^2\right)\omega_i = \left(\tfrac{1}{2}MR^2 + mr^2\right)\omega_f$$

$$\omega_f = \left(\frac{\tfrac{1}{2}MR^2 + mR^2}{\tfrac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$\omega_f = \left(\frac{200 + 240}{200 + 15}\right)(2.0 \text{ rad/s}) = \boxed{4.1 \text{ rad/s}}$$

As expected, the angular speed has increased.

Exercise Calculate the initial and final rotational energies of the system.

Answer $K_i = 880$ J; $K_f = 1.8 \times 10^3$ J.

Quick Quiz 11.5

Note that the rotational energy of the system described in Example 11.9 increases. What accounts for this increase in energy?

EXAMPLE 11.10 The Spinning Bicycle Wheel

In a favorite classroom demonstration, a student holds the axle of a spinning bicycle wheel while seated on a stool that is free to rotate (Fig. 11.17). The student and stool are initially at rest while the wheel is spinning in a horizontal plane with an initial angular momentum $\mathbf{L}_i$ that points upward. When the wheel is inverted about its center by 180°, the student and stool start rotating. In terms of $\mathbf{L}_i$, what are the magnitude and the direction of $\mathbf{L}$ for the student plus stool?

Figure 11.17 The wheel is initially spinning when the student is at rest. What happens when the wheel is inverted?

Solution The system consists of the student, the wheel, and the stool. Initially, the total angular momentum of the system $\mathbf{L}_i$ comes entirely from the spinning wheel. As the wheel is inverted, the student applies a torque to the wheel, but this torque is internal to the system. No external torque is acting on the system about the vertical axis. Therefore, the angular momentum of the system is conserved. Initially, we have

$$\mathbf{L}_{\text{system}} = \mathbf{L}_i = \mathbf{L}_{\text{wheel}} \qquad \text{(upward)}$$

After the wheel is inverted, we have $\mathbf{L}_{\text{inverted wheel}} = -\mathbf{L}_i$. For angular momentum to be conserved, some other part of the system has to start rotating so that the total angular momentum remains the initial angular momentum $\mathbf{L}_i$. That other part of the system is the student plus the stool she is sitting on. So, we can now state that

$$\mathbf{L}_f = \mathbf{L}_i = \mathbf{L}_{\text{student+stool}} - \mathbf{L}_i$$

$$\mathbf{L}_{\text{student+stool}} = \boxed{2\mathbf{L}_i}$$

EXAMPLE 11.11 Disk and Stick

A 2.0-kg disk traveling at 3.0 m/s strikes a 1.0-kg stick that is lying flat on nearly frictionless ice, as shown in Figure 11.18. Assume that the collision is elastic. Find the translational speed of the disk, the translational speed of the stick, and the rotational speed of the stick after the collision. The moment of inertia of the stick about its center of mass is $1.33\ \text{kg}\cdot\text{m}^2$.

Solution Because the disk and stick form an isolated system, we can assume that total energy, linear momentum, and angular momentum are all conserved. We have three unknowns, and so we need three equations to solve simultaneously. The first comes from the law of the conservation of linear momentum:

$$p_i = p_f$$

$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$(2.0\ \text{kg})(3.0\ \text{m/s}) = (2.0\ \text{kg})v_{df} + (1.0\ \text{kg})v_s$$

$$(1) \qquad 6.0\ \text{kg}\cdot\text{m/s} - (2.0\ \text{kg})v_{df} = (1.0\ \text{kg})v_s$$

Now we apply the law of conservation of angular momentum, using the initial position of the center of the stick as our reference point. We know that the component of angular momentum of the disk along the axis perpendicular to the plane of the ice is negative (the right-hand rule shows that $\mathbf{L}_d$ points into the ice).

$$L_i = L_f$$

$$-rm_d v_{di} = -rm_d v_{df} + I\omega$$

$$-(2.0\ \text{m})(2.0\ \text{kg})(3.0\ \text{m/s}) = -(2.0\ \text{m})(2.0\ \text{kg})v_{df} + (1.33\ \text{kg}\cdot\text{m}^2)\omega$$

$$-12\ \text{kg}\cdot\text{m}^2/\text{s} = -(4.0\ \text{kg}\cdot\text{m})v_{df} + (1.33\ \text{kg}\cdot\text{m}^2)\omega$$

$$(2) \qquad -9.0\ \text{rad/s} + (3.0\ \text{rad/m})v_{df} = \omega$$

We used the fact that radians are dimensionless to ensure consistent units for each term.

Finally, the elastic nature of the collision reminds us that kinetic energy is conserved; in this case, the kinetic energy consists of translational and rotational forms:

$$K_i = K_f$$

$$\tfrac{1}{2}m_d v_{di}^2 = \tfrac{1}{2}m_d v_{df}^2 + \tfrac{1}{2}m_s v_s^2 + \tfrac{1}{2}I\omega^2$$

$$\tfrac{1}{2}(2.0\ \text{kg})(3.0\ \text{m/s})^2 = \tfrac{1}{2}(2.0\ \text{kg})v_{df}^2 + \tfrac{1}{2}(1.0\ \text{kg})v_s^2 + \tfrac{1}{2}(1.33\ \text{kg}\cdot\text{m}^2/\text{s})\omega^2$$

$$(3) \qquad 54\ \text{m}^2/\text{s}^2 = 6.0v_{df}^2 + 3.0v_s^2 + (4.0\ \text{m}^2)\omega^2$$

In solving Equations (1), (2), and (3) simultaneously, we find that $v_{df} = 2.3$ m/s, $v_s = 1.3$ m/s, and $\omega = -2.0$ rad/s. These values seem reasonable. The disk is moving more slowly than it was before the collision, and the stick has a small translational speed. Table 11.1 summarizes the initial and final values of variables for the disk and the stick and verifies the conservation of linear momentum, angular momentum, and kinetic energy.

Exercise Verify the values in Table 11.1.

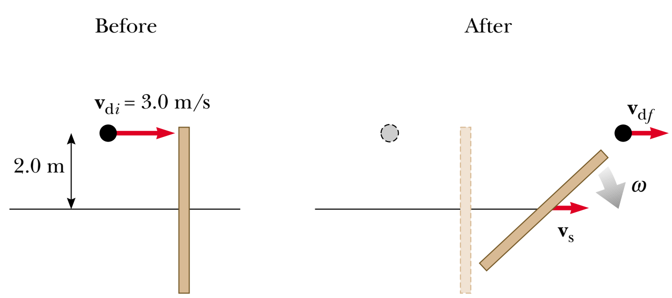


Figure 11.18 Overhead view of a disk striking a stick in an elastic collision, which causes the stick to rotate.

TABLE 11.1 Comparison of Values in Example 11.11 Before and After the Collision[a]

	v (m/s)	ω (rad/s)	p (kg·m/s)	L (kg·m²/s)	K_{trans} (J)	K_{rot} (J)
Before						
Disk	3.0	—	6.0	−12	9.0	—
Stick	0	0	0	0	0	0
Total	—	—	6.0	−12	9.0	0
After						
Disk	2.3	—	4.7	−9.3	5.4	—
Stick	1.3	−2.0	1.3	−2.7	0.9	2.7
Total	—	—	6.0	−12	6.3	2.7

[a] Notice that linear momentum, angular momentum, and total kinetic energy are conserved.

Optional Section

11.6 THE MOTION OF GYROSCOPES AND TOPS

A very unusual and fascinating type of motion you probably have observed is that of a top spinning about its axis of symmetry, as shown in Figure 11.19a. If the top spins very rapidly, the axis rotates about the z axis, sweeping out a cone (see Fig. 11.19b). The motion of the axis about the vertical—known as **precessional motion**—is usually slow relative to the spin motion of the top.

Precessional motion

It is quite natural to wonder why the top does not fall over. Because the center of mass is not directly above the pivot point O, a net torque is clearly acting on the top about O—a torque resulting from the force of gravity $M\mathbf{g}$. The top would certainly fall over if it were not spinning. Because it is spinning, however, it has an angular momentum $\mathbf{L}$ directed along its symmetry axis. As we shall show, the motion of this symmetry axis about the z axis (the precessional motion) occurs because the torque produces a change in the *direction* of the symmetry axis. This is an excellent example of the importance of the directional nature of angular momentum.

The two forces acting on the top are the downward force of gravity $M\mathbf{g}$ and the normal force $\mathbf{n}$ acting upward at the pivot point O. The normal force produces no torque about the pivot because its moment arm through that point is zero. However, the force of gravity produces a torque $\boldsymbol{\tau} = \mathbf{r} \times M\mathbf{g}$ about O, where the direction of $\boldsymbol{\tau}$ is perpendicular to the plane formed by $\mathbf{r}$ and $M\mathbf{g}$. By necessity, the vector $\boldsymbol{\tau}$ lies in a horizontal xy plane perpendicular to the angular momentum vector. The net torque and angular momentum of the top are related through Equation 11.19:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

From this expression, we see that the nonzero torque produces a change in angular momentum $d\mathbf{L}$—a change that is in the same direction as $\boldsymbol{\tau}$. Therefore, like the torque vector, $d\mathbf{L}$ must also be at right angles to $\mathbf{L}$. Figure 11.19b illustrates the resulting precessional motion of the symmetry axis of the top. In a time Δt, the change in angular momentum is $\Delta\mathbf{L} = \mathbf{L}_f - \mathbf{L}_i = \boldsymbol{\tau}\,\Delta t$. Because $\Delta\mathbf{L}$ is perpendicular to $\mathbf{L}$, the magnitude of $\mathbf{L}$ does not change ($|\mathbf{L}_i| = |\mathbf{L}_f|$). Rather, what is changing is the *direction* of $\mathbf{L}$. Because the change in angular momentum $\Delta\mathbf{L}$ is in the direction of $\boldsymbol{\tau}$, which lies in the xy plane, the top undergoes precessional motion.

The essential features of precessional motion can be illustrated by considering the simple gyroscope shown in Figure 11.20a. This device consists of a wheel free to spin about an axle that is pivoted at a distance h from the center of mass of the wheel. When given an angular velocity $\boldsymbol{\omega}$ about the axle, the wheel has an angular momentum $\mathbf{L} = I\boldsymbol{\omega}$ directed along the axle as shown. Let us consider the torque acting on the wheel about the pivot O. Again, the force $\mathbf{n}$ exerted by the support on the axle produces no torque about O, and the force of gravity $M\mathbf{g}$ produces a torque of magnitude Mgh about O, where the axle is perpendicular to the support. The direction of this torque is perpendicular to the axle (and perpendicular to $\mathbf{L}$), as shown in Figure 11.20a. This torque causes the angular momentum to change in the direction perpendicular to the axle. Hence, the axle moves in the direction of the torque—that is, in the horizontal plane.

To simplify the description of the system, we must make an assumption: The total angular momentum of the precessing wheel is the sum of the angular momentum $I\boldsymbol{\omega}$ due to the spinning and the angular momentum due to the motion of

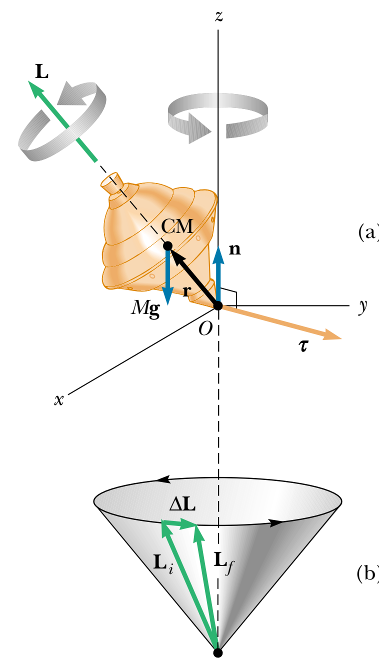


Figure 11.19 Precessional motion of a top spinning about its symmetry axis. (a) The only external forces acting on the top are the normal force $\mathbf{n}$ and the force of gravity $M\mathbf{g}$. The direction of the angular momentum $\mathbf{L}$ is along the axis of symmetry. The right-hand rule indicates that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times M\mathbf{g}$ is in the xy plane. (b). The direction of $\Delta\mathbf{L}$ is parallel to that of $\boldsymbol{\tau}$ in part (a). The fact that $\mathbf{L}_f = \Delta\mathbf{L} + \mathbf{L}_i$ indicates that the top precesses about the z axis.

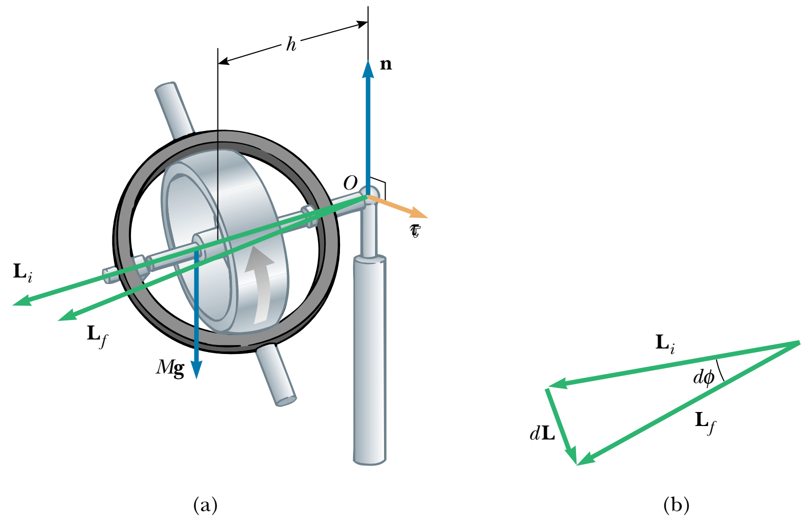
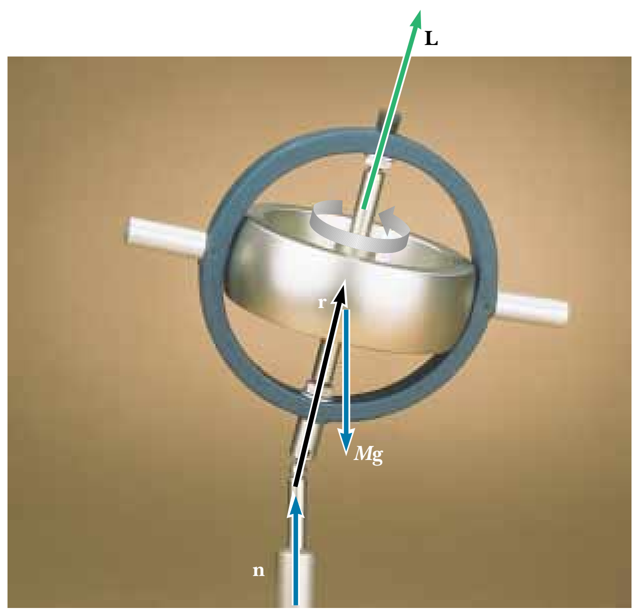


Figure 11.20 (a) The motion of a simple gyroscope pivoted a distance h from its center of mass. The force of gravity $M\mathbf{g}$ produces a torque about the pivot, and this torque is perpendicular to the axle. (b) This torque results in a change in angular momentum $d\mathbf{L}$ in a direction perpendicular to the axle. The axle sweeps out an angle $d\phi$ in a time dt.



This toy gyroscope undergoes precessional motion about the vertical axis as it spins about its axis of symmetry. The only forces acting on it are the force of gravity $M\mathbf{g}$ and the upward force of the pivot $\mathbf{n}$. The direction of its angular momentum $\mathbf{L}$ is along the axis of symmetry. The torque and $\Delta\mathbf{L}$ are directed into the page. *(Courtesy of Central Scientific Company)*

the center of mass about the pivot. In our treatment, we shall neglect the contribution from the center-of-mass motion and take the total angular momentum to be just $I\boldsymbol{\omega}$. In practice, this is a good approximation if $\boldsymbol{\omega}$ is made very large.

In a time dt, the torque due to the gravitational force changes the angular momentum of the system by $d\mathbf{L} = \boldsymbol{\tau}\, dt$. When added vectorially to the original total angular momentum $I\boldsymbol{\omega}$, this additional angular momentum causes a shift in the direction of the total angular momentum.

The vector diagram in Figure 11.20b shows that in the time dt, the angular momentum vector rotates through an angle $d\phi$, which is also the angle through which the axle rotates. From the vector triangle formed by the vectors $\mathbf{L}_i$, $\mathbf{L}_f$, and $d\mathbf{L}$, we see that

$$\sin(d\phi) \approx d\phi = \frac{dL}{L} = \frac{(Mgh)\,dt}{L}$$

where we have used the fact that, for small values of any angle θ, $\sin\theta \approx \theta$. Dividing through by dt and using the relationship $L = I\omega$, we find that the rate at which the axle rotates about the vertical axis is

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgh}{I\omega} \tag{11.28}$$

The angular speed ω_p is called the **precessional frequency.** This result is valid only when $\omega_p \ll \omega$. Otherwise, a much more complicated motion is involved. As you can see from Equation 11.28, the condition $\omega_p \ll \omega$ is met when $I\omega$ is great compared with Mgh. Furthermore, note that the precessional frequency decreases as ω increases—that is, as the wheel spins faster about its axis of symmetry.

Precessional frequency

Quick Quiz 11.6

How much work is done by the force of gravity when a top precesses through one complete circle?

Optional Section

11.7 ANGULAR MOMENTUM AS A FUNDAMENTAL QUANTITY

We have seen that the concept of angular momentum is very useful for describing the motion of macroscopic systems. However, the concept also is valid on a submicroscopic scale and has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it was found that the angular momentum of a system is a fundamental quantity. The word *fundamental* in this context implies that angular momentum is an intrinsic property of atoms, molecules, and their constituents, a property that is a part of their very nature.

To explain the results of a variety of experiments on atomic and molecular systems, we rely on the fact that the angular momentum has discrete values. These discrete values are multiples of the fundamental unit of angular momentum $\hbar = h/2\pi$, where h is called Planck's constant:

$$\text{Fundamental unit of angular momentum} = \hbar = 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Let us accept this postulate without proof for the time being and show how it can be used to estimate the angular speed of a diatomic molecule. Consider the O_2 molecule as a rigid rotor, that is, two atoms separated by a fixed distance d and rotating about the center of mass (Fig. 11.21). Equating the angular momentum to the fundamental unit $\hbar$, we can estimate the lowest angular speed:

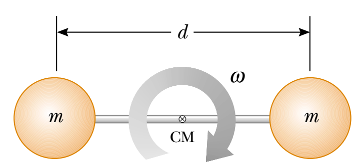


Figure 11.21 The rigid-rotor model of a diatomic molecule. The rotation occurs about the center of mass in the plane of the page.

$$I_{CM}\omega \approx \hbar \qquad \text{or} \qquad \omega \approx \frac{\hbar}{I_{CM}}$$

In Example 10.3, we found that the moment of inertia of the O_2 molecule about this axis of rotation is $1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2$. Therefore,

$$\omega \approx \frac{\hbar}{I_{\text{CM}}} = \frac{1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2} = 5.41 \times 10^{11}\ \text{rad/s}$$

Actual angular speeds are multiples of this smallest possible value.

This simple example shows that certain classical concepts and models, when properly modified, might be useful in describing some features of atomic and molecular systems. A wide variety of phenomena on the submicroscopic scale can be explained only if we assume discrete values of the angular momentum associated with a particular type of motion.

The Danish physicist Niels Bohr (1885–1962) accepted and adopted this radical idea of discrete angular momentum values in developing his theory of the hydrogen atom. Strictly classical models were unsuccessful in describing many properties of the hydrogen atom. Bohr postulated that the electron could occupy only those circular orbits about the proton for which the orbital angular momentum was equal to $n\hbar$, where n is an integer. That is, he made the bold assumption that orbital angular momentum is quantized. From this simple model, the rotational frequencies of the electron in the various orbits can be estimated (see Problem 43).

SUMMARY

The **total kinetic energy** of a rigid object rolling on a rough surface without slipping equals the rotational kinetic energy about its center of mass, $\frac{1}{2}I_{\text{CM}}\omega^2$, plus the translational kinetic energy of the center of mass, $\frac{1}{2}Mv_{\text{CM}}^2$:

$$K = \tfrac{1}{2}I_{\text{CM}}\omega^2 + \tfrac{1}{2}Mv_{\text{CM}}^2 \qquad \textbf{(11.4)}$$

The **torque** $\boldsymbol{\tau}$ due to a force **F** about an origin in an inertial frame is defined to be

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \qquad \textbf{(11.7)}$$

Given two vectors **A** and **B**, the **cross product A × B** is a vector **C** having a magnitude

$$C \equiv AB \sin\theta \qquad \textbf{(11.9)}$$

where θ is the angle between **A** and **B**. The direction of the vector **C** = **A** × **B** is perpendicular to the plane formed by **A** and **B**, and this direction is determined by the right-hand rule.

The **angular momentum L** of a particle having linear momentum **p** = m**v** is

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \qquad \textbf{(11.15)}$$

where **r** is the vector position of the particle relative to an origin in an inertial frame.

The **net external torque** acting on a particle or rigid object is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt} \qquad \textbf{(11.20)}$$

The z *component* of **angular momentum** of a rigid object rotating about a fixed z axis is

$$L_z = I\omega \qquad \textbf{(11.21)}$$

where I is the moment of inertia of the object about the axis of rotation and ω is its angular speed.

The **net external torque** acting on a rigid object equals the product of its moment of inertia about the axis of rotation and its angular acceleration:

$$\sum \tau_{\text{ext}} = I\alpha \qquad \textbf{(11.23)}$$

If the net external torque acting on a system is zero, then the total angular momentum of the system is constant. Applying this **law of conservation of angular momentum** to a system whose moment of inertia changes gives

$$I_i\omega_i = I_f\omega_f = \text{constant} \qquad \textbf{(11.27)}$$

Questions

1. Is it possible to calculate the torque acting on a rigid body without specifying a center of rotation? Is the torque independent of the location of the center of rotation?

2. Is the triple product defined by $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ a scalar or a vector quantity? Explain why the operation $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ has no meaning.

3. In some motorcycle races, the riders drive over small hills, and the motorcycles become airborne for a short time. If a motorcycle racer keeps the throttle open while leaving the hill and going into the air, the motorcycle tends to nose upward. Why does this happen?

4. If the torque acting on a particle about a certain origin is zero, what can you say about its angular momentum about that origin?

5. Suppose that the velocity vector of a particle is completely specified. What can you conclude about the direction of its angular momentum vector with respect to the direction of motion?

6. If a single force acts on an object, and the torque caused by that force is nonzero about some point, is there any other point about which the torque is zero?

7. If a system of particles is in motion, is it possible for the total angular momentum to be zero about some origin? Explain.

8. A ball is thrown in such a way that it does not spin about its own axis. Does this mean that the angular momentum is zero about an arbitrary origin? Explain.

9. In a tape recorder, the tape is pulled past the read-and-write heads at a constant speed by the drive mechanism. Consider the reel from which the tape is pulled—as the tape is pulled off it, the radius of the roll of remaining tape decreases. How does the torque on the reel change with time? How does the angular speed of the reel change with time? If the tape mechanism is suddenly turned on so that the tape is quickly pulled with a great force, is the tape more likely to break when pulled from a nearly full reel or a nearly empty reel?

10. A scientist at a hotel sought assistance from a bellhop to carry a mysterious suitcase. When the unaware bellhop rounded a corner carrying the suitcase, it suddenly moved away from him for some unknown reason. At this point, the alarmed bellhop dropped the suitcase and ran off. What do you suppose might have been in the suitcase?

11. When a cylinder rolls on a horizontal surface as in Figure 11.3, do any points on the cylinder have only a vertical component of velocity at some instant? If so, where are they?

12. Three objects of uniform density—a solid sphere, a solid cylinder, and a hollow cylinder—are placed at the top of an incline (Fig. Q11.12). If they all are released from rest at the same elevation and roll without slipping, which object reaches the bottom first? Which reaches it last? You should try this at home and note that the result is independent of the masses and the radii of the objects.

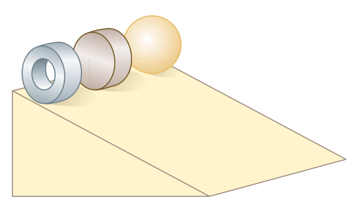


Figure Q11.12 Which object wins the race?

13. A mouse is initially at rest on a horizontal turntable mounted on a frictionless vertical axle. If the mouse begins to walk around the perimeter, what happens to the turntable? Explain.

14. Stars originate as large bodies of slowly rotating gas. Because of gravity, these regions of gas slowly decrease in size. What happens to the angular speed of a star as it shrinks? Explain.

15. Often, when a high diver wants to execute a flip in midair, she draws her legs up against her chest. Why does this make her rotate faster? What should she do when she wants to come out of her flip?

16. As a tether ball winds around a thin pole, what happens to its angular speed? Explain.

17. Two solid spheres—a large, massive sphere and a small sphere with low mass—are rolled down a hill. Which sphere reaches the bottom of the hill first? Next, a large, low-density sphere and a small, high-density sphere having the same mass are rolled down the hill. Which one reaches the bottom first in this case?

18. Suppose you are designing a car for a coasting race—the cars in this race have no engines; they simply coast down a hill. Do you want to use large wheels or small wheels? Do you want to use solid, disk-like wheels or hoop-like wheels? Should the wheels be heavy or light?

19. Why do tightrope walkers carry a long pole to help themselves keep their balance?

20. Two balls have the same size and mass. One is hollow, whereas the other is solid. How would you determine which is which without breaking them apart?

21. A particle is moving in a circle with constant speed. Locate one point about which the particle's angular momentum is constant and another about which it changes with time.

22. If global warming occurs over the next century, it is likely that some polar ice will melt and the water will be distributed closer to the equator. How would this change the moment of inertia of the Earth? Would the length of the day (one revolution) increase or decrease?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Section 11.1 Rolling Motion of a Rigid Object

WEB 1. A cylinder of mass 10.0 kg rolls without slipping on a horizontal surface. At the instant its center of mass has a speed of 10.0 m/s, determine (a) the translational kinetic energy of its center of mass, (b) the rotational energy about its center of mass, and (c) its total energy.

2. A bowling ball has a mass of 4.00 kg, a moment of inertia of 1.60×10^{-2} kg·m^2, and a radius of 0.100 m. If it rolls down the lane without slipping at a linear speed of 4.00 m/s, what is its total energy?

3. A bowling ball has a mass M, a radius R, and a moment of inertia $\frac{2}{5}MR^2$. If it starts from rest, how much work must be done on it to set it rolling without slipping at a linear speed v? Express the work in terms of M and v.

4. A uniform solid disk and a uniform hoop are placed side by side at the top of an incline of height h. If they are released from rest and roll without slipping, determine their speeds when they reach the bottom. Which object reaches the bottom first?

5. (a) Determine the acceleration of the center of mass of a uniform solid disk rolling down an incline making an angle θ with the horizontal. Compare this acceleration with that of a uniform hoop. (b) What is the minimum coefficient of friction required to maintain pure rolling motion for the disk?

6. A ring of mass 2.40 kg, inner radius 6.00 cm, and outer radius 8.00 cm rolls (without slipping) up an inclined plane that makes an angle of $\theta = 36.9°$ (Fig. P11.6). At the moment the ring is at position $x = 2.00$ m up the plane, its speed is 2.80 m/s. The ring continues up the plane for some additional distance and then rolls back down. It does not roll off the top end. How far up the plane does it go?

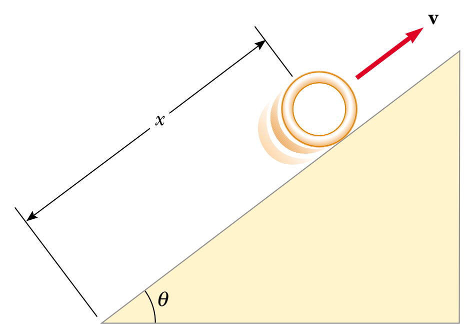


Figure P11.6

7. A metal can containing condensed mushroom soup has a mass of 215 g, a height of 10.8 cm, and a diameter of 6.38 cm. It is placed at rest on its side at the top of a 3.00-m-long incline that is at an angle of 25.0° to the horizontal and is then released to roll straight down. Assuming energy conservation, calculate the moment of inertia of the can if it takes 1.50 s to reach the bottom of the incline. Which pieces of data, if any, are unnecessary for calculating the solution?

8. A tennis ball is a hollow sphere with a thin wall. It is set rolling without slipping at 4.03 m/s on the horizontal section of a track, as shown in Figure P11.8. It rolls around the inside of a vertical circular loop 90.0 cm in diameter and finally leaves the track at a point 20.0 cm below the horizontal section. (a) Find the speed of the ball at the top of the loop. Demonstrate that it will not fall from the track. (b) Find its speed as it leaves the track. (c) Suppose that static friction between the ball and the track was negligible, so that the ball slid instead of rolling. Would its speed

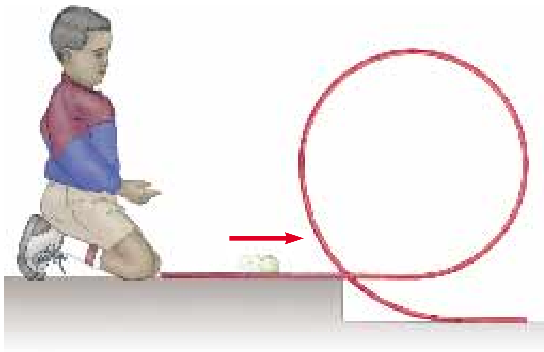


Figure P11.8

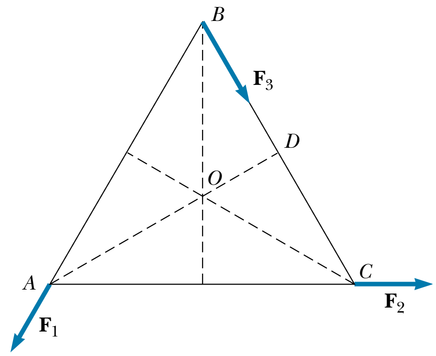


Figure P11.18

then be higher, lower, or the same at the top of the loop? Explain.

Section 11.2 The Vector Product and Torque

9. Given $\mathbf{M} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{N} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, calculate the vector product $\mathbf{M} \times \mathbf{N}$.

10. The vectors 42.0 cm at 15.0° and 23.0 cm at 65.0° both start from the origin. Both angles are measured counterclockwise from the x axis. The vectors form two sides of a parallelogram. (a) Find the area of the parallelogram. (b) Find the length of its longer diagonal.

WEB **11.** Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between $\mathbf{A}$ and $\mathbf{B}$.

12. For the vectors $\mathbf{A} = -3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 10\mathbf{j} + 9\mathbf{k}$, evaluate the expressions (a) $\cos^{-1}(\mathbf{A}\cdot\mathbf{B}/AB)$ and (b) $\sin^{-1}(|\mathbf{A} \times \mathbf{B}|/AB)$. (c) Which give(s) the angle between the vectors?

13. A force of $\mathbf{F} = 2.00\mathbf{i} + 3.00\mathbf{j}$ N is applied to an object that is pivoted about a fixed axis aligned along the z coordinate axis. If the force is applied at the point $\mathbf{r} = (4.00\mathbf{i} + 5.00\mathbf{j} + 0\mathbf{k})$ m, find (a) the magnitude of the net torque about the z axis and (b) the direction of the torque vector $\boldsymbol{\tau}$.

14. A student claims that she has found a vector $\mathbf{A}$ such that $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{A} = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. Do you believe this claim? Explain.

15. Vector $\mathbf{A}$ is in the negative y direction, and vector $\mathbf{B}$ is in the negative x direction. What are the directions of (a) $\mathbf{A} \times \mathbf{B}$ and (b) $\mathbf{B} \times \mathbf{A}$?

16. A particle is located at the vector position $\mathbf{r} = (\mathbf{i} + 3\mathbf{j})$ m, and the force acting on it is $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j})$ N. What is the torque about (a) the origin and (b) the point having coordinates (0, 6) m?

17. If $|\mathbf{A} \times \mathbf{B}| = \mathbf{A}\cdot\mathbf{B}$, what is the angle between $\mathbf{A}$ and $\mathbf{B}$?

18. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act along the two sides of an equilateral triangle, as shown in Figure P11.18. Point O is the intersection of the altitudes of the triangle. Find a third force $\mathbf{F}_3$ to be applied at B and along BC that will make the total torque about the point O be zero. Will the total torque change if $\mathbf{F}_3$ is applied not at B, but rather at any other point along BC?

Section 11.3 Angular Momentum of a Particle

19. A light, rigid rod 1.00 m in length joins two particles—with masses 4.00 kg and 3.00 kg—at its ends. The combination rotates in the xy plane about a pivot through the center of the rod (Fig. P11.19). Determine the angular momentum of the system about the origin when the speed of each particle is 5.00 m/s.

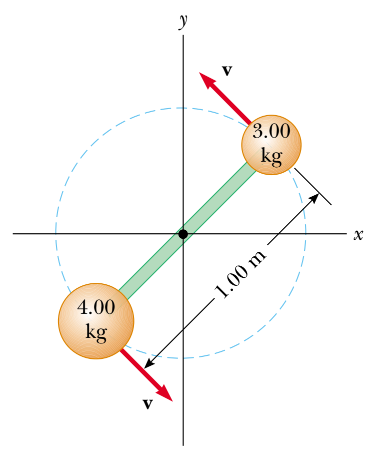


Figure P11.19

20. A 1.50-kg particle moves in the xy plane with a velocity of $\mathbf{v} = (4.20\mathbf{i} - 3.60\mathbf{j})$ m/s. Determine the particle's angular momentum when its position vector is $\mathbf{r} = (1.50\mathbf{i} + 2.20\mathbf{j})$ m.

WEB **21.** The position vector of a particle of mass 2.00 kg is given as a function of time by $\mathbf{r} = (6.00\mathbf{i} + 5.00t\mathbf{j})$ m. Determine the angular momentum of the particle about the origin as a function of time.

22. A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane, as shown in Figure P11.22. During the motion, the supporting wire of length ℓ maintains the constant angle θ with the vertical. Show that the magnitude of the angular momentum of the mass about the center of the circle is

$$L = (m^2 g \ell^3 \sin^4 \theta / \cos \theta)^{1/2}$$

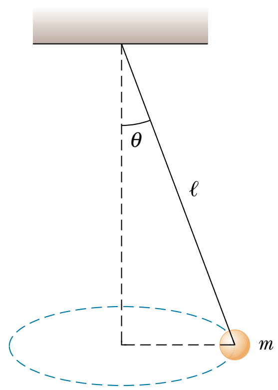


Figure P11.22

23. A particle of mass m moves in a circle of radius R at a constant speed v, as shown in Figure P11.23. If the motion begins at point Q, determine the angular momentum of the particle about point P as a function of time.

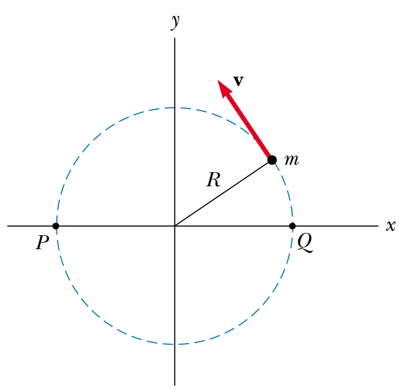


Figure P11.23

24. A 4.00-kg mass is attached to a light cord that is wound around a pulley (see Fig. 10.20). The pulley is a uniform solid cylinder with a radius of 8.00 cm and a mass of 2.00 kg. (a) What is the net torque on the system about the point O? (b) When the mass has a speed v, the pulley has an angular speed $\omega = v/R$. Determine the total angular momentum of the system about O. (c) Using the fact that $\boldsymbol{\tau} = d\mathbf{L}/dt$ and your result from part (b), calculate the acceleration of the mass.

25. A particle of mass m is shot with an initial velocity $\mathbf{v}_i$ and makes an angle θ with the horizontal, as shown in Figure P11.25. The particle moves in the gravitational field of the Earth. Find the angular momentum of the particle about the origin when the particle is (a) at the origin, (b) at the highest point of its trajectory, and (c) just about to hit the ground. (d) What torque causes its angular momentum to change?

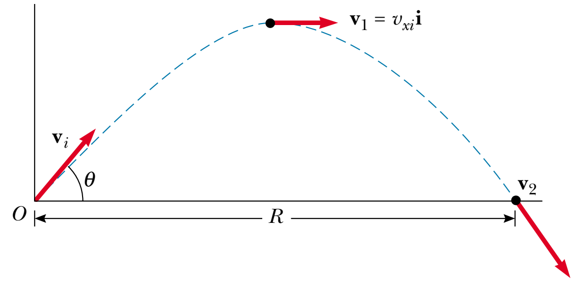


Figure P11.25

26. Heading straight toward the summit of Pike's Peak, an airplane of mass 12 000 kg flies over the plains of Kansas at a nearly constant altitude of 4.30 km and with a constant velocity of 175 m/s westward. (a) What is the airplane's vector angular momentum relative to a wheat farmer on the ground directly below the airplane? (b) Does this value change as the airplane continues its motion along a straight line? (c) What is its angular momentum relative to the summit of Pike's Peak?

27. A ball of mass m is fastened at the end of a flagpole connected to the side of a tall building at point P, as shown in Figure P11.27. The length of the flagpole is ℓ, and θ is the angle the flagpole makes with the horizontal. Suppose that the ball becomes loose and starts to fall. Determine the angular momentum (as a function of time) of the ball about point P. Neglect air resistance.

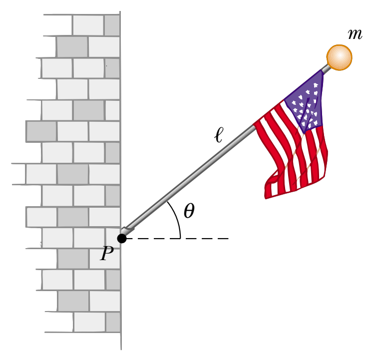


Figure P11.27

28. A fireman clings to a vertical ladder and directs the nozzle of a hose horizontally toward a burning building. The rate of water flow is 6.31 kg/s, and the nozzle speed is 12.5 m/s. The hose passes between the fireman's feet, which are 1.30 m vertically below the nozzle. Choose the origin to be inside the hose between the fireman's feet. What torque must the fireman exert on the hose? That is, what is the rate of change of angular momentum of the water?

Section 11.4 Angular Momentum of a Rotating Rigid Object

29. A uniform solid sphere with a radius of 0.500 m and a mass of 15.0 kg turns counterclockwise about a vertical axis through its center. Find its vector angular momentum when its angular speed is 3.00 rad/s.

30. A uniform solid disk with a mass of 3.00 kg and a radius of 0.200 m rotates about a fixed axis perpendicular to its face. If the angular speed is 6.00 rad/s, calculate the angular momentum of the disk when the axis of rotation (a) passes through its center of mass and (b) passes through a point midway between the center and the rim.

31. A particle with a mass of 0.400 kg is attached to the 100-cm mark of a meter stick with a mass of 0.100 kg. The meter stick rotates on a horizontal, frictionless table with an angular speed of 4.00 rad/s. Calculate the angular momentum of the system when the stick is pivoted about an axis (a) perpendicular to the table through the 50.0-cm mark and (b) perpendicular to the table through the 0-cm mark.

32. The hour and minute hands of Big Ben, the famous Parliament Building tower clock in London, are 2.70 m and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively. Calculate the total angular momentum of these hands about the center point. Treat the hands as long thin rods.

Section 11.5 Conservation of Angular Momentum

33. A cylinder with a moment of inertia of I_1 rotates about a vertical, frictionless axle with angular velocity ω_i. A second cylinder that has a moment of inertia of I_2 and initially is not rotating drops onto the first cylinder (Fig. P11.33). Because of friction between the surfaces, the two eventually reach the same angular speed ω_f. (a) Calculate ω_f. (b) Show that the kinetic energy of the system decreases in this interaction and calculate the ratio of the final rotational energy to the initial rotational energy.

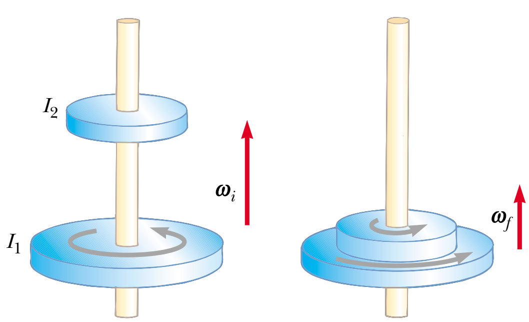


Figure P11.33

34. A playground merry-go-round of radius $R = 2.00$ m has a moment of inertia of $I = 250$ kg$\cdot$m^2 and is rotating at 10.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0-kg child hops onto the merry-go-round and manages to sit down on its edge. What is the new angular speed of the merry-go-round?

35. A student sits on a freely rotating stool holding two weights, each of which has a mass of 3.00 kg. When his arms are extended horizontally, the weights are 1.00 m from the axis of rotation and he rotates with an angular speed of 0.750 rad/s. The moment of inertia of the student plus stool is 3.00 kg$\cdot$m^2 and is assumed to be constant. The student pulls the weights inward horizontally to a position 0.300 m from the rotation axis. (a) Find the new angular speed of the student. (b) Find the kinetic energy of the rotating system before and after he pulls the weights inward.

36. A uniform rod with a mass of 100 g and a length of 50.0 cm rotates in a horizontal plane about a fixed, vertical, frictionless pin passing through its center. Two small beads, each having a mass 30.0 g, are mounted on the rod so that they are able to slide without friction along its length. Initially, the beads are held by catches at positions 10.0 cm on each side of center; at this time, the system rotates at an angular speed of 20.0 rad/s. Suddenly, the catches are released, and the small beads slide outward along the rod. Find (a) the angular speed of the system at the instant the beads reach the ends of the rod and (b) the angular speed of the rod after the beads fly off the rod's ends.

WEB **37.** A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of 500 kg$\cdot$m^2 and a radius of 2.00 m. The turntable is initially at rest and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth. (a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

38. A puck with a mass of 80.0 g and a radius of 4.00 cm slides along an air table at a speed of 1.50 m/s, as shown in Figure P11.38a. It makes a glancing collision

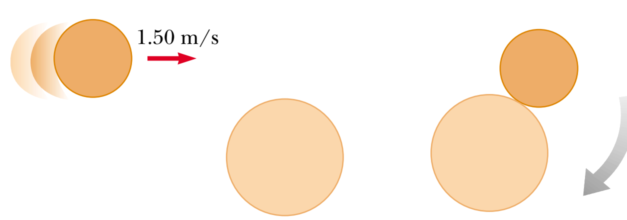


Figure P11.38

with a second puck having a radius of 6.00 cm and a mass of 120 g (initially at rest) such that their rims just touch. Because their rims are coated with instant-acting glue, the pucks stick together and spin after the collision (Fig. P11.38b). (a) What is the angular momentum of the system relative to the center of mass? (b) What is the angular speed about the center of mass?

39. A wooden block of mass M resting on a frictionless horizontal surface is attached to a rigid rod of length ℓ and of negligible mass (Fig. P11.39). The rod is pivoted at the other end. A bullet of mass m traveling parallel to the horizontal surface and normal to the rod with speed v hits the block and becomes embedded in it. (a) What is the angular momentum of the bullet–block system? (b) What fraction of the original kinetic energy is lost in the collision?

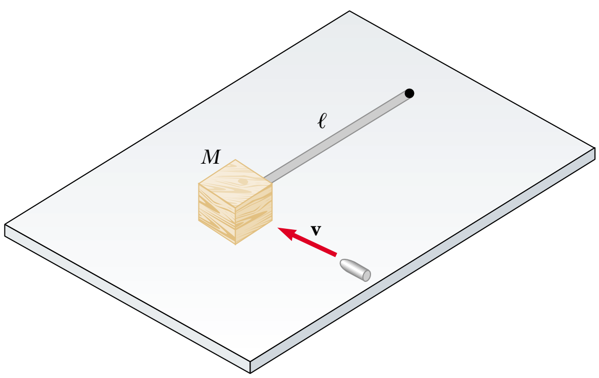


Figure P11.39

40. A space station shaped like a giant wheel has a radius of 100 m and a moment of inertia of 5.00×10^8 kg·m². A crew of 150 are living on the rim, and the station's rotation causes the crew to experience an acceleration of $1g$ (Fig. P11.40). When 100 people move to the center of the station for a union meeting, the angular speed changes. What acceleration is experienced by the managers remaining at the rim? Assume that the average mass of each inhabitant is 65.0 kg.

41. A wad of sticky clay of mass m and velocity $\mathbf{v}_i$ is fired at a solid cylinder of mass M and radius R (Fig. P11.41). The cylinder is initially at rest and is mounted on a fixed horizontal axle that runs through the center of mass. The line of motion of the projectile is perpendicular to the axle and at a distance d, less than R, from the center. (a) Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. (b) Is mechanical energy conserved in this process? Explain your answer.

42. Suppose a meteor with a mass of 3.00×10^{13} kg is moving at 30.0 km/s relative to the center of the Earth and strikes the Earth. What is the order of magnitude of the maximum possible decrease in the angular speed of the Earth due to this collision? Explain your answer.

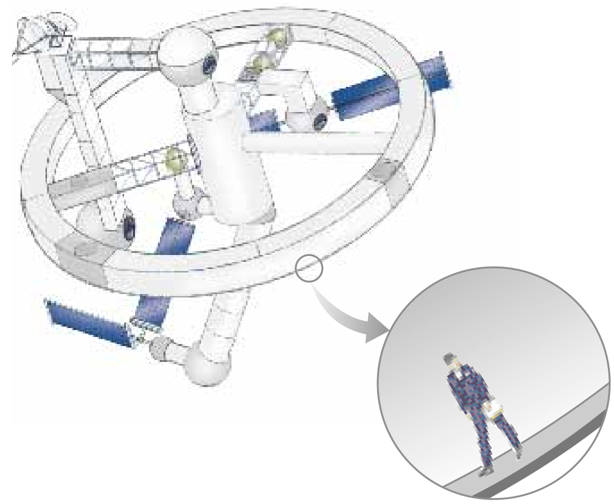


Figure P11.40

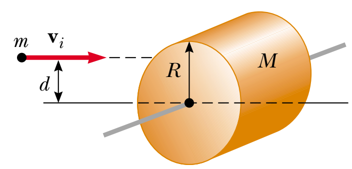


Figure P11.41

(Optional)

Section 11.7 Angular Momentum as a Fundamental Quantity

43. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming that the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular speed of the electron's motion.

ADDITIONAL PROBLEMS

44. Review Problem. A rigid, massless rod has three equal masses attached to it, as shown in Figure P11.44. The rod is free to rotate in a vertical plane about a frictionless axle perpendicular to the rod through the point P, and it is released from rest in the horizontal position at $t = 0$. Assuming m and d are known, find (a) the moment of inertia of the system about the pivot, (b) the torque acting on the system at $t = 0$, (c) the angular acceleration of the system at $t = 0$, (d) the linear acceleration of the mass labeled "3" at $t = 0$, (e) the maximum kinetic energy of the system, (f) the maximum angular speed attained by the rod, (g) the maximum angular momentum of the system, and (h) the maximum speed attained by the mass labeled "2."

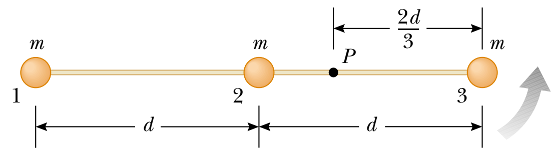


Figure P11.44

45. A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl having a much greater radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Fig. P11.45). Determine the angular speed of the sphere when it reaches the bottom of the bowl.

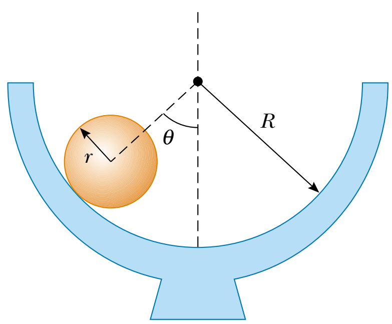


Figure P11.45

46. A 100-kg uniform horizontal disk of radius 5.50 m turns without friction at 2.50 rev/s on a vertical axis through its center, as shown in Figure P11.46. A feedback mechanism senses the angular speed of the disk, and a drive motor at A ensures that the angular speed remains constant. While the disk turns, a 1.20-kg mass on top of the disk slides outward in a radial slot. The 1.20-kg mass starts at the center of the disk at time $t = 0$ and moves outward with a constant speed of 1.25 cm/s relative to the disk until it reaches the edge at $t = 440$ s. The sliding mass experiences no friction. Its motion is constrained by a brake at B so that its radial speed remains constant. The constraint produces tension in a light string tied to the mass. (a) Find the torque as a function of time that the drive motor must provide while the mass is sliding. (b) Find the value of this torque at $t = 440$ s, just before the sliding mass finishes its motion. (c) Find the power that the drive motor must deliver as a function of time. (d) Find the value of the power when the sliding mass is just reaching the end of the slot. (e) Find the string tension as a function of time. (f) Find the work done by the drive motor during the 440-s motion. (g) Find the work done by the string brake on the sliding mass. (h) Find the total work done on the system consisting of the disk and the sliding mass.

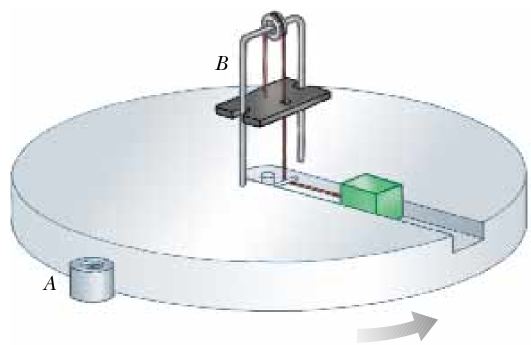


Figure P11.46

47. A string is wound around a uniform disk of radius R and mass M. The disk is released from rest when the string is vertical and its top end is tied to a fixed bar (Fig. P11.47). Show that (a) the tension in the string is one-third the weight of the disk, (b) the magnitude of the acceleration of the center of mass is $2g/3$, and (c) the speed of the center of mass is $(4gh/3)^{1/2}$ as the disk descends. Verify your answer to part (c) using the energy approach.

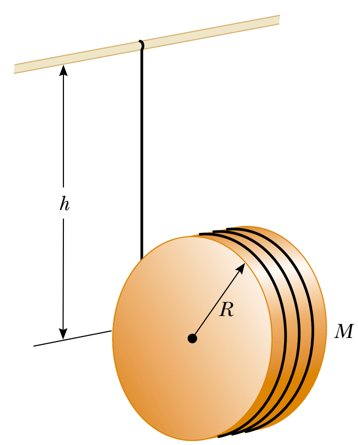


Figure P11.47

48. Comet Halley moves about the Sun in an elliptical orbit, with its closest approach to the Sun being about 0.590 AU and its greatest distance from the Sun being 35.0 AU (1 AU = the average Earth–Sun distance). If the comet's speed at its closest approach is 54.0 km/s, what is its speed when it is farthest from the Sun? The angular momentum of the comet about the Sun is conserved because no torque acts on the comet. The gravitational force exerted by the Sun on the comet has a moment arm of zero.

49. A constant horizontal force **F** is applied to a lawn roller having the form of a uniform solid cylinder of radius R and mass M (Fig. P11.49). If the roller rolls without slipping on the horizontal surface, show that (a) the acceleration of the center of mass is $2\mathbf{F}/3M$ and that (b) the minimum coefficient of friction necessary to prevent slipping is $F/3Mg$. (*Hint:* Consider the torque with respect to the center of mass.)

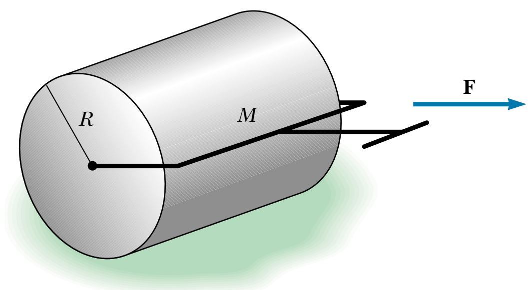


Figure P11.49

50. A light rope passes over a light, frictionless pulley. A bunch of bananas of mass M is fastened at one end, and a monkey of mass M clings to the other (Fig. P11.50).

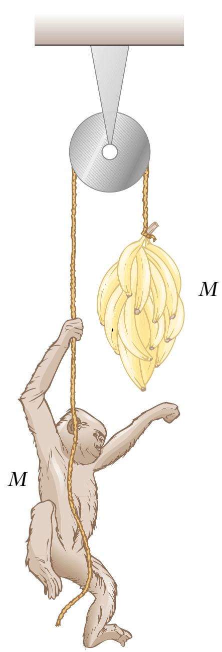


Figure P11.50

The monkey climbs the rope in an attempt to reach the bananas. (a) Treating the system as consisting of the monkey, bananas, rope, and pulley, evaluate the net torque about the pulley axis. (b) Using the results to part (a), determine the total angular momentum about the pulley axis and describe the motion of the system. Will the monkey reach the bananas?

51. A solid sphere of mass m and radius r rolls without slipping along the track shown in Figure P11.51. The sphere starts from rest with its lowest point at height h above the bottom of a loop of radius R, which is much larger than r. (a) What is the minimum value that h can have (in terms of R) if the sphere is to complete the loop? (b) What are the force components on the sphere at point P if $h = 3R$?

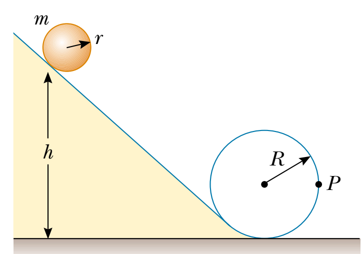


Figure P11.51

52. A thin rod with a mass of 0.630 kg and a length of 1.24 m is at rest, hanging vertically from a strong fixed hinge at its top end. Suddenly, a horizontal impulsive force (14.7**i**) N is applied to it. (a) Suppose that the force acts at the bottom end of the rod. Find the acceleration of the rod's center of mass and the horizontal force that the hinge exerts. (b) Suppose that the force acts at the midpoint of the rod. Find the acceleration of this point and the horizontal hinge reaction. (c) Where can the impulse be applied so that the hinge exerts no horizontal force? (This point is called the *center of percussion.*)

53. At one moment, a bowling ball is both sliding and spinning on a horizontal surface such that its rotational kinetic energy equals its translational kinetic energy. Let v_{CM} represent the ball's center-of-mass speed relative to the surface. Let v_r represent the speed of the topmost point on the ball's surface relative to the center of mass. Find the ratio v_{CM}/v_r.

54. A projectile of mass m moves to the right with speed v_i (Fig. P11.54a). The projectile strikes and sticks to the end of a stationary rod of mass M and length d that is pivoted about a frictionless axle through its center (Fig. P11.54b). (a) Find the angular speed of the system right after the collision. (b) Determine the fractional loss in mechanical energy due to the collision.

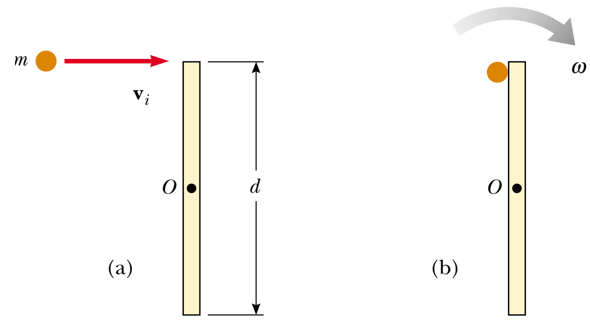


Figure P11.54

55. A mass m is attached to a cord passing through a small hole in a frictionless, horizontal surface (Fig. P11.55). The mass is initially orbiting with speed v_i in a circle of radius r_i. The cord is then slowly pulled from below, and the radius of the circle decreases to r. (a) What is the speed of the mass when the radius is r? (b) Find the tension in the cord as a function of r. (c) How much work W is done in moving m from r_i to r? (*Note:* The tension depends on r.) (d) Obtain numerical values for v, T, and W when $r = 0.100$ m, $m = 50.0$ g, $r_i = 0.300$ m, and $v_i = 1.50$ m/s.

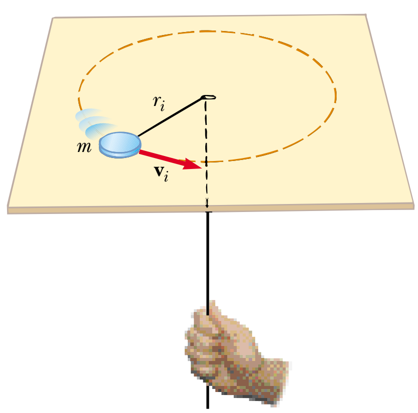


Figure P11.55

56. A bowler releases a bowling ball with no spin, sending it sliding straight down the alley toward the pins. The ball continues to slide for some distance before its motion becomes rolling without slipping; of what order of magnitude is this distance? State the quantities you take as data, the values you measure or estimate for them, and your reasoning.

57. Following Thanksgiving dinner, your uncle falls into a deep sleep while sitting straight up and facing the television set. A naughty grandchild balances a small spherical grape at the top of his bald head, which itself has the shape of a sphere. After all of the children have had time to giggle, the grape starts from rest and rolls down your uncle's head without slipping. It loses contact with your uncle's scalp when the radial line joining it to the center of curvature makes an angle θ with the vertical. What is the measure of angle θ?

58. A thin rod of length h and mass M is held vertically with its lower end resting on a frictionless horizontal surface. The rod is then released to fall freely. (a) Determine the speed of its center of mass just before it hits the horizontal surface. (b) Now suppose that the rod has a fixed pivot at its lower end. Determine the speed of the rod's center of mass just before it hits the surface.

WEB 59. Two astronauts (Fig. P11.59), each having a mass of 75.0 kg, are connected by a 10.0-m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of 5.00 m/s. (a) Treating the astronauts as particles, calculate the magnitude of the angular momentum and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to 5.00 m. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronaut in shortening the rope?

60. Two astronauts (see Fig. P11.59), each having a mass M, are connected by a rope of length d having negligible mass. They are isolated in space, orbiting their center of mass at speeds v. Treating the astronauts as particles, calculate (a) the magnitude of the angular momentum and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to $d/2$. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronaut in shortening the rope?

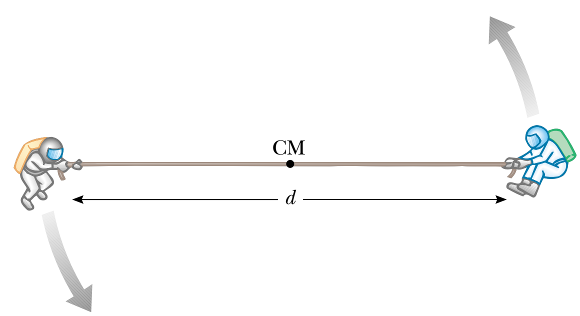


Figure P11.59 Problems 59 and 60.

61. A solid cube of wood of side $2a$ and mass M is resting on a horizontal surface. The cube is constrained to ro-

tate about an axis AB (Fig. P11.61). A bullet of mass m and speed v is shot at the face opposite $ABCD$ at a height of $4a/3$. The bullet becomes embedded in the cube. Find the minimum value of v required to tip the cube so that it falls on face $ABCD$. Assume $m \ll M$.

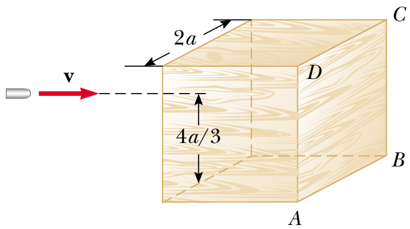


Figure P11.61

62. A large, cylindrical roll of paper of initial radius R lies on a long, horizontal surface with the open end of the paper nailed to the surface. The roll is given a slight shove ($v_i \approx 0$) and begins to unroll. (a) Determine the speed of the center of mass of the roll when its radius has diminished to r. (b) Calculate a numerical value for this speed at $r = 1.00$ mm, assuming $R = 6.00$ m. (c) What happens to the energy of the system when the paper is completely unrolled? (*Hint:* Assume that the roll has a uniform density and apply energy methods.)

63. A spool of wire of mass M and radius R is unwound under a constant force $\mathbf{F}$ (Fig. P11.63). Assuming that the spool is a uniform solid cylinder that does not slip, show that (a) the acceleration of the center of mass is $4\mathbf{F}/3M$ and that (b) the force of friction is to the *right* and is equal in magnitude to $F/3$. (c) If the cylinder starts from rest and rolls without slipping, what is the speed of its center of mass after it has rolled through a distance d?

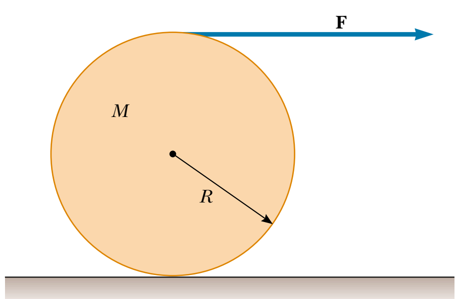


Figure P11.63

64. A uniform solid disk is set into rotation with an angular speed ω_i about an axis through its center. While still rotating at this speed, the disk is placed into contact with

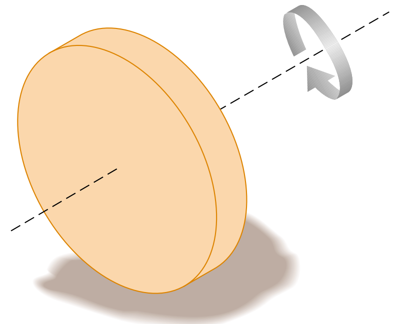


Figure P11.64 Problems 64 and 65.

a horizontal surface and released, as shown in Figure P11.64. (a) What is the angular speed of the disk once pure rolling takes place? (b) Find the fractional loss in kinetic energy from the time the disk is released until the time pure rolling occurs. (*Hint:* Consider torques about the center of mass.)

65. Suppose a solid disk of radius R is given an angular speed ω_i about an axis through its center and is then lowered to a horizontal surface and released, as shown in Problem 64 (see Fig. P11.64). Furthermore, assume that the coefficient of friction between the disk and the surface is μ. (a) Show that the time it takes for pure rolling motion to occur is $R\omega_i/3\mu g$. (b) Show that the distance the disk travels before pure rolling occurs is $R^2\omega_i^2/18\mu g$.

66. A solid cube of side $2a$ and mass M is sliding on a frictionless surface with uniform velocity $\mathbf{v}$, as shown in Figure P11.66a. It hits a small obstacle at the end of the table; this causes the cube to tilt, as shown in Figure

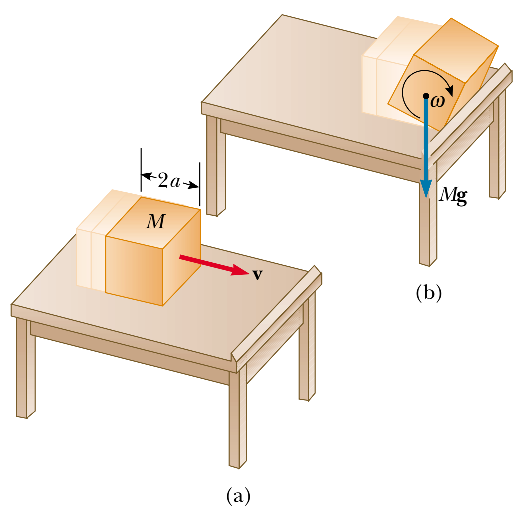


Figure P11.66

P11.66b. Find the minimum value of **v** such that the cube falls off the table. Note that the moment of inertia of the cube about an axis along one of its edges is $8Ma^2/3$. (*Hint:* The cube undergoes an inelastic collision at the edge.)

67. A plank with a mass $M = 6.00$ kg rides on top of two identical solid cylindrical rollers that have $R = 5.00$ cm and $m = 2.00$ kg (Fig. P11.67). The plank is pulled by a constant horizontal force of magnitude $F = 6.00$ N applied to the end of the plank and perpendicular to the axes of the cylinders (which are parallel). The cylinders roll without slipping on a flat surface. Also, no slipping occurs between the cylinders and the plank. (a) Find the acceleration of the plank and that of the rollers. (b) What frictional forces are acting?

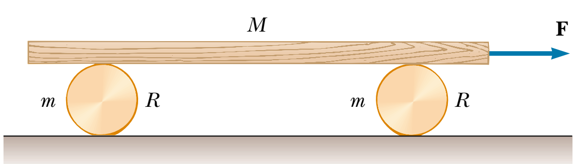


Figure P11.67

68. A spool of wire rests on a horizontal surface as in Figure P11.68. As the wire is pulled, the spool does not slip at the contact point P. On separate trials, each one of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is applied to the spool. For each one of these forces, determine the direction in which the spool will roll. Note that the line of action of $\mathbf{F}_2$ passes through P.

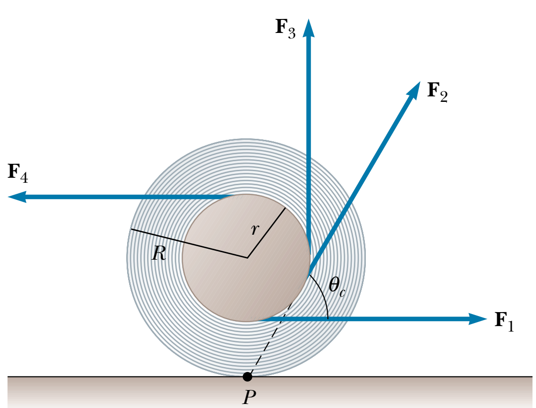


Figure P11.68 Problems 68 and 69.

69. The spool of wire shown in Figure P11.68 has an inner radius r and an outer radius R. The angle θ between the applied force and the horizontal can be varied. Show that the critical angle for which the spool does not slip and remains stationary is

$$\cos \theta_c = \frac{r}{R}$$

(*Hint:* At the critical angle, the line of action of the applied force passes through the contact point.)

70. In a demonstration that employs a ballistics cart, a ball is projected vertically upward from a cart moving with constant velocity along the horizontal direction. The ball lands in the catching cup of the cart because both the cart and the ball have the same horizontal component of velocity. Now consider a ballistics cart on an incline making an angle θ with the horizontal, as shown in Figure P11.70. The cart (including its wheels) has a mass M, and the moment of inertia of each of the two wheels is $mR^2/2$. (a) Using conservation of energy considerations (assuming that there is no friction between the cart and the axles) and assuming pure rolling motion (that is, no slipping), show that the acceleration of the cart along the incline is

$$a_x = \left(\frac{M}{M + 2m}\right) g \sin \theta$$

(b) Note that the x component of acceleration of the ball released by the cart is $g \sin \theta$. Thus, the x component of the cart's acceleration is *smaller* than that of the ball by the factor $M/(M + 2m)$. Use this fact and kinematic equations to show that the ball overshoots the cart by an amount Δx, where

$$\Delta x = \left(\frac{4m}{M + 2m}\right)\left(\frac{\sin \theta}{\cos^2 \theta}\right) \frac{{v_{yi}}^2}{g}$$

and v_{yi} is the initial speed of the ball imparted to it by the spring in the cart. (c) Show that the distance d that the ball travels measured along the incline is

$$d = \frac{2{v_{yi}}^2}{g} \frac{\sin \theta}{\cos^2 \theta}$$

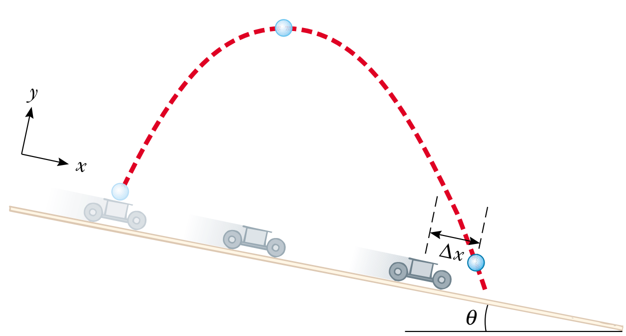


Figure P11.70

Answers to Quick Quizzes

11.1 There is very little resistance to motion that can reduce the kinetic energy of the rolling ball. Even though there is friction between the ball and the floor (if there were not, then no rotation would occur, and the ball would slide), there is no relative motion of the two surfaces (by the definition of "rolling"), and so kinetic friction cannot reduce K. (Air drag and friction associated with deformation of the ball eventually stop the ball.)

11.2 The box. Because none of the box's initial potential energy is converted to rotational kinetic energy, at any time $t > 0$ its translational kinetic energy is greater than that of the rolling ball.

11.3 Zero. If she were moving directly toward the pole, **r** and **p** would be antiparallel to each other, and the sine of the angle between them is zero; therefore, $L = 0$.

11.4 Both (a) and (b) are false. The net force is not necessarily zero. If the line of action of the net force passes through the point, then the net torque about an axis passing through that point is zero even though the net force is not zero. Because the net force is not necessarily zero, you cannot conclude that the particle's velocity is constant.

11.5 The student does work as he walks from the rim of the platform toward its center.

11.6 Because it is perpendicular to the precessional motion of the top, the force of gravity does no work. This is a situation in which a force causes motion but does no work.

PUZZLER

This one-bottle wine holder is an interesting example of a mechanical system that seems to defy gravity. The system (holder plus bottle) is balanced when its center of gravity is directly over the lowest support point. What two conditions are necessary for an object to exhibit this kind of stability? *(Charles D. Winters)*

c h a p t e r

12

Static Equilibrium and Elasticity

Chapter Outline

In Chapters 10 and 11 we studied the dynamics of rigid objects—that is, objects whose parts remain at a fixed separation with respect to each other when subjected to external forces. Part of this chapter addresses the conditions under which a rigid object is in equilibrium. The term *equilibrium* implies either that the object is at rest or that its center of mass moves with constant velocity. We deal here only with the former case, in which the object is described as being in *static equilibrium.* Static equilibrium represents a common situation in engineering practice, and the principles it involves are of special interest to civil engineers, architects, and mechanical engineers. If you are an engineering student you will undoubtedly take an advanced course in statics in the future.

The last section of this chapter deals with how objects deform under load conditions. Such deformations are usually elastic and do not affect the conditions for equilibrium. An *elastic* object returns to its original shape when the deforming forces are removed. Several elastic constants are defined, each corresponding to a different type of deformation.

12.1 THE CONDITIONS FOR EQUILIBRIUM

In Chapter 5 we stated that one necessary condition for equilibrium is that the net force acting on an object be zero. If the object is treated as a particle, then this is the only condition that must be satisfied for equilibrium. The situation with real (extended) objects is more complex, however, because these objects cannot be treated as particles. For an extended object to be in static equilibrium, a second condition must be satisfied. This second condition involves the net torque acting on the extended object. Note that equilibrium does not require the absence of motion. For example, a rotating object can have constant angular velocity and still be in equilibrium.

Consider a single force **F** acting on a rigid object, as shown in Figure 12.1. The effect of the force depends on its point of application *P*. If **r** is the position vector of this point relative to *O*, the torque associated with the force **F** about *O* is given by Equation 11.7:

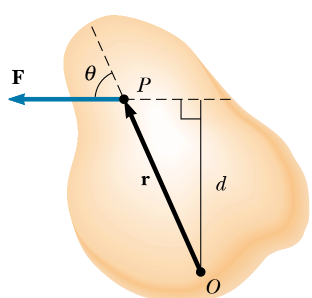


Figure 12.1 A single force **F** acts on a rigid object at the point *P*.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Recall from the discussion of the vector product in Section 11.2 that the vector $\boldsymbol{\tau}$ is perpendicular to the plane formed by **r** and **F**. You can use the right-hand rule to determine the direction of $\boldsymbol{\tau}$: Curl the fingers of your right hand in the direction of rotation that **F** tends to cause about an axis through *O*: your thumb then points in the direction of $\boldsymbol{\tau}$. Hence, in Figure 12.1 $\boldsymbol{\tau}$ is directed toward you out of the page.

As you can see from Figure 12.1, the tendency of **F** to rotate the object about an axis through *O* depends on the moment arm *d*, as well as on the magnitude of **F**. Recall that the magnitude of $\boldsymbol{\tau}$ is *Fd* (see Eq. 10.19). Now suppose a rigid object is acted on first by force $\mathbf{F}_1$ and later by force $\mathbf{F}_2$. If the two forces have the same magnitude, they will produce the same effect on the object only if they have the same direction and the same line of action. In other words,

Equivalent forces

two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are **equivalent** if and only if $F_1 = F_2$ and if and only if the two produce the same torque about any axis.

The two forces shown in Figure 12.2 are equal in magnitude and opposite in direction. They are *not* equivalent. The force directed to the right tends to rotate the object clockwise about an axis perpendicular to the diagram through O, whereas the force directed to the left tends to rotate it counterclockwise about that axis.

Suppose an object is pivoted about an axis through its center of mass, as shown in Figure 12.3. Two forces of equal magnitude act in opposite directions along parallel lines of action. A pair of forces acting in this manner form what is called a **couple.** (The two forces shown in Figure 12.2 also form a couple.) Do not make the mistake of thinking that the forces in a couple are a result of Newton's third law. They cannot be third-law forces because they act on the same object. Third-law force pairs act on different objects. Because each force produces the same torque Fd, the net torque has a magnitude of $2Fd$. Clearly, the object rotates clockwise and undergoes an angular acceleration about the axis. With respect to rotational motion, this is a nonequilibrium situation. The net torque on the object gives rise to an angular acceleration α according to the relationship $\Sigma\tau = 2Fd = I\alpha$ (see Eq. 10.21).

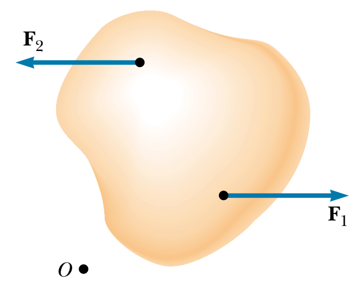


Figure 12.2 The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are not equivalent because they do not produce the same torque about some axis, even though they are equal in magnitude and opposite in direction.

In general, an object is in rotational equilibrium only if its angular acceleration $\alpha = 0$. Because $\Sigma\tau = I\alpha$ for rotation about a fixed axis, our second necessary condition for equilibrium is that **the net torque about any axis must be zero.** We now have two necessary conditions for equilibrium of an object:

Conditions for equilibrium

1. The resultant external force must equal zero. $$\sum \mathbf{F} = 0 \quad \textbf{(12.1)}$$

2. The resultant external torque about *any* axis must be zero. $$\sum \boldsymbol{\tau} = 0 \quad \textbf{(12.2)}$$

The first condition is a statement of translational equilibrium; it tells us that the linear acceleration of the center of mass of the object must be zero when viewed from an inertial reference frame. The second condition is a statement of rotational equilibrium and tells us that the angular acceleration about any axis must be zero. In the special case of **static equilibrium,** which is the main subject of this chapter, the object is at rest and so has no linear or angular speed (that is, $v_{CM} = 0$ and $\omega = 0$).

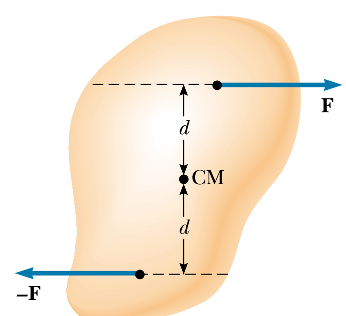


Figure 12.3 Two forces of equal magnitude form a couple if their lines of action are different parallel lines. In this case, the object rotates clockwise. The net torque about any axis is $2Fd$.

Quick Quiz 12.1

(a) Is it possible for a situation to exist in which Equation 12.1 is satisfied while Equation 12.2 is not? (b) Can Equation 12.2 be satisfied while Equation 12.1 is not?

The two vector expressions given by Equations 12.1 and 12.2 are equivalent, in general, to six scalar equations: three from the first condition for equilibrium, and three from the second (corresponding to x, y, and z components). Hence, in a complex system involving several forces acting in various directions, you could be faced with solving a set of equations with many unknowns. Here, we restrict our discussion to situations in which all the forces lie in the xy plane. (Forces whose vector representations are in the same plane are said to be *coplanar.*) With this restriction, we must deal with only three scalar equations. Two of these come from balancing the forces in the x and y directions. The third comes from the torque equation—namely, that the net torque about *any* point in the xy plane must be zero. Hence, the two conditions of equilibrium provide the equations

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau_z = 0 \quad \textbf{(12.3)}$$

where the axis of the torque equation is arbitrary, as we now show.

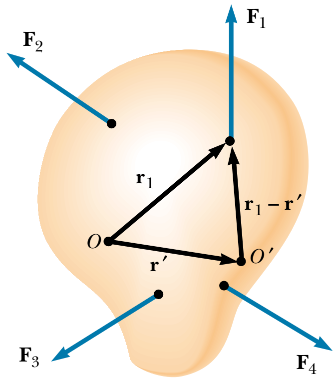


Figure 12.4 Construction showing that if the net torque is zero about origin O, it is also zero about any other origin, such as O'.

Regardless of the number of forces that are acting, if an object is in translational equilibrium and if the net torque is zero about one axis, then the net torque must also be zero about any other axis. The point can be inside or outside the boundaries of the object. Consider an object being acted on by several forces such that the resultant force $\Sigma \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0$. Figure 12.4 describes this situation (for clarity, only four forces are shown). The point of application of $\mathbf{F}_1$ relative to O is specified by the position vector $\mathbf{r}_1$. Similarly, the points of application of $\mathbf{F}_2$, $\mathbf{F}_3$, . . . are specified by $\mathbf{r}_2$, $\mathbf{r}_3$, . . . (not shown). The net torque about an axis through O is

$$\sum \boldsymbol{\tau}_O = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \cdots$$

Now consider another arbitrary point O' having a position vector $\mathbf{r}'$ relative to O. The point of application of $\mathbf{F}_1$ relative to O' is identified by the vector $\mathbf{r}_1 - \mathbf{r}'$. Likewise, the point of application of $\mathbf{F}_2$ relative to O' is $\mathbf{r}_2 - \mathbf{r}'$, and so forth. Therefore, the torque about an axis through O' is

$$\begin{aligned}\sum \boldsymbol{\tau}_{O'} &= (\mathbf{r}_1 - \mathbf{r}') \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}') \times \mathbf{F}_2 + (\mathbf{r}_3 - \mathbf{r}') \times \mathbf{F}_3 + \cdots \\ &= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \cdots - \mathbf{r}' \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots)\end{aligned}$$

Because the net force is assumed to be zero (given that the object is in translational equilibrium), the last term vanishes, and we see that the torque about O' is equal to the torque about O. Hence, **if an object is in translational equilibrium and the net torque is zero about one point, then the net torque must be zero about any other point.**

12.2 MORE ON THE CENTER OF GRAVITY

We have seen that the point at which a force is applied can be critical in determining how an object responds to that force. For example, two equal-magnitude but oppositely directed forces result in equilibrium if they are applied at the same point on an object. However, if the point of application of one of the forces is moved, so that the two forces no longer act along the same line of action, then a force couple results and the object undergoes an angular acceleration. (This is the situation shown in Figure 12.3.)

Whenever we deal with a rigid object, one of the forces we must consider is the force of gravity acting on it, and we must know the point of application of this force. As we learned in Section 9.6, on every object is a special point called its center of gravity. All the various gravitational forces acting on all the various mass elements of the object are equivalent to a single gravitational force acting through this point. Thus, to compute the torque due to the gravitational force on an object of mass M, we need only consider the force $M\mathbf{g}$ acting at the center of gravity of the object.

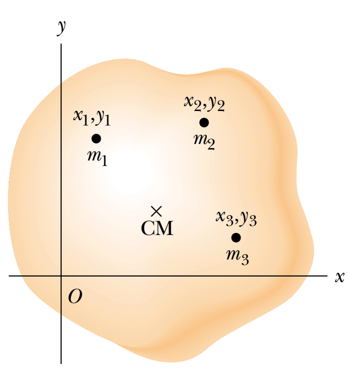


Figure 12.5 An object can be divided into many small particles each having a specific mass and specific coordinates. These particles can be used to locate the center of mass.

How do we find this special point? As we mentioned in Section 9.6, if we assume that $\mathbf{g}$ is uniform over the object, then the center of gravity of the object coincides with its center of mass. To see that this is so, consider an object of arbitrary shape lying in the xy plane, as illustrated in Figure 12.5. Suppose the object is divided into a large number of particles of masses m_1, m_2, m_3, . . . having coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), ... In

Equation 9.28 we defined the x coordinate of the center of mass of such an object to be

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

We use a similar equation to define the y coordinate of the center of mass, replacing each x with its y counterpart.

Let us now examine the situation from another point of view by considering the force of gravity exerted on each particle, as shown in Figure 12.6. Each particle contributes a torque about the origin equal in magnitude to the particle's weight mg multiplied by its moment arm. For example, the torque due to the force $m_1\mathbf{g}_1$ is $m_1g_1x_1$, where g_1 is the magnitude of the gravitational field at the position of the particle of mass m_1. We wish to locate the center of gravity, the point at which application of the single gravitational force $M\mathbf{g}$ (where $M = m_1 + m_2 + m_3 + \cdots$ is the total mass of the object) has the same effect on rotation as does the combined effect of all the individual gravitational forces $m_i\mathbf{g}_i$. Equating the torque resulting from $M\mathbf{g}$ acting at the center of gravity to the sum of the torques acting on the individual particles gives

$$(m_1g_1 + m_2g_2 + m_3g_3 + \cdots)x_{CG} = m_1g_1x_1 + m_2g_2x_2 + m_3g_3x_3 + \cdots$$

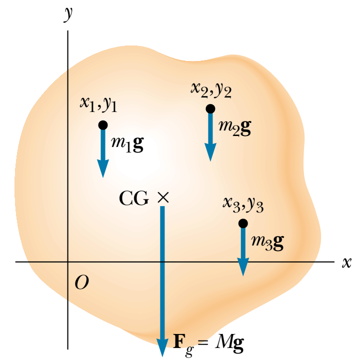


Figure 12.6 The center of gravity of an object is located at the center of mass if **g** is constant over the object.

This expression accounts for the fact that the gravitational field strength g can in general vary over the object. If we assume uniform g over the object (as is usually the case), then the g terms cancel and we obtain

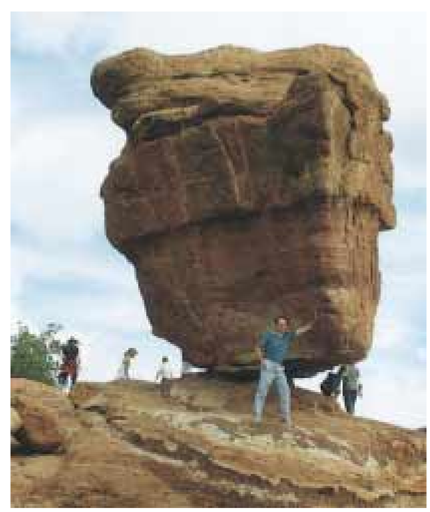
$$x_{CG} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad \textbf{(12.4)}$$

Comparing this result with Equation 9.28, we see that **the center of gravity is located at the center of mass as long as the object is in a uniform gravitational field.**

In several examples presented in the next section, we are concerned with homogeneous, symmetric objects. The center of gravity for any such object coincides with its geometric center.

12.3 EXAMPLES OF RIGID OBJECTS IN STATIC EQUILIBRIUM

The photograph of the one-bottle wine holder on the first page of this chapter shows one example of a balanced mechanical system that seems to defy gravity. For the system (wine holder plus bottle) to be in equilibrium, the net external force must be zero (see Eq. 12.1) and the net external torque must be zero (see Eq. 12.2). The second condition can be satisfied only when the center of gravity of the system is directly over the support point.



A large balanced rock at the Garden of the Gods in Colorado Springs, Colorado—an example of stable equilibrium.

In working static equilibrium problems, it is important to recognize all the external forces acting on the object. Failure to do so results in an incorrect analysis. When analyzing an object in equilibrium under the action of several external forces, use the following procedure.

Problem-Solving Hints

Objects in Static Equilibrium

- Draw a simple, neat diagram of the system.
- Isolate the object being analyzed. Draw a free-body diagram and then show and label all external forces acting on the object, indicating where those forces are applied. Do not include forces exerted by the object on its surroundings. (For systems that contain more than one object, draw a *separate* free-body diagram for each one.) Try to guess the correct direction for each force. If the direction you select leads to a negative force, do not be alarmed; this merely means that the direction of the force is the opposite of what you guessed.
- Establish a convenient coordinate system for the object and find the components of the forces along the two axes. Then apply the first condition for equilibrium. Remember to keep track of the signs of all force components.
- Choose a convenient axis for calculating the net torque on the object. Remember that the choice of origin for the torque equation is arbitrary; therefore, choose an origin that simplifies your calculation as much as possible. Note that a force that acts along a line passing through the point chosen as the origin gives zero contribution to the torque and thus can be ignored.

The first and second conditions for equilibrium give a set of linear equations containing several unknowns, and these equations can be solved simultaneously.

EXAMPLE 12.1 The Seesaw

A uniform 40.0-N board supports a father and daughter weighing 800 N and 350 N, respectively, as shown in Figure 12.7. If the support (called the *fulcrum*) is under the center of gravity of the board and if the father is 1.00 m from the center, (a) determine the magnitude of the upward force **n** exerted on the board by the support.

Solution First note that, in addition to **n**, the external forces acting on the board are the downward forces exerted by each person and the force of gravity acting on the board. We know that the board's center of gravity is at its geometric center because we were told the board is uniform. Because the system is in static equilibrium, the upward force **n** must balance all the downward forces. From $\Sigma F_y = 0$, we have, once we define upward as the positive y direction,

$$n - 800\text{ N} - 350\text{ N} - 40.0\text{ N} = 0$$

$$n = \boxed{1\,190\text{ N}}$$

(The equation $\Sigma F_x = 0$ also applies, but we do not need to consider it because no forces act horizontally on the board.)

(b) Determine where the child should sit to balance the system.

Solution To find this position, we must invoke the second condition for equilibrium. Taking an axis perpendicular to the page through the center of gravity of the board as the axis for our torque equation (this means that the torques

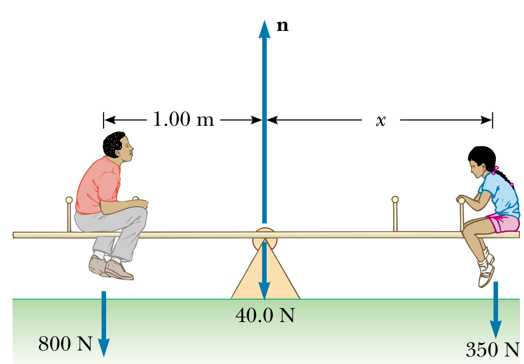


Figure 12.7 A balanced system.

produced by **n** and the force of gravity acting on the board about this axis are zero), we see from $\sum \tau = 0$ that

$$(800\ \text{N})(1.00\ \text{m}) - (350\ \text{N})x = 0$$

$$x = \boxed{2.29\ \text{m}}$$

(c) Repeat part (b) for another axis.

Solution To illustrate that the choice of axis is arbitrary, let us choose an axis perpendicular to the page and passing through the location of the father. Recall that the sign of the torque associated with a force is positive if that force tends to rotate the system counterclockwise, while the sign of the torque is negative if the force tends to rotate the system clockwise. In this case, $\sum \tau = 0$ yields

$$n(1.00\ \text{m}) - (40.0\ \text{N})(1.00\ \text{m}) - (350\ \text{N})(1.00\ \text{m} + x) = 0$$

From part (a) we know that $n = 1\ 190$ N. Thus, we can solve for x to find $\boxed{x = 2.29\ \text{m.}}$ This result is in agreement with the one we obtained in part (b).

Quick Quiz 12.2

In Example 12.1, if the fulcrum did not lie under the board's center of gravity, what other information would you need to solve the problem?

EXAMPLE 12.2 A Weighted Hand

A person holds a 50.0-N sphere in his hand. The forearm is horizontal, as shown in Figure 12.8a. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0 cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of the forearm.

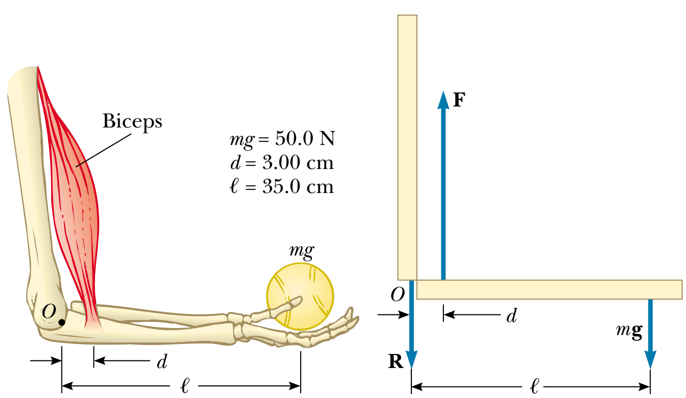


Figure 12.8 (a) The biceps muscle pulls upward with a force **F** that is essentially at right angles to the forearm. (b) The mechanical model for the system described in part (a).

Solution We simplify the situation by modeling the forearm as a bar as shown in Figure 12.8b, where **F** is the upward force exerted by the biceps and **R** is the downward force exerted by the upper arm at the joint. From the first condition for equilibrium, we have, with upward as the positive y direction,

$$(1) \qquad \sum F_y = F - R - 50.0\ \text{N} = 0$$

From the second condition for equilibrium, we know that the sum of the torques about any point must be zero. With the joint O as the axis, we have

$$Fd - mg\ell = 0$$

$$F(3.00\ \text{cm}) - (50.0\ \text{N})(35.0\ \text{cm}) = 0$$

$$F = \boxed{583\ \text{N}}$$

This value for F can be substituted into Equation (1) to give $R = 533$ N. As this example shows, the forces at joints and in muscles can be extremely large.

Exercise In reality, the biceps makes an angle of 15.0° with the vertical; thus, **F** has both a vertical and a horizontal component. Find the magnitude of **F** and the components of **R** when you include this fact in your analysis.

Answer $F = 604$ N, $R_x = 156$ N, $R_y = 533$ N.

EXAMPLE 12.3 Standing on a Horizontal Beam

A uniform horizontal beam with a length of 8.00 m and a weight of 200 N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0° with the horizontal (Fig. 12.9a). If a 600-N person stands 2.00 m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.

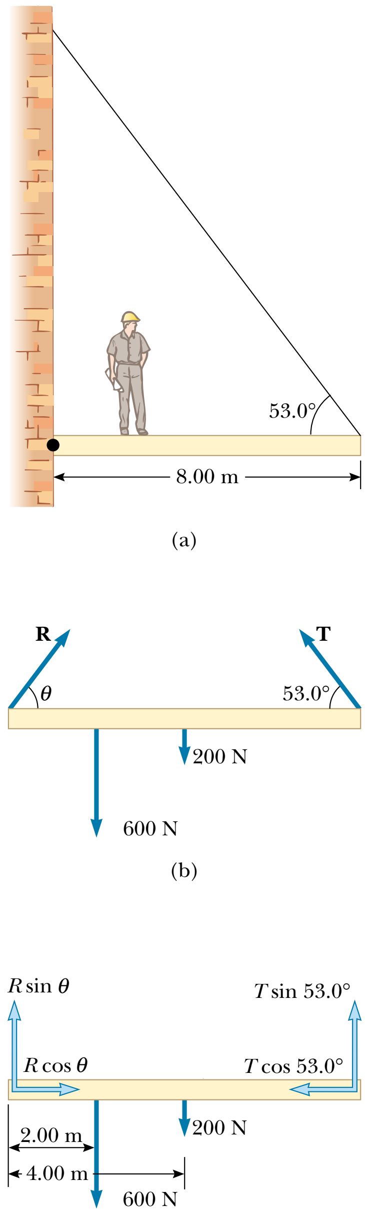


Figure 12.9 (a) A uniform beam supported by a cable. (b) The free-body diagram for the beam. (c) The free-body diagram for the beam showing the components of **R** and **T**.

Solution First we must identify all the external forces acting on the beam: They are the 200-N force of gravity, the force **T** exerted by the cable, the force **R** exerted by the wall at the pivot, and the 600-N force that the person exerts on the beam. These forces are all indicated in the free-body diagram for the beam shown in Figure 12.9b. When we consider directions for forces, it sometimes is helpful if we imagine what would happen if a force were suddenly removed. For example, if the wall were to vanish suddenly, the left end of the beam would probably move to the left as it begins to fall. This tells us that the wall is not only holding the beam up but is also pressing outward against it. Thus, we draw the vector **R** as shown in Figure 12.9b. If we resolve **T** and **R** into horizontal and vertical components, as shown in Figure 12.9c, and apply the first condition for equilibrium, we obtain

$$(1) \qquad \sum F_x = R\cos\theta - T\cos 53.0° = 0$$

$$(2) \qquad \sum F_y = R\sin\theta + T\sin 53.0° - 600\text{ N} - 200\text{ N} = 0$$

where we have chosen rightward and upward as our positive directions. Because R, T, and θ are all unknown, we cannot obtain a solution from these expressions alone. (The number of simultaneous equations must equal the number of unknowns for us to be able to solve for the unknowns.)

Now let us invoke the condition for rotational equilibrium. A convenient axis to choose for our torque equation is the one that passes through the pin connection. The feature that makes this point so convenient is that the force **R** and the horizontal component of **T** both have a moment arm of zero; hence, these forces provide no torque about this point. Recalling our counterclockwise-equals-positive convention for the sign of the torque about an axis and noting that the moment arms of the 600-N, 200-N, and $T\sin 53.0°$ forces are 2.00 m, 4.00 m, and 8.00 m, respectively, we obtain

$$\sum \tau = (T\sin 53.0°)(8.00\text{ m}) - (600\text{ N})(2.00\text{ m}) - (200\text{ N})(4.00\text{ m}) = 0$$

$$T = 313\text{ N}$$

Thus, the torque equation with this axis gives us one of the unknowns directly! We now substitute this value into Equations (1) and (2) and find that

$$R\cos\theta = 188\text{ N}$$

$$R\sin\theta = 550\text{ N}$$

We divide the second equation by the first and, recalling the trigonometric identity $\sin\theta/\cos\theta = \tan\theta$, we obtain

$$\tan\theta = \frac{550\text{ N}}{188\text{ N}} = 2.93$$

$$\theta = 71.1°$$

This positive value indicates that our estimate of the direction of **R** was accurate.

Finally,

$$R = \frac{188\text{ N}}{\cos\theta} = \frac{188\text{ N}}{\cos 71.1°} = 580\text{ N}$$

If we had selected some other axis for the torque equation, the solution would have been the same. For example, if we had chosen an axis through the center of gravity of the beam, the torque equation would involve both T and R. However, this equation, coupled with Equations (1) and (2), could still be solved for the unknowns. Try it!

When many forces are involved in a problem of this nature, it is convenient to set up a table. For instance, for the example just given, we could construct the following table. Setting the sum of the terms in the last column equal to zero represents the condition of rotational equilibrium.

Force Component	**Moment Arm Relative to O (m)**	**Torque About O (N·m)**
$T \sin 53.0°$	8.00	$(8.00)T \sin 53.0°$
$T \cos 53.0°$	0	0
200 N	4.00	$-(4.00)(200)$
600 N	2.00	$-(2.00)(600)$
$R \sin \theta$	0	0
$R \cos \theta$	0	0

EXAMPLE 12.4 The Leaning Ladder

A uniform ladder of length ℓ and weight $mg = 50$ N rests against a smooth, vertical wall (Fig. 12.10a). If the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$, find the minimum angle $\theta_{\min}$ at which the ladder does not slip.

Solution The free-body diagram showing all the external forces acting on the ladder is illustrated in Figure 12.10b. The reaction force **R** exerted by the ground on the ladder is the vector sum of a normal force **n** and the force of static friction $\mathbf{f}_s$. The reaction force **P** exerted by the wall on the ladder is horizontal because the wall is frictionless. Notice how we have included only forces that act on the ladder. For example, the forces exerted by the ladder on the ground and on the wall are not part of the problem and thus do not appear in the free-body diagram. Applying the first condition for equilibrium to the ladder, we have

$$\sum F_x = f - P = 0$$

$$\sum F_y = n - mg = 0$$

From the second equation we see that $n = mg = 50$ N. Furthermore, when the ladder is on the verge of slipping, the force of friction must be a maximum, which is given by $f_{s,\max} = \mu_s n = 0.40(50 \text{ N}) = 20$ N. (Recall Eq. 5.8: $f_s \le \mu_s n$.) Thus, at this angle, $P = 20$ N.

To find $\theta_{\min}$, we must use the second condition for equilibrium. When we take the torques about an axis through the origin O at the bottom of the ladder, we have

$$\sum \tau_O = P\ell \sin\theta - mg\frac{\ell}{2}\cos\theta = 0$$

Because $P = 20$ N when the ladder is about to slip, and because $mg = 50$ N, this expression gives

$$\tan\theta_{\min} = \frac{mg}{2P} = \frac{50 \text{ N}}{40 \text{ N}} = 1.25$$

$$\theta_{\min} = 51°$$

An alternative approach is to consider the intersection O' of the lines of action of forces $m\mathbf{g}$ and **P**. Because the torque about any origin must be zero, the torque about O' must be zero. This requires that the line of action of **R** (the resultant of **n** and **f**) pass through O'. In other words, because the ladder is stationary, the three forces acting on it must all pass through some common point. (We say that such forces are *concurrent.*) With this condition, you could then obtain the angle ϕ that **R** makes with the horizontal (where ϕ is greater than θ). Because this approach depends on the length of the ladder, you would have to know the value of ℓ to obtain a value for $\theta_{\min}$.

Exercise For the angles labeled in Figure 12.10, show that $\tan\phi = 2\tan\theta$.

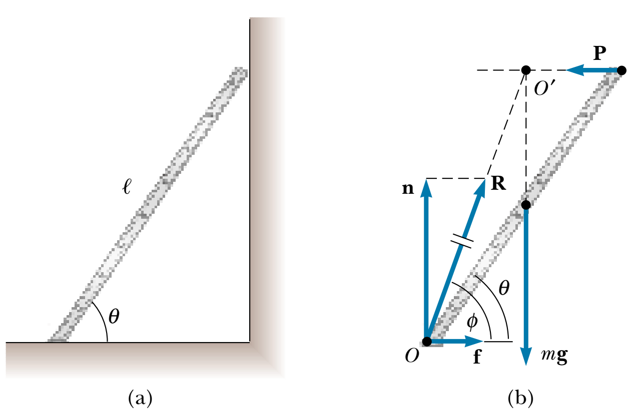


Figure 12.10 (a) A uniform ladder at rest, leaning against a smooth wall. The ground is rough. (b) The free-body diagram for the ladder. Note that the forces **R**, $m\mathbf{g}$, and **P** pass through a common point O'.

EXAMPLE 12.5 Negotiating a Curb

(a) Estimate the magnitude of the force **F** a person must apply to a wheelchair's main wheel to roll up over a sidewalk curb (Fig. 12.11a). This main wheel, which is the one that comes in contact with the curb, has a radius *r*, and the height of the curb is *h*.

Solution Normally, the person's hands supply the required force to a slightly smaller wheel that is concentric with the main wheel. We assume that the radius of the smaller wheel is the same as the radius of the main wheel, and so we can use *r* for our radius. Let us estimate a combined weight of $mg = 1\,400$ N for the person and the wheelchair and choose a wheel radius of $r = 30$ cm, as shown in Figure 12.11b. We also pick a curb height of $h = 10$ cm. We assume that the wheelchair and occupant are symmetric, and that each wheel supports a weight of 700 N. We then proceed to analyze only one of the wheels.

When the wheel is just about to be raised from the street, the reaction force exerted by the ground on the wheel at point *Q* goes to zero. Hence, at this time only three forces act on the wheel, as shown in Figure 12.11c. However, the force **R**, which is the force exerted on the wheel by the curb, acts at point *P*, and so if we choose to have our axis of rotation pass through point *P*, we do not need to include **R** in our torque equation. From the triangle *OPQ* shown in Figure 12.11b, we see that the moment arm *d* of the gravitational force $m\mathbf{g}$ acting on the wheel relative to point *P* is

$$d = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

The moment arm of **F** relative to point *P* is $2r - h$. Therefore, the net torque acting on the wheel about point *P* is

$$mgd - F(2r - h) = 0$$

$$mg\sqrt{2rh - h^2} - F(2r - h) = 0$$

$$F = \frac{mg\sqrt{2rh - h^2}}{2r - h}$$

$$F = \frac{(700\text{ N})\sqrt{2(0.3\text{ m})(0.1\text{ m}) - (0.1\text{ m})^2}}{2(0.3\text{ m}) - 0.1\text{ m}} = \boxed{300\text{ N}}$$

(Notice that we have kept only one digit as significant.) This result indicates that the force that must be applied to each wheel is substantial. You may want to estimate the force required to roll a wheelchair up a typical sidewalk accessibility ramp for comparison.

(b) Determine the magnitude and direction of **R**.

Solution We use the first condition for equilibrium to determine the direction:

$$\sum F_x = F - R\cos\theta = 0$$

$$\sum F_y = R\sin\theta - mg = 0$$

Dividing the second equation by the first gives

$$\tan\theta = \frac{mg}{F} = \frac{700\text{ N}}{300\text{ N}}; \quad \theta = \boxed{70°}$$

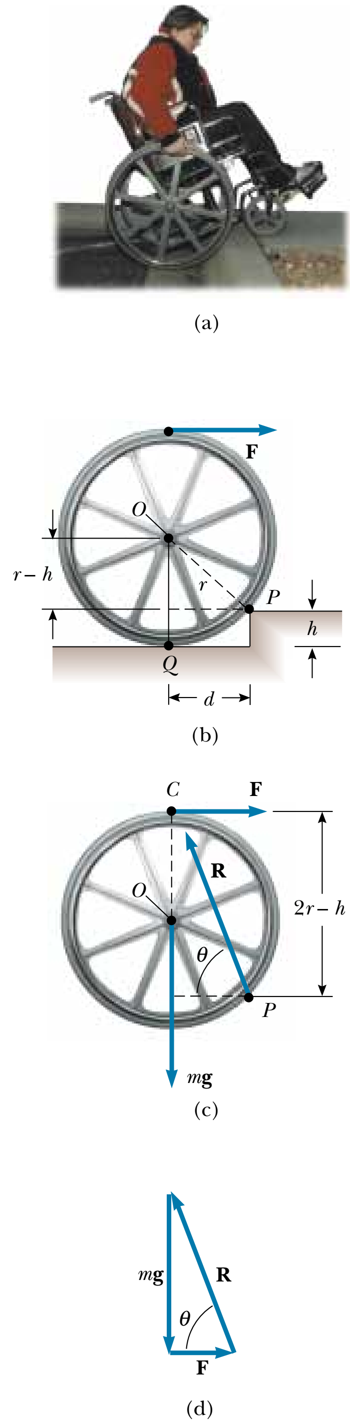


Figure 12.11 (a) A wheelchair and person of total weight mg being raised over a curb by a force **F**. (b) Details of the wheel and curb. (c) The free-body diagram for the wheel when it is just about to be raised. Three forces act on the wheel at this instant: **F**, which is exerted by the hand; **R**, which is exerted by the curb; and the gravitational force $m\mathbf{g}$. (d) The vector sum of the three external forces acting on the wheel is zero.

We can use the right triangle shown in Figure 12.11d to obtain R:

$$R = \sqrt{(mg)^2 + F^2} = \sqrt{(700\text{ N})^2 + (300\text{ N})^2} = \boxed{800\text{ N}}$$

Exercise Solve this problem by noting that the three forces acting on the wheel are concurrent (that is, that all three pass through the point C). The three forces form the sides of the triangle shown in Figure 12.11d.

APPLICATION Analysis of a Truss

Roofs, bridges, and other structures that must be both strong and lightweight often are made of trusses similar to the one shown in Figure 12.12a. Imagine that this truss structure represents part of a bridge. To approach this problem, we assume that the structural components are connected by pin joints. We also assume that the entire structure is free to slide horizontally because it sits on "rockers" on each end, which allow it to move back and forth as it undergoes thermal expansion and contraction. Assuming the mass of the bridge structure is negligible compared with the load, let us calculate the forces of tension or compression in all the structural components when it is supporting a 7 200-N load at the center (see Problem 58).

The force notation that we use here is not of our usual format. Until now, we have used the notation F_{AB} to mean "the force exerted by A on B." For this application, however, all double-letter subscripts on F indicate only the body exerting the force. The body on which a given force acts is not named in the subscript. For example, in Figure 12.12, F_{AB} is the force exerted by strut AB on the pin at A.

First, we apply Newton's second law to the truss as a whole in the vertical direction. Internal forces do not enter into this accounting. We balance the weight of the load with the normal forces exerted at the two ends by the supports on which the bridge rests:

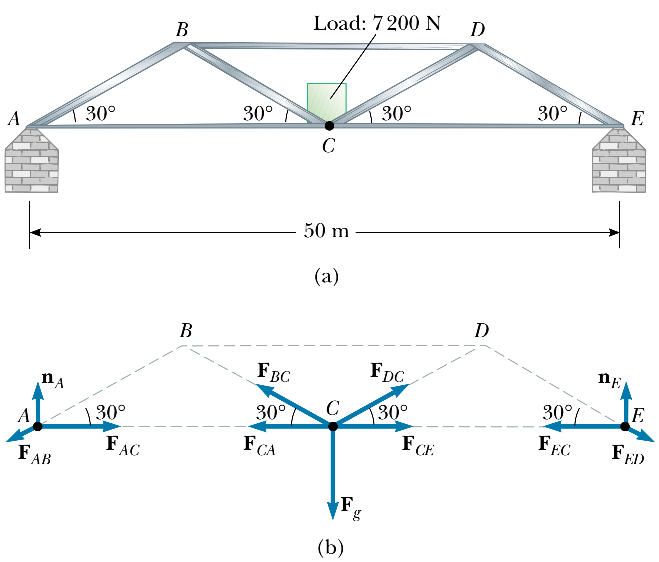


Figure 12.12 (a) Truss structure for a bridge. (b) The forces acting on the pins at points A, C, and E. As an exercise, you should diagram the forces acting on the pin at point B.

$$\sum F_y = n_A + n_E - F_g = 0$$

$$n_A + n_E = 7\,200\text{ N}$$

Next, we calculate the torque about A, noting that the overall length of the bridge structure is $L = 50$ m:

$$\sum \tau = Ln_E - (L/2)F_g = 0$$

$$n_E = F_g/2 = 3\,600\text{ N}$$

Although we could repeat the torque calculation for the right end (point E), it should be clear from symmetry arguments that $n_A = 3\,600$ N.

Now let us balance the vertical forces acting on the pin at point A. If we assume that strut AB is in compression, then the force F_{AB} that the strut exerts on the pin at point A has a negative y component. (If the strut is actually in tension, our calculations will result in a negative value for the magnitude of the force, still of the correct size):

$$\sum F_y = n_A - F_{AB}\sin 30° = 0$$

$$F_{AB} = 7\,200\text{ N}$$

The positive result shows that our assumption of compression was correct.

We can now find the forces acting in the strut between A and C by considering the horizontal forces acting on the pin at point A. Because point A is not accelerating, we can safely assume that F_{AC} must point toward the right (Fig. 12.12b); this indicates that the bar between points A and C is under tension:

$$\sum F_x = F_{AC} - F_{AB}\cos 30° = 0$$

$$F_{AC} = (7\,200\text{ N})\cos 30° = 6\,200\text{ N}$$

Now let us consider the vertical forces acting on the pin at point C. We shall assume that strut BC is in tension. (Imagine the subsequent motion of the pin at point C if strut BC were to break suddenly.) On the basis of symmetry, we assert that $F_{BC} = F_{DC}$ and that $F_{AC} = F_{EC}$:

$$\sum F_y = 2\,F_{BC}\sin 30° - 7\,200\text{ N} = 0$$

$$F_{BC} = 7\,200\text{ N}$$

Finally, we balance the horizontal forces on B, assuming that strut BD is in compression:

$$\sum F_x = F_{AB}\cos 30° + F_{BC}\cos 30° - F_{BD} = 0$$

$$(7\,200\text{ N})\cos 30° + (7\,200\text{ N})\cos 30° - F_{BD} = 0$$

$$F_{BD} = 12\,000\text{ N}$$

Thus, the top bar in a bridge of this design must be very strong.

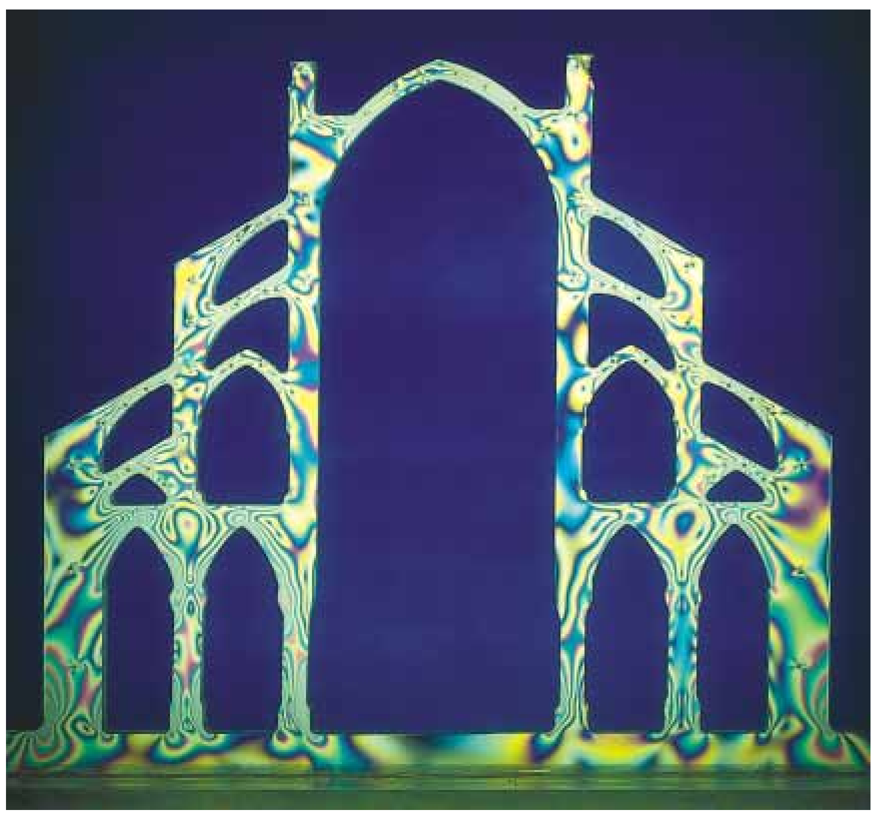
12.4 ELASTIC PROPERTIES OF SOLIDS

In our study of mechanics thus far, we have assumed that objects remain undeformed when external forces act on them. In reality, all objects are deformable. That is, it is possible to change the shape or the size of an object (or both) by applying external forces. As these changes take place, however, internal forces in the object resist the deformation.

We shall discuss the deformation of solids in terms of the concepts of stress and strain. **Stress** is a quantity that is proportional to the force causing a deformation; more specifically, stress is the external force acting on an object per unit cross-sectional area. **Strain** is a measure of the degree of deformation. It is found that, for sufficiently small stresses, **strain is proportional to stress;** the constant of proportionality depends on the material being deformed and on the nature of the deformation. We call this proportionality constant the **elastic modulus.** The elastic modulus is therefore the ratio of the stress to the resulting strain:

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}} \tag{12.5}$$

In a very real sense it is a comparison of what is done to a solid object (a force is applied) and how that object responds (it deforms to some extent).



A plastic model of an arch structure under load conditions. The wavy lines indicate regions where the stresses are greatest. Such models are useful in designing architectural components.

We consider three types of deformation and define an elastic modulus for each:

1. **Young's modulus,** which measures the resistance of a solid to a change in its length
2. **Shear modulus,** which measures the resistance to motion of the planes of a solid sliding past each other
3. **Bulk modulus,** which measures the resistance of solids or liquids to changes in their volume

Young's Modulus: Elasticity in Length

Consider a long bar of cross-sectional area A and initial length L_i that is clamped at one end, as in Figure 12.13. When an external force is applied perpendicular to the cross section, internal forces in the bar resist distortion ("stretching"), but the bar attains an equilibrium in which its length L_f is greater than L_i and in which the external force is exactly balanced by internal forces. In such a situation, the bar is said to be stressed. We define the **tensile stress** as the ratio of the magnitude of the external force F to the cross-sectional area A. The **tensile strain** in this case is defined as the ratio of the change in length ΔL to the original length L_i. We define **Young's modulus** by a combination of these two ratios:

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L_i} \qquad \textbf{(12.6)}$$

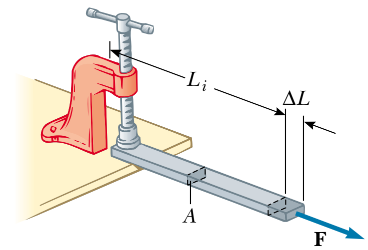


Figure 12.13 A long bar clamped at one end is stretched by an amount ΔL under the action of a force **F**.

Young's modulus

Young's modulus is typically used to characterize a rod or wire stressed under either tension or compression. Note that because strain is a dimensionless quantity, Y has units of force per unit area. Typical values are given in Table 12.1. Experiments show (a) that for a fixed applied force, the change in length is proportional to the original length and (b) that the force necessary to produce a given strain is proportional to the cross-sectional area. Both of these observations are in accord with Equation 12.6.

The **elastic limit** of a substance is defined as the maximum stress that can be applied to the substance before it becomes permanently deformed. It is possible to exceed the elastic limit of a substance by applying a sufficiently large stress, as seen in Figure 12.14. Initially, a stress–strain curve is a straight line. As the stress increases, however, the curve is no longer straight. When the stress exceeds the elas-

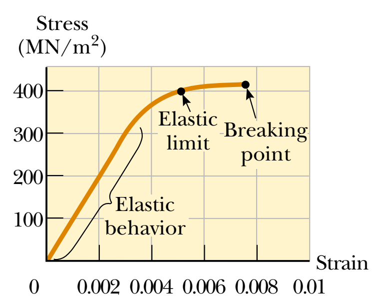


Figure 12.14 Stress-versus-strain curve for an elastic solid.

TABLE 12.1 Typical Values for Elastic Modulus

Substance	Young's Modulus (N/m^2)	Shear Modulus (N/m^2)	Bulk Modulus (N/m^2)
Tungsten	35×10^{10}	14×10^{10}	20×10^{10}
Steel	20×10^{10}	8.4×10^{10}	6×10^{10}
Copper	11×10^{10}	4.2×10^{10}	14×10^{10}
Brass	9.1×10^{10}	3.5×10^{10}	6.1×10^{10}
Aluminum	7.0×10^{10}	2.5×10^{10}	7.0×10^{10}
Glass	$6.5–7.8 \times 10^{10}$	$2.6–3.2 \times 10^{10}$	$5.0–5.5 \times 10^{10}$
Quartz	5.6×10^{10}	2.6×10^{10}	2.7×10^{10}
Water	—	—	0.21×10^{10}
Mercury	—	—	2.8×10^{10}

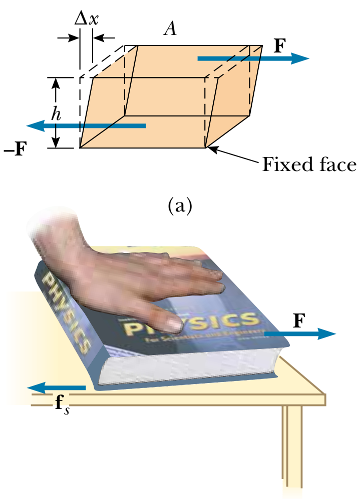


Figure 12.15 (a) A shear deformation in which a rectangular block is distorted by two forces of equal magnitude but opposite directions applied to two parallel faces. (b) A book under shear stress.

tic limit, the object is permanently distorted and does not return to its original shape after the stress is removed. Hence, the shape of the object is permanently changed. As the stress is increased even further, the material ultimately breaks.

Quick Quiz 12.3

What is Young's modulus for the elastic solid whose stress–strain curve is depicted in Figure 12.14?

Quick Quiz 12.4

A material is said to be *ductile* if it can be stressed well beyond its elastic limit without breaking. A *brittle* material is one that breaks soon after the elastic limit is reached. How would you classify the material in Figure 12.14?

Shear Modulus: Elasticity of Shape

Another type of deformation occurs when an object is subjected to a force tangential to one of its faces while the opposite face is held fixed by another force (Fig. 12.15a). The stress in this case is called a shear stress. If the object is originally a rectangular block, a shear stress results in a shape whose cross-section is a parallelogram. A book pushed sideways, as shown in Figure 12.15b, is an example of an object subjected to a shear stress. To a first approximation (for small distortions), no change in volume occurs with this deformation.

We define the **shear stress** as F/A, the ratio of the tangential force to the area A of the face being sheared. The **shear strain** is defined as the ratio $\Delta x/h$, where Δx is the horizontal distance that the sheared face moves and h is the height of the object. In terms of these quantities, the **shear modulus** is

Shear modulus

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h} \qquad \textbf{(12.7)}$$

Values of the shear modulus for some representative materials are given in Table 12.1. The unit of shear modulus is force per unit area.

Bulk Modulus: Volume Elasticity

QuickLab

Estimate the shear modulus for the pages of your textbook. Does the thickness of the book have any effect on the modulus value?

Bulk modulus characterizes the response of a substance to uniform squeezing or to a reduction in pressure when the object is placed in a partial vacuum. Suppose that the external forces acting on an object are at right angles to all its faces, as shown in Figure 12.16, and that they are distributed uniformly over all the faces. As we shall see in Chapter 15, such a uniform distribution of forces occurs when an object is immersed in a fluid. An object subject to this type of deformation undergoes a change in volume but no change in shape. The **volume stress** is defined as the ratio of the magnitude of the normal force F to the area A. The quantity $P = F/A$ is called the **pressure.** If the pressure on an object changes by an amount $\Delta P = \Delta F/A$, then the object will experience a volume change ΔV. The **volume strain** is equal to the change in volume ΔV divided by the initial volume V_i. Thus, from Equation 12.5, we can characterize a volume ("bulk") compression in terms of the **bulk modulus,** which is defined as

Bulk modulus

$$B \equiv \frac{\text{volume stress}}{\text{volume strain}} = -\frac{\Delta F/A}{\Delta V/V_i} = -\frac{\Delta P}{\Delta V/V_i} \qquad \textbf{(12.8)}$$

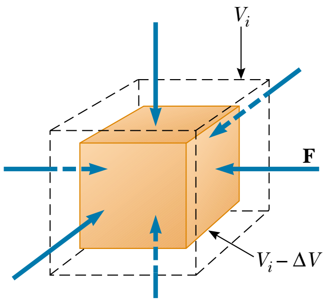


Figure 12.16 When a solid is under uniform pressure, it undergoes a change in volume but no change in shape. This cube is compressed on all sides by forces normal to its six faces.

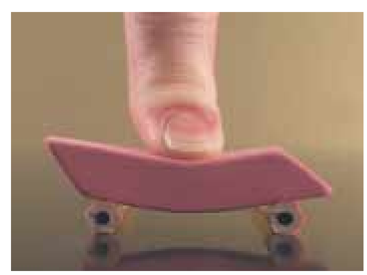
A negative sign is inserted in this defining equation so that B is a positive number. This maneuver is necessary because an increase in pressure (positive ΔP) causes a decrease in volume (negative ΔV) and vice versa.

Table 12.1 lists bulk moduli for some materials. If you look up such values in a different source, you often find that the reciprocal of the bulk modulus is listed. The reciprocal of the bulk modulus is called the **compressibility** of the material.

Note from Table 12.1 that both solids and liquids have a bulk modulus. However, no shear modulus and no Young's modulus are given for liquids because a liquid does not sustain a shearing stress or a tensile stress (it flows instead).

QuickLab

Support a new flat eraser (art gum or Pink Pearl will do) on two parallel pencils at least 3 cm apart. Press down on the middle of the top surface just enough to make the top face of the eraser curve a bit. Is the top face under tension or compression? How about the bottom? Why does a flat slab of concrete supported at the ends tend to crack on the bottom face and not the top?



Prestressed Concrete

If the stress on a solid object exceeds a certain value, the object fractures. The maximum stress that can be applied before fracture occurs depends on the nature of the material and on the type of applied stress. For example, concrete has a tensile strength of about 2×10^6 N/m^2, a compressive strength of 20×10^6 N/m^2, and a shear strength of 2×10^6 N/m^2. If the applied stress exceeds these values, the concrete fractures. It is common practice to use large safety factors to prevent failure in concrete structures.

Concrete is normally very brittle when it is cast in thin sections. Thus, concrete slabs tend to sag and crack at unsupported areas, as shown in Figure 12.17a. The slab can be strengthened by the use of steel rods to reinforce the concrete, as illustrated in Figure 12.17b. Because concrete is much stronger under compression (squeezing) than under tension (stretching) or shear, vertical columns of concrete can support very heavy loads, whereas horizontal beams of concrete tend to sag and crack. However, a significant increase in shear strength is achieved if the reinforced concrete is prestressed, as shown in Figure 12.17c. As the concrete is being poured, the steel rods are held under tension by external forces. The external

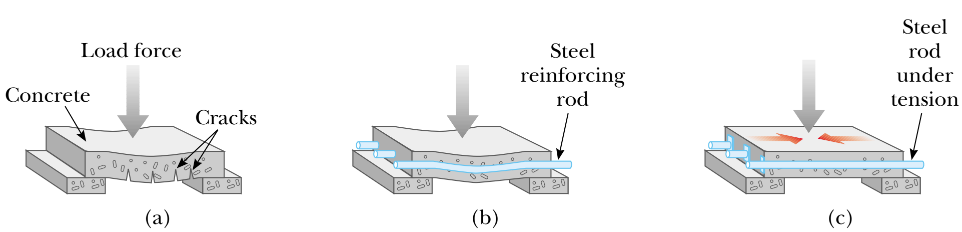


Figure 12.17 (a) A concrete slab with no reinforcement tends to crack under a heavy load. (b) The strength of the concrete is increased by using steel reinforcement rods. (c) The concrete is further strengthened by prestressing it with steel rods under tension.

forces are released after the concrete cures; this results in a permanent tension in the steel and hence a compressive stress on the concrete. This enables the concrete slab to support a much heavier load.

EXAMPLE 12.6 Stage Design

Recall Example 8.10, in which we analyzed a cable used to support an actor as he swung onto the stage. The tension in the cable was 940 N. What diameter should a 10-m-long steel wire have if we do not want it to stretch more than 0.5 cm under these conditions?

Solution From the definition of Young's modulus, we can solve for the required cross-sectional area. Assuming that the cross section is circular, we can determine the diameter of the wire. From Equation 12.6, we have

$$Y = \frac{F/A}{\Delta L/L_i}$$

$$A = \frac{FL_i}{Y\Delta L} = \frac{(940\text{ N})(10\text{ m})}{(20 \times 10^{10}\text{ N/m}^2)(0.005\text{ m})} = 9.4 \times 10^{-6}\text{ m}^2$$

The radius of the wire can be found from $A = \pi r^2$:

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{9.4 \times 10^{-6}\text{ m}^2}{\pi}} = 1.7 \times 10^{-3}\text{ m} = 1.7\text{ mm}$$

$$d = 2r = 2(1.7\text{ mm}) = \boxed{3.4\text{ mm}}$$

To provide a large margin of safety, we would probably use a flexible cable made up of many smaller wires having a total cross-sectional area substantially greater than our calculated value.

EXAMPLE 12.7 Squeezing a Brass Sphere

A solid brass sphere is initially surrounded by air, and the air pressure exerted on it is 1.0×10^5 N/m^2 (normal atmospheric pressure). The sphere is lowered into the ocean to a depth at which the pressure is 2.0×10^7 N/m^2. The volume of the sphere in air is 0.50 m^3. By how much does this volume change once the sphere is submerged?

Solution From the definition of bulk modulus, we have

$$B = -\frac{\Delta P}{\Delta V/V_i}$$

$$\Delta V = -\frac{V_i \Delta P}{B}$$

Because the final pressure is so much greater than the initial pressure, we can neglect the initial pressure and state that $\Delta P = P_f - P_i \approx P_f = 2.0 \times 10^7$ N/m^2. Therefore,

$$\Delta V = -\frac{(0.50\text{ m}^3)(2.0 \times 10^7\text{ N/m}^2)}{6.1 \times 10^{10}\text{ N/m}^2} = \boxed{-1.6 \times 10^{-4}\text{ m}^3}$$

The negative sign indicates a decrease in volume.

SUMMARY

A rigid object is in **equilibrium** if and only if **the resultant external force acting on it is zero and the resultant external torque on it is zero about any axis:**

$$\sum \mathbf{F} = 0 \qquad \textbf{(12.1)}$$

$$\sum \boldsymbol{\tau} = 0 \qquad \textbf{(12.2)}$$

The first condition is the condition for translational equilibrium, and the second is the condition for rotational equilibrium. These two equations allow you to analyze a great variety of problems. Make sure you can identify forces unambiguously, create a free-body diagram, and then apply Equations 12.1 and 12.2 and solve for the unknowns.

The force of gravity exerted on an object can be considered as acting at a single point called the **center of gravity.** The center of gravity of an object coincides with its center of mass if the object is in a uniform gravitational field.

We can describe the elastic properties of a substance using the concepts of stress and strain. **Stress** is a quantity proportional to the force producing a deformation; **strain** is a measure of the degree of deformation. Strain is proportional to stress, and the constant of proportionality is the **elastic modulus:**

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}} \tag{12.5}$$

Three common types of deformation are (1) the resistance of a solid to elongation under a load, characterized by **Young's modulus** Y; (2) the resistance of a solid to the motion of internal planes sliding past each other, characterized by the **shear modulus** S; and (3) the resistance of a solid or fluid to a volume change, characterized by the **bulk modulus** B.

QUESTIONS

1. Can a body be in equilibrium if only one external force acts on it? Explain.
2. Can a body be in equilibrium if it is in motion? Explain.
3. Locate the center of gravity for the following uniform objects: (a) sphere, (b) cube, (c) right circular cylinder.
4. The center of gravity of an object may be located outside the object. Give a few examples for which this is the case.
5. You are given an arbitrarily shaped piece of plywood, together with a hammer, nail, and plumb bob. How could you use these items to locate the center of gravity of the plywood? (*Hint:* Use the nail to suspend the plywood.)
6. For a chair to be balanced on one leg, where must the center of gravity of the chair be located?
7. Can an object be in equilibrium if the only torques acting on it produce clockwise rotation?
8. A tall crate and a short crate of equal mass are placed side by side on an incline (without touching each other). As the incline angle is increased, which crate will topple first? Explain.
9. When lifting a heavy object, why is it recommended to keep the back as vertical as possible, lifting from the knees, rather than bending over and lifting from the waist?
10. Give a few examples in which several forces are acting on a system in such a way that their sum is zero but the system is not in equilibrium.
11. If you measure the net torque and the net force on a system to be zero, (a) could the system still be rotating with respect to you? (b) Could it be translating with respect to you?
12. A ladder is resting inclined against a wall. Would you feel safer climbing up the ladder if you were told that the ground is frictionless but the wall is rough or that the wall is frictionless but the ground is rough? Justify your answer.
13. What kind of deformation does a cube of Jell-O exhibit when it "jiggles"?
14. Ruins of ancient Greek temples often have intact vertical columns, but few horizontal slabs of stone are still in place. Can you think of a reason why this is so?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Section 12.1 The Conditions for Equilibrium

1. A baseball player holds a 36-oz bat (weight = 10.0 N) with one hand at the point O (Fig. P12.1). The bat is in equilibrium. The weight of the bat acts along a line 60.0 cm to the right of O. Determine the force and the torque exerted on the bat by the player.

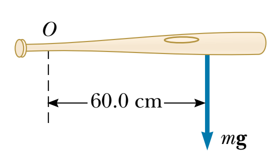


Figure P12.1

2. Write the necessary conditions of equilibrium for the body shown in Figure P12.2. Take the origin of the torque equation at the point O.

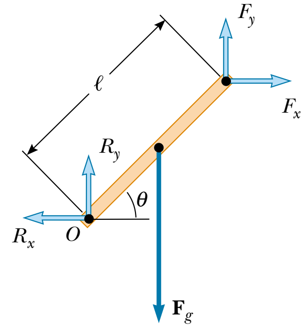


Figure P12.2

WEB **3.** A uniform beam of mass m_b and length ℓ supports blocks of masses m_1 and m_2 at two positions, as shown in Figure P12.3. The beam rests on two points. For what value of x will the beam be balanced at P such that the normal force at O is zero?

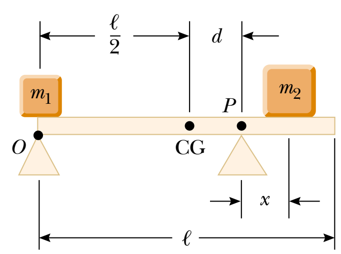


Figure P12.3

4. A student gets his car stuck in a snow drift. Not at a loss, having studied physics, he attaches one end of a stout rope to the vehicle and the other end to the trunk of a nearby tree, allowing for a very small amount of slack. The student then exerts a force $\mathbf{F}$ on the center of the rope in the direction perpendicular to the car–tree line, as shown in Figure P12.4. If the rope is inextensible and if the magnitude of the applied force is 500 N, what is the force on the car? (Assume equilibrium conditions.)

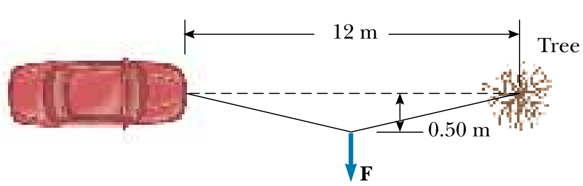


Figure P12.4

Section 12.2 More on the Center of Gravity

5. A 3.00-kg particle is located on the x axis at $x = -5.00$ m, and a 4.00-kg particle is located on the x axis at $x = 3.00$ m. Find the center of gravity of this two-particle system.

6. A circular pizza of radius R has a circular piece of radius $R/2$ removed from one side, as shown in Figure P12.6. Clearly, the center of gravity has moved from C to C' along the x axis. Show that the distance from C to C' is $R/6$. (Assume that the thickness and density of the pizza are uniform throughout.)

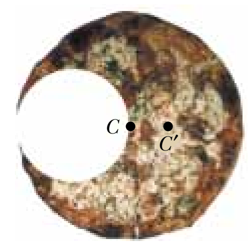


Figure P12.6

7. A carpenter's square has the shape of an L, as shown in Figure P12.7. Locate its center of gravity.

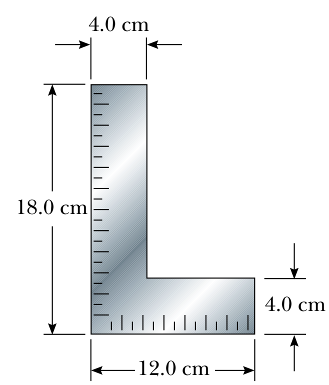


Figure P12.7

8. Pat builds a track for his model car out of wood, as illustrated in Figure P12.8. The track is 5.00 cm wide, 1.00 m high, and 3.00 m long, and it is solid. The runway is cut so that it forms a parabola described by the equation $y = (x - 3)^2/9$. Locate the horizontal position of the center of gravity of this track.

WEB **9.** Consider the following mass distribution: 5.00 kg at (0, 0) m, 3.00 kg at (0, 4.00) m, and 4.00 kg at (3.00, 0) m. Where should a fourth mass of 8.00 kg be placed so that the center of gravity of the four-mass arrangement will be at (0, 0)?

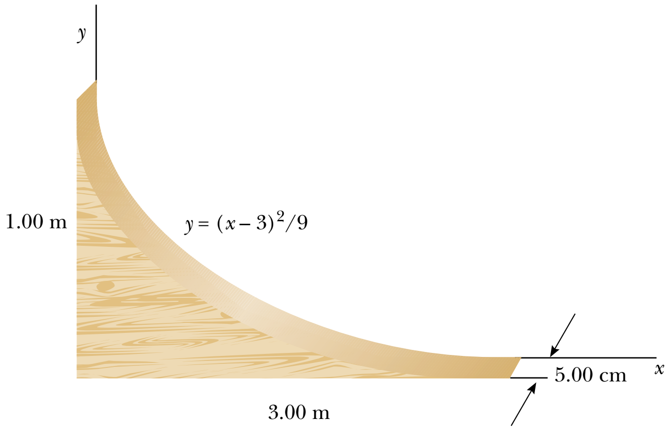


Figure P12.8

10. Figure P12.10 shows three uniform objects: a rod, a right triangle, and a square. Their masses in kilograms and their coordinates in meters are given. Determine the center of gravity for the three-object system.

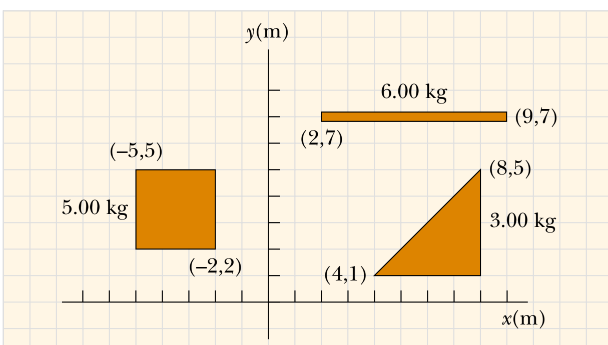


Figure P12.10

Section 12.3 Examples of Rigid Objects in Static Equilibrium

11. Stephen is pushing his sister Joyce in a wheelbarrow when it is stopped by a brick 8.00 cm high (Fig. P12.11). The handles make an angle of 15.0° from the horizontal. A downward force of 400 N is exerted on the wheel, which has a radius of 20.0 cm. (a) What force must Stephen apply along the handles to just start the wheel over the brick? (b) What is the force (magnitude and direction) that the brick exerts on the wheel just as the wheel begins to lift over the brick? Assume in both parts (a) and (b) that the brick remains fixed and does not slide along the ground.

12. Two pans of a balance are 50.0 cm apart. The fulcrum of the balance has been shifted 1.00 cm away from the center by a dishonest shopkeeper. By what percentage is the true weight of the goods being marked up by the shopkeeper? (Assume that the balance has negligible mass.)

Figure P12.11

13. A 15.0-m uniform ladder weighing 500 N rests against a frictionless wall. The ladder makes a 60.0° angle with the horizontal. (a) Find the horizontal and vertical forces that the ground exerts on the base of the ladder when an 800-N firefighter is 4.00 m from the bottom. (b) If the ladder is just on the verge of slipping when the firefighter is 9.00 m up, what is the coefficient of static friction between the ladder and the ground?

14. A uniform ladder of length L and mass m_1 rests against a frictionless wall. The ladder makes an angle θ with the horizontal. (a) Find the horizontal and vertical forces that the ground exerts on the base of the ladder when a firefighter of mass m_2 is a distance x from the bottom. (b) If the ladder is just on the verge of slipping when the firefighter is a distance d from the bottom, what is the coefficient of static friction between the ladder and the ground?

15. Figure P12.15 shows a claw hammer as it is being used to pull a nail out of a horizontal surface. If a force of magnitude 150 N is exerted horizontally as shown, find

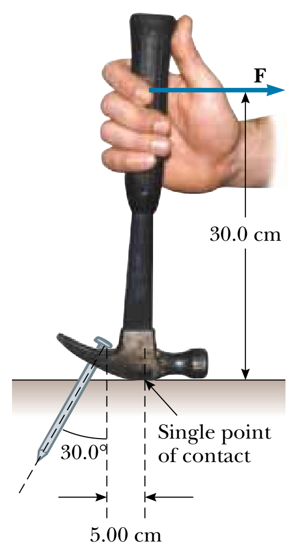


Figure P12.15

(a) the force exerted by the hammer claws on the nail and (b) the force exerted by the surface on the point of contact with the hammer head. Assume that the force the hammer exerts on the nail is parallel to the nail.

16. A uniform plank with a length of 6.00 m and a mass of 30.0 kg rests horizontally across two horizontal bars of a scaffold. The bars are 4.50 m apart, and 1.50 m of the plank hangs over one side of the scaffold. Draw a free-body diagram for the plank. How far can a painter with a mass of 70.0 kg walk on the overhanging part of the plank before it tips?

17. A 1 500-kg automobile has a wheel base (the distance between the axles) of 3.00 m. The center of mass of the automobile is on the center line at a point 1.20 m behind the front axle. Find the force exerted by the ground on each wheel.

18. A vertical post with a square cross section is 10.0 m tall. Its bottom end is encased in a base 1.50 m tall that is precisely square but slightly loose. A force of 5.50 N to the right acts on the top of the post. The base maintains the post in equilibrium. Find the force that the top of the right sidewall of the base exerts on the post. Find the force that the bottom of the left sidewall of the base exerts on the post.

19. A flexible chain weighing 40.0 N hangs between two hooks located at the same height (Fig. P12.19). At each hook, the tangent to the chain makes an angle $\theta = 42.0°$ with the horizontal. Find (a) the magnitude of the force each hook exerts on the chain and (b) the tension in the chain at its midpoint. (*Hint:* For part (b), make a free-body diagram for half the chain.)

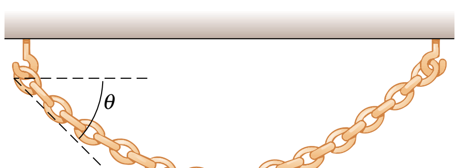


Figure P12.19

20. A hemispherical sign 1.00 m in diameter and of uniform mass density is supported by two strings, as shown in Figure P12.20. What fraction of the sign's weight is supported by each string?

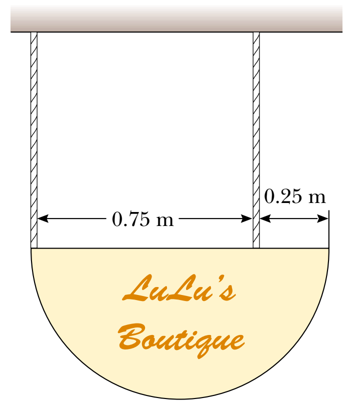


Figure P12.20

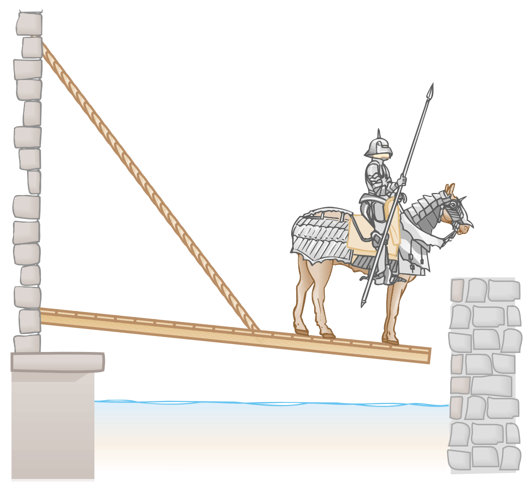


Figure P12.21

21. Sir Lost-a-Lot dons his armor and sets out from the castle on his trusty steed in his quest to improve communication between damsels and dragons (Fig. P12.21). Unfortunately, his squire lowered the draw bridge too far and finally stopped lowering it when it was 20.0° below the horizontal. Lost-a-Lot and his horse stop when their combined center of mass is 1.00 m from the end of the bridge. The bridge is 8.00 m long and has a mass of 2 000 kg. The lift cable is attached to the bridge 5.00 m from the hinge at the castle end and to a point on the castle wall 12.0 m above the bridge. Lost-a-Lot's mass combined with that of his armor and steed is 1 000 kg. Determine (a) the tension in the cable, as well as (b) the horizontal and (c) the vertical force components acting on the bridge at the hinge.

22. Two identical, uniform bricks of length L are placed in a stack over the edge of a horizontal surface such that the maximum possible overhang without falling is achieved, as shown in Figure P12.22. Find the distance x.

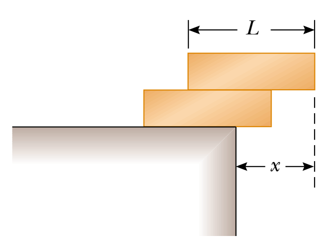


Figure P12.22

23. A vaulter holds a 29.4-N pole in equilibrium by exerting an upward force **U** with her leading hand and a downward force **D** with her trailing hand, as shown in Figure P12.23. Point C is the center of gravity of the pole. What are the magnitudes of **U** and **D**?

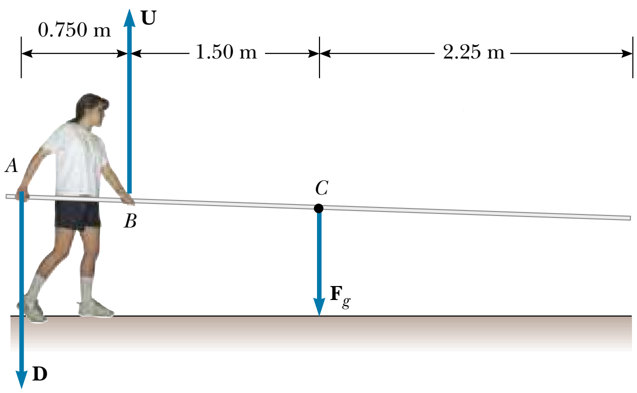


Figure P12.23

Section 12.4 Elastic Properties of Solids

24. Assume that Young's modulus for bone is 1.50×10^{10} N/m^2 and that a bone will fracture if more than 1.50×10^8 N/m^2 is exerted. (a) What is the maximum force that can be exerted on the femur bone in the leg if it has a minimum effective diameter of 2.50 cm? (b) If a force of this magnitude is applied compressively, by how much does the 25.0-cm-long bone shorten?

25. A 200-kg load is hung on a wire with a length of 4.00 m, a cross-sectional area of 0.200×10^{-4} m^2, and a Young's modulus of 8.00×10^{10} N/m^2. What is its increase in length?

26. A steel wire 1 mm in diameter can support a tension of 0.2 kN. Suppose you need a cable made of these wires to support a tension of 20 kN. The cable's diameter should be of what order of magnitude?

27. A child slides across a floor in a pair of rubber-soled shoes. The frictional force acting on each foot is 20.0 N. The footprint area of each shoe's sole is 14.0 cm^2, and the thickness of each sole is 5.00 mm. Find the horizontal distance by which the upper and lower surfaces of each sole are offset. The shear modulus of the rubber is 3.00×10^6 N/m^2.

28. **Review Problem.** A 30.0-kg hammer strikes a steel spike 2.30 cm in diameter while moving with a speed of 20.0 m/s. The hammer rebounds with a speed of 10.0 m/s after 0.110 s. What is the average strain in the spike during the impact?

29. If the elastic limit of copper is 1.50×10^8 N/m^2, determine the minimum diameter a copper wire can have under a load of 10.0 kg if its elastic limit is not to be exceeded.

30. **Review Problem.** A 2.00-m-long cylindrical steel wire with a cross-sectional diameter of 4.00 mm is placed over a light frictionless pulley, with one end of the wire connected to a 5.00-kg mass and the other end connected to a 3.00-kg mass. By how much does the wire stretch while the masses are in motion?

31. **Review Problem.** A cylindrical steel wire of length L_i with a cross-sectional diameter d is placed over a light frictionless pulley, with one end of the wire connected to a mass m_1 and the other end connected to a mass m_2. By how much does the wire stretch while the masses are in motion?

32. Calculate the density of sea water at a depth of 1 000 m, where the water pressure is about 1.00×10^7 N/m^2. (The density of sea water is 1.030×10^3 kg/m^3 at the surface.)

WEB 33. If the shear stress exceeds about 4.00×10^8 N/m^2, steel ruptures. Determine the shearing force necessary (a) to shear a steel bolt 1.00 cm in diameter and (b) to punch a 1.00-cm-diameter hole in a steel plate 0.500 cm thick.

34. (a) Find the minimum diameter of a steel wire 18.0 m long that elongates no more than 9.00 mm when a load of 380 kg is hung on its lower end. (b) If the elastic limit for this steel is 3.00×10^8 N/m^2, does permanent deformation occur with this load?

35. When water freezes, it expands by about 9.00%. What would be the pressure increase inside your automobile's engine block if the water in it froze? (The bulk modulus of ice is 2.00×10^9 N/m^2.)

36. For safety in climbing, a mountaineer uses a 50.0-m nylon rope that is 10.0 mm in diameter. When supporting the 90.0-kg climber on one end, the rope elongates by 1.60 m. Find Young's modulus for the rope material.

ADDITIONAL PROBLEMS

37. A bridge with a length of 50.0 m and a mass of 8.00×10^4 kg is supported on a smooth pier at each end, as illustrated in Figure P12.37. A truck of mass 3.00×10^4 kg

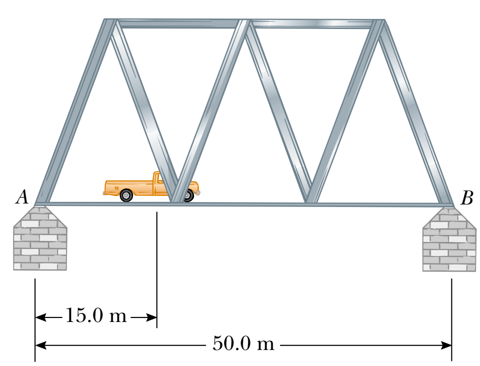


Figure P12.37

is located 15.0 m from one end. What are the forces on the bridge at the points of support?

38. A frame in the shape of the letter A is formed from two uniform pieces of metal, each of weight 26.0 N and length 1.00 m. They are hinged at the top and held together by a horizontal wire 1.20 m in length (Fig. P12.38). The structure rests on a frictionless surface. If the wire is connected at points a distance of 0.650 m from the top of the frame, determine the tension in the wire.

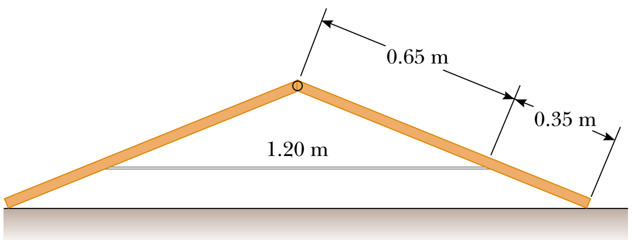


Figure P12.38

39. Refer to Figure 12.17c. A lintel of prestressed reinforced concrete is 1.50 m long. The cross-sectional area of the concrete is 50.0 cm^2. The concrete encloses one steel reinforcing rod with a cross-sectional area of 1.50 cm^2. The rod joins two strong end plates. Young's modulus for the concrete is $30.0 \times 10^9\ \text{N/m}^2$. After the concrete cures and the original tension T_1 in the rod is released, the concrete will be under a compressive stress of $8.00 \times 10^6\ \text{N/m}^2$. (a) By what distance will the rod compress the concrete when the original tension in the rod is released? (b) Under what tension T_2 will the rod still be? (c) How much longer than its unstressed length will the rod then be? (d) When the concrete was poured, the rod should have been stretched by what extension distance from its unstressed length? (e) Find the required original tension T_1 in the rod.

40. A solid sphere of radius R and mass M is placed in a trough, as shown in Figure P12.40. The inner surfaces of the trough are frictionless. Determine the forces exerted by the trough on the sphere at the two contact points.

41. A 10.0-kg monkey climbs up a 120-N uniform ladder of length L, as shown in Figure P12.41. The upper and lower ends of the ladder rest on frictionless surfaces. The lower end is fastened to the wall by a horizontal rope that can support a maximum tension of 110 N. (a) Draw a free-body diagram for the ladder. (b) Find the tension in the rope when the monkey is one third the way up the ladder. (c) Find the maximum distance d that the monkey can climb up the ladder before the rope breaks. Express your answer as a fraction of L.

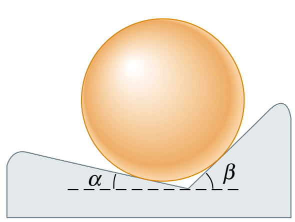


Figure P12.40

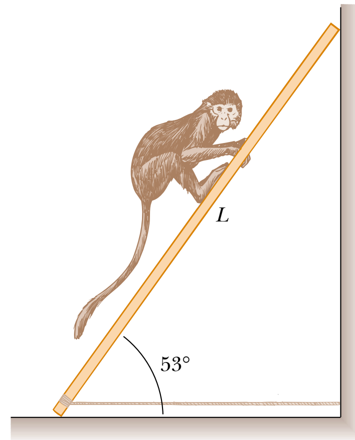


Figure P12.41

42. A hungry bear weighing 700 N walks out on a beam in an attempt to retrieve a basket of food hanging at the end of the beam (Fig. P12.42). The beam is uniform, weighs 200 N, and is 6.00 m long; the basket weighs 80.0 N. (a) Draw a free-body diagram for the beam. (b) When the bear is at $x = 1.00$ m, find the tension in the wire and the components of the force exerted by the wall on the left end of the beam. (c) If the wire can withstand a maximum tension of 900 N, what is the maximum distance that the bear can walk before the wire breaks?

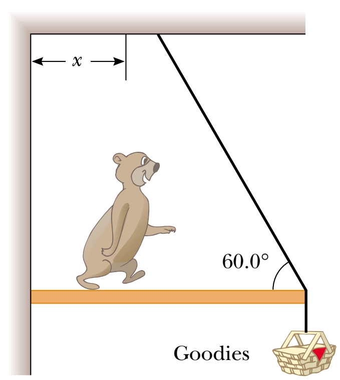


Figure P12.42

43. Old MacDonald had a farm, and on that farm he had a gate (Fig. P12.43). The gate is 3.00 m wide and 1.80 m

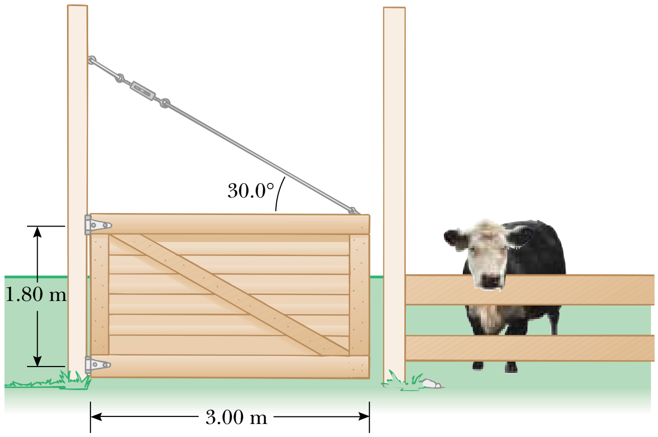


Figure P12.43

high, with hinges attached to the top and bottom. The guy wire makes an angle of 30.0° with the top of the gate and is tightened by a turn buckle to a tension of 200 N. The mass of the gate is 40.0 kg. (a) Determine the horizontal force exerted on the gate by the bottom hinge. (b) Find the horizontal force exerted by the upper hinge. (c) Determine the combined vertical force exerted by both hinges. (d) What must the tension in the guy wire be so that the horizontal force exerted by the upper hinge is zero?

44. A 1 200-N uniform boom is supported by a cable, as illustrated in Figure P12.44. The boom is pivoted at the bottom, and a 2 000-N object hangs from its top. Find the tension in the cable and the components of the reaction force exerted on the boom by the floor.

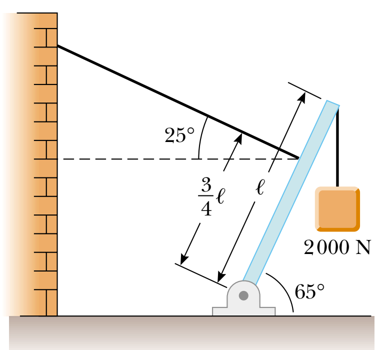


Figure P12.44

WEB **45.** A uniform sign of weight F_g and width $2L$ hangs from a light, horizontal beam hinged at the wall and supported by a cable (Fig. P12.45). Determine (a) the tension in the cable and (b) the components of the reaction force exerted by the wall on the beam in terms of F_g, d, L, and θ.

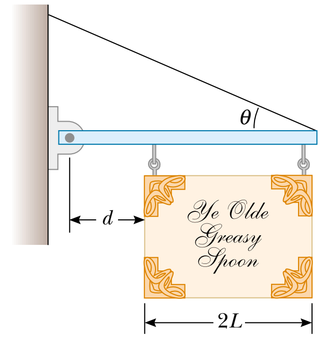


Figure P12.45

46. A crane of mass 3 000 kg supports a load of 10 000 kg as illustrated in Figure P12.46. The crane is pivoted with a frictionless pin at A and rests against a smooth support at B. Find the reaction forces at A and B.

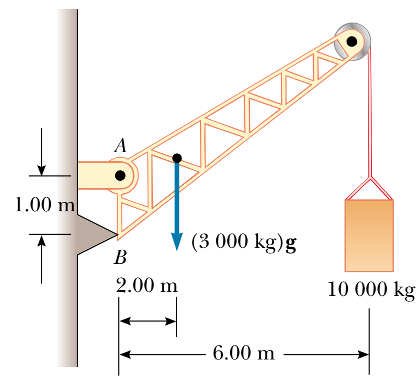


Figure P12.46

47. A ladder having a uniform density and a mass m rests against a frictionless vertical wall, making an angle 60.0° with the horizontal. The lower end rests on a flat surface, where the coefficient of static friction is $\mu_s = 0.400$. A window cleaner having a mass $M = 2m$ attempts to climb the ladder. What fraction of the length L of the ladder will the worker have reached when the ladder begins to slip?

48. A uniform ladder weighing 200 N is leaning against a wall (see Fig. 12.10). The ladder slips when $\theta = 60.0°$. Assuming that the coefficients of static friction at the wall and the ground are the same, obtain a value for μ_s.

49. A 10 000-N shark is supported by a cable attached to a 4.00-m rod that can pivot at its base. Calculate the tension in the tie-rope between the wall and the rod if it is holding the system in the position shown in Figure P12.49. Find the horizontal and vertical forces exerted on the base of the rod. (Neglect the weight of the rod.)

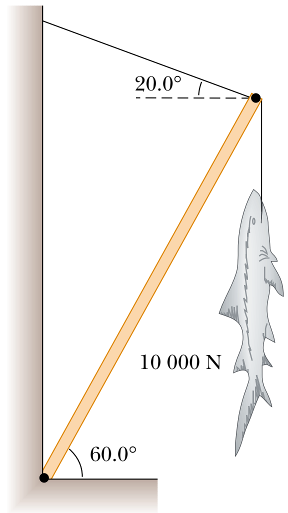


Figure P12.49

50. When a person stands on tiptoe (a strenuous position), the position of the foot is as shown in Figure P12.50a. The total weight of the body $\mathbf{F}_g$ is supported by the force $\mathbf{n}$ exerted by the floor on the toe. A mechanical model for the situation is shown in Figure P12.50b, where $\mathbf{T}$ is the force exerted by the Achilles tendon on the foot and $\mathbf{R}$ is the force exerted by the tibia on the foot. Find the values of T, R, and θ when $F_g = 700$ N.

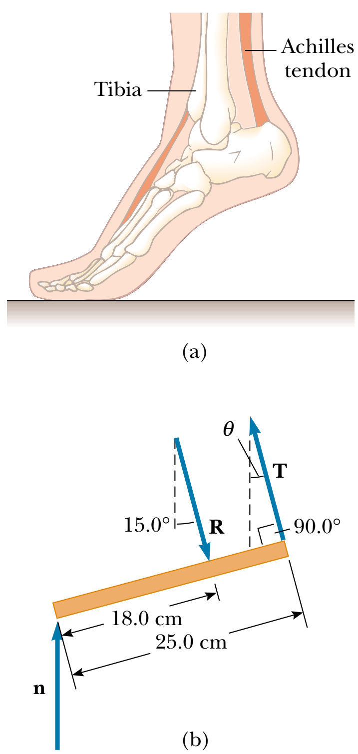


Figure P12.50

51. A person bends over and lifts a 200-N object as shown in Figure P12.51a, with his back in a horizontal position (a terrible way to lift an object). The back muscle attached at a point two thirds the way up the spine maintains the position of the back, and the angle between the spine and this muscle is 12.0°. Using the mechanical model shown in Figure P12.51b and taking the weight of the upper body to be 350 N, find the tension in the back muscle and the compressional force in the spine.

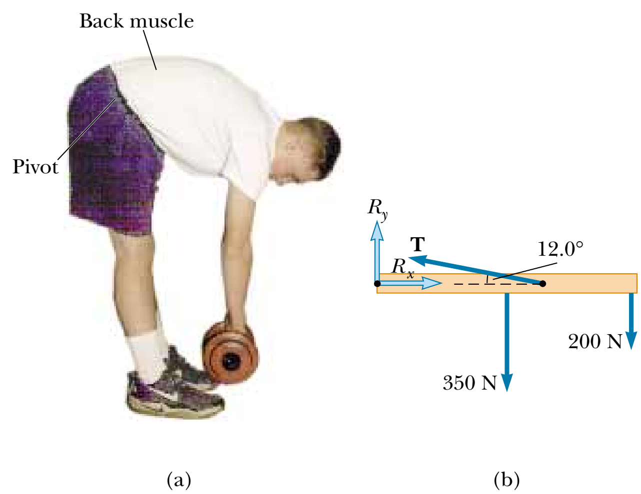


Figure P12.51

52. Two 200-N traffic lights are suspended from a single cable, as shown in Figure 12.52. Neglecting the cable's weight, (a) prove that if $\theta_1 = \theta_2$, then $T_1 = T_2$. (b) Determine the three tensions T_1, T_2, and T_3 if $\theta_1 = \theta_2 = 8.00°$.

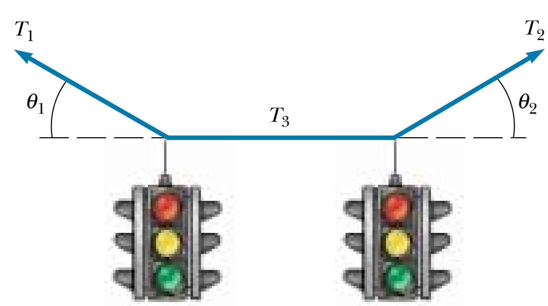


Figure P12.52

53. A force acts on a rectangular cabinet weighing 400 N, as illustrated in Figure P12.53. (a) If the cabinet slides with constant speed when $F = 200$ N and $h = 0.400$ m, find the coefficient of kinetic friction and the position of the resultant normal force. (b) If $F = 300$ N, find the value of h for which the cabinet just begins to tip.

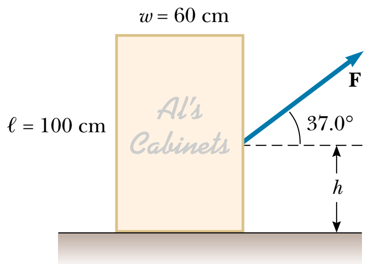


Figure P12.53 Problems 53 and 54.

54. Consider the rectangular cabinet of Problem 53, but with a force **F** applied horizontally at its upper edge. (a) What is the minimum force that must be applied for the cabinet to start tipping? (b) What is the minimum coefficient of static friction required to prevent the cabinet from sliding with the application of a force of this magnitude? (c) Find the magnitude and direction of the minimum force required to tip the cabinet if the point of application can be chosen anywhere on it.

55. A uniform rod of weight F_g and length L is supported at its ends by a frictionless trough, as shown in Figure P12.55. (a) Show that the center of gravity of the rod is directly over point O when the rod is in equilibrium. (b) Determine the equilibrium value of the angle θ.

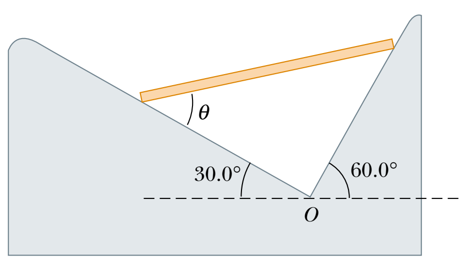


Figure P12.55

56. Review Problem. A cue stick strikes a cue ball and delivers a horizontal impulse in such a way that the ball rolls without slipping as it starts to move. At what height above the ball's center (in terms of the radius of the ball) was the blow struck?

57. A uniform beam of mass m is inclined at an angle θ to the horizontal. Its upper end produces a 90° bend in a very rough rope tied to a wall, and its lower end rests on a rough floor (Fig. P12.57). (a) If the coefficient of static friction between the beam and the floor is μ_s, determine an expression for the maximum mass M that can be suspended from the top before the beam slips. (b) Determine the magnitude of the reaction force at the floor and the magnitude of the force exerted by the beam on the rope at P in terms of m, M, and μ_s.

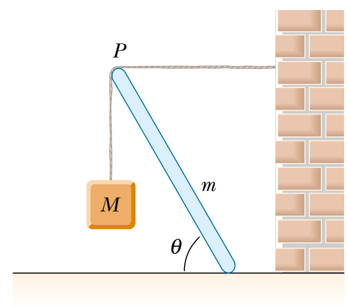


Figure P12.57

58. Figure P12.58 shows a truss that supports a downward force of 1 000 N applied at the point B. The truss has negligible weight. The piers at A and C are smooth. (a) Apply the conditions of equilibrium to prove that $n_A = 366$ N and that $n_C = 634$ N. (b) Show that, because forces act on the light truss only at the hinge joints, each bar of the truss must exert on each hinge pin only a force along the length of that bar—a force of tension or compression. (c) Find the force of tension or compression in each of the three bars.

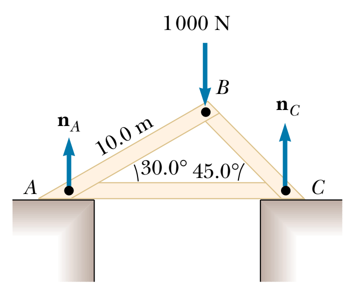


Figure P12.58

59. A stepladder of negligible weight is constructed as shown in Figure P12.59. A painter with a mass of 70.0 kg stands on the ladder 3.00 m from the bottom. Assuming that the floor is frictionless, find (a) the tension in the horizontal bar connecting the two halves of the ladder, (b) the normal forces at A and B, and (c) the components of the reaction force at the single hinge C that the left half of the ladder exerts on the right half. (*Hint:* Treat each half of the ladder separately.)

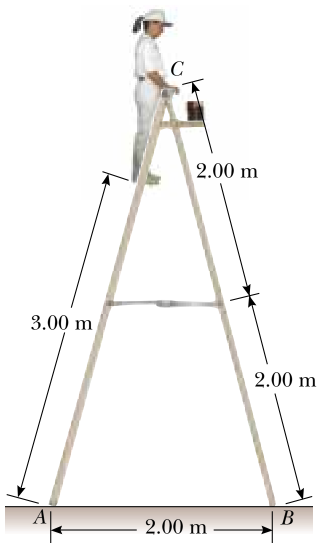


Figure P12.59

60. A flat dance floor of dimensions 20.0 m by 20.0 m has a mass of 1 000 kg. Three dance couples, each of mass 125 kg, start in the top left, top right, and bottom left corners. (a) Where is the initial center of gravity? (b) The couple in the bottom left corner moves 10.0 m to the right. Where is the new center of gravity? (c) What was the average velocity of the center of gravity if it took that couple 8.00 s to change position?

61. A shelf bracket is mounted on a vertical wall by a single screw, as shown in Figure P12.61. Neglecting the weight of the bracket, find the horizontal component of the force that the screw exerts on the bracket when an 80.0-N vertical force is applied as shown. (*Hint:* Imagine that the bracket is slightly loose.)

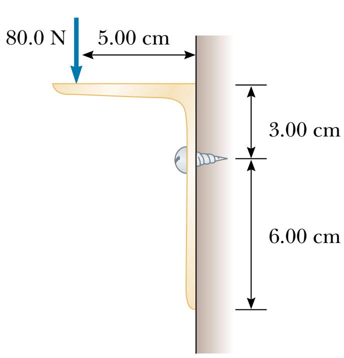


Figure P12.61

62. Figure P12.62 shows a vertical force applied tangentially to a uniform cylinder of weight F_g. The coefficient of

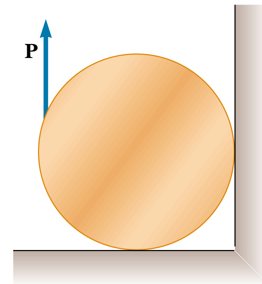


Figure P12.62

static friction between the cylinder and all surfaces is 0.500. In terms of F_g, find the maximum force **P** that can be applied that does not cause the cylinder to rotate. (*Hint:* When the cylinder is on the verge of slipping, both friction forces are at their maximum values. Why?)

WEB **63.** **Review Problem.** A wire of length L_i, Young's modulus Y, and cross-sectional area A is stretched elastically by an amount ΔL. According to Hooke's law, the restoring force is $-k\,\Delta L$. (a) Show that $k = YA/L_i$. (b) Show that the work done in stretching the wire by an amount ΔL is $W = YA(\Delta L)^2/2L_i$.

64. Two racquetballs are placed in a glass jar, as shown in Figure P12.64. Their centers and the point A lie on a straight line. (a) Assuming that the walls are frictionless, determine P_1, P_2, and P_3. (b) Determine the magnitude of the force exerted on the right ball by the left ball. Assume each ball has a mass of 170 g.

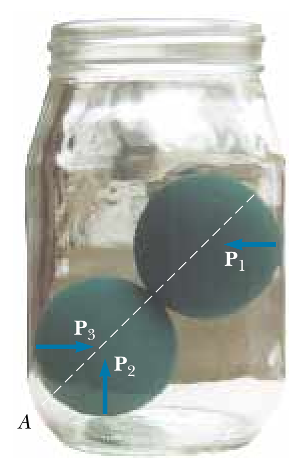


Figure P12.64

65. In Figure P12.65, the scales read $F_{g1} = 380$ N and $F_{g2} = 320$ N. Neglecting the weight of the supporting plank,

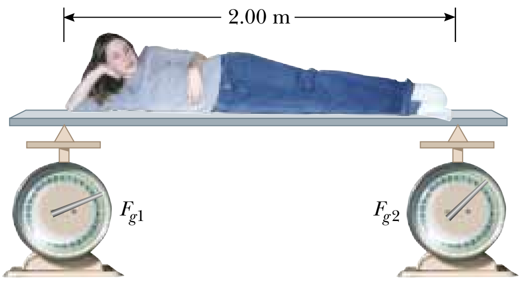


Figure P12.65

how far from the woman's feet is her center of mass, given that her height is 2.00 m?

66. A steel cable 3.00 cm^2 in cross-sectional area has a mass of 2.40 kg per meter of length. If 500 m of the cable is hung over a vertical cliff, how much does the cable stretch under its own weight? (For Young's modulus for steel, refer to Table 12.1.)

67. (a) Estimate the force with which a karate master strikes a board if the hand's speed at time of impact is 10.0 m/s and decreases to 1.00 m/s during a 0.002 00-s time-of-contact with the board. The mass of coordinated hand-and-arm is 1.00 kg. (b) Estimate the shear stress if this force is exerted on a 1.00-cm-thick pine board that is 10.0 cm wide. (c) If the maximum shear stress a pine board can receive before breaking is 3.60×10^6 N/m^2, will the board break?

68. A bucket is made from thin sheet metal. The bottom and top of the bucket have radii of 25.0 cm and 35.0 cm, respectively. The bucket is 30.0 cm high and filled with water. Where is the center of gravity? (Ignore the weight of the bucket itself.)

69. **Review Problem.** A trailer with a loaded weight of F_g is being pulled by a vehicle with a force **P**, as illustrated in Figure P12.69. The trailer is loaded such that its center of mass is located as shown. Neglect the force of rolling friction and let a represent the x component of the acceleration of the trailer. (a) Find the vertical component of **P** in terms of the given parameters. (b) If $a = 2.00$ m/s^2 and $h = 1.50$ m, what must be the value of d so that $P_y = 0$ (that is, no vertical load on the vehicle)? (c) Find the values of P_x and P_y given that $F_g = 1\,500$ N, $d = 0.800$ m, $L = 3.00$ m, $h = 1.50$ m, and $a = -2.00$ m/s^2.

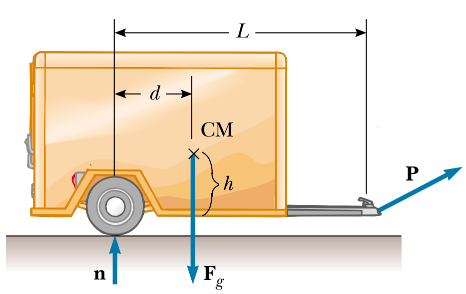


Figure P12.69

70. **Review Problem.** An aluminum wire is 0.850 m long and has a circular cross section of diameter 0.780 mm. Fixed at the top end, the wire supports a 1.20-kg mass that swings in a horizontal circle. Determine the angular velocity required to produce strain 1.00×10^{-3}.

71. A 200-m-long bridge truss extends across a river (Fig. P12.71). Calculate the force of tension or compression in each structural component when a 1 360-kg car is at the center of the bridge. Assume that the structure is free to slide horizontally to permit thermal expansion and contraction, that the structural components are connected by pin joints, and that the masses of the structural components are small compared with the mass of the car.

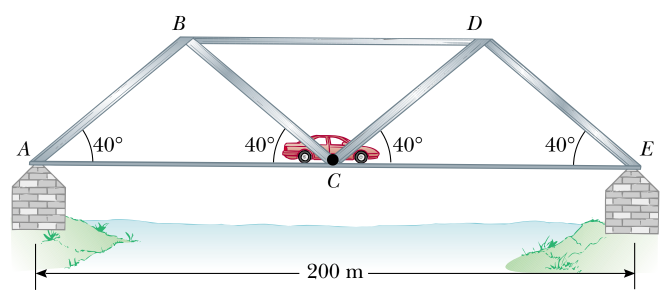


Figure P12.71

72. A 100-m-long bridge truss is supported at its ends so that it can slide freely (Fig. P12.72). A 1 500-kg car is halfway between points A and C. Show that the weight of the car is evenly distributed between points A and C, and calculate the force in each structural component. Specify whether each structural component is under tension or compression. Assume that the structural components are connected by pin joints and that the masses of the components are small compared with the mass of the car.

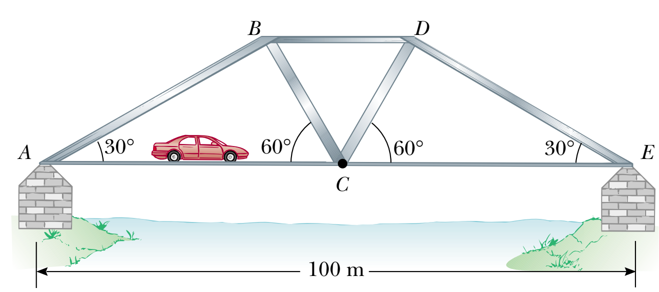


Figure P12.72

ANSWERS TO QUICK QUIZZES

12.1 (a) Yes, as Figure 12.3 shows. The unbalanced torques cause an angular acceleration even though the linear acceleration is zero. (b) Yes, again. This happens when the lines of action of all the forces intersect at a common point. If a net force acts on the object, then the object has a translational acceleration. However, because there is no net torque on the object, the object has no angular acceleration. There are other instances in which torques cancel but the forces do not. You should be able to draw at least two.

12.2 The location of the board's center of gravity relative to the fulcrum.

12.3 Young's modulus is given by the ratio of stress to strain, which is the slope of the elastic behavior section of the graph in Figure 12.14. Reading from the graph, we note that a stress of approximately $3 \times 10^8\ \text{N/m}^2$ results in a strain of 0.003. The slope, and hence Young's modulus, are therefore $10 \times 10^{10}\ \text{N/m}^2$.

12.4 A substantial part of the graph extends beyond the elastic limit, indicating permanent deformation. Thus, the material is ductile.

PUZZLER

Inside the pocket watch is a small disk (called a torsional pendulum) that oscillates back and forth at a very precise rate and controls the watch gears. A grandfather clock keeps accurate time because of its pendulum. The tall wooden case provides the space needed by the long pendulum as it advances the clock gears with each swing. In both of these timepieces, the vibration of a carefully shaped component is critical to accurate operation. What properties of oscillating objects make them so useful in timing devices? *(Photograph of pocket watch, George Semple; photograph of grandfather clock, Charles D. Winters)*

chapter

13

Oscillatory Motion

Chapter Outline

A very special kind of motion occurs when the force acting on a body is proportional to the displacement of the body from some equilibrium position. If this force is always directed toward the equilibrium position, repetitive back-and-forth motion occurs about this position. Such motion is called *periodic motion, harmonic motion, oscillation,* or *vibration* (the four terms are completely equivalent).

You are most likely familiar with several examples of periodic motion, such as the oscillations of a block attached to a spring, the swinging of a child on a playground swing, the motion of a pendulum, and the vibrations of a stringed musical instrument. In addition to these everyday examples, numerous other systems exhibit periodic motion. For example, the molecules in a solid oscillate about their equilibrium positions; electromagnetic waves, such as light waves, radar, and radio waves, are characterized by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage, current, and electrical charge vary periodically with time.

Most of the material in this chapter deals with *simple harmonic motion,* in which an object oscillates such that its position is specified by a sinusoidal function of time with no loss in mechanical energy. In real mechanical systems, damping (frictional) forces are often present. These forces are considered in optional Section 13.6 at the end of this chapter.

13.1 SIMPLE HARMONIC MOTION

8.10 Consider a physical system that consists of a block of mass m attached to the end of a spring, with the block free to move on a horizontal, frictionless surface (Fig. 13.1). When the spring is neither stretched nor compressed, the block is at the position $x = 0$, called the *equilibrium position* of the system. We know from experience that such a system oscillates back and forth if disturbed from its equilibrium position.

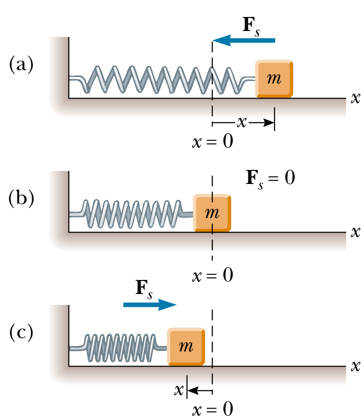


Figure 13.1 A block attached to a spring moving on a frictionless surface. (a) When the block is displaced to the right of equilibrium ($x > 0$), the force exerted by the spring acts to the left. (b) When the block is at its equilibrium position ($x = 0$), the force exerted by the spring is zero. (c) When the block is displaced to the left of equilibrium ($x < 0$), the force exerted by the spring acts to the right.

We can understand the motion in Figure 13.1 qualitatively by first recalling that when the block is displaced a small distance x from equilibrium, the spring exerts on the block a force that is proportional to the displacement and given by Hooke's law (see Section 7.3):

$$F_s = -kx \qquad \textbf{(13.1)}$$

We call this a **restoring force** because it is is always directed toward the equilibrium position and therefore *opposite* the displacement. That is, when the block is displaced to the right of $x = 0$ in Figure 13.1, then the displacement is positive and the restoring force is directed to the left. When the block is displaced to the left of $x = 0$, then the displacement is negative and the restoring force is directed to the right.

Applying Newton's second law to the motion of the block, together with Equation 13.1, we obtain

$$F_s = -kx = ma$$

$$a = -\frac{k}{m}x \qquad \textbf{(13.2)}$$

That is, the acceleration is proportional to the displacement of the block, and its direction is opposite the direction of the displacement. Systems that behave in this way are said to exhibit **simple harmonic motion. An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.**

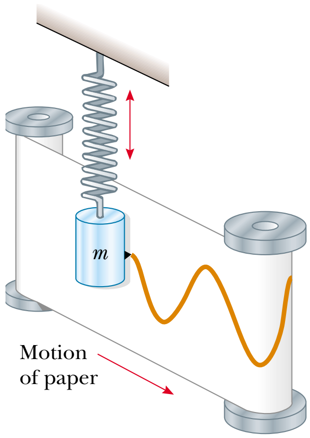


Figure 13.2 An experimental apparatus for demonstrating simple harmonic motion. A pen attached to the oscillating mass traces out a wavelike pattern on the moving chart paper.

An experimental arrangement that exhibits simple harmonic motion is illustrated in Figure 13.2. A mass oscillating vertically on a spring has a pen attached to it. While the mass is oscillating, a sheet of paper is moved perpendicular to the direction of motion of the spring, and the pen traces out a wavelike pattern.

8.2 & 8.3

In general, a particle moving along the x axis exhibits simple harmonic motion when x, the particle's displacement from equilibrium, varies in time according to the relationship

Displacement versus time for simple harmonic motion

$$x = A\cos(\omega t + \phi) \tag{13.3}$$

where A, ω, and ϕ are constants. To give physical significance to these constants, we have labeled a plot of x as a function of t in Figure 13.3a. This is just the pattern that is observed with the experimental apparatus shown in Figure 13.2. The **amplitude** A of the motion is the maximum displacement of the particle in either the positive or negative x direction. The constant ω is called the **angular frequency** of the motion and has units of radians per second. (We shall discuss the geometric significance of ω in Section 13.2.) The constant angle ϕ, called the **phase constant** (or phase angle), is determined by the initial displacement and velocity of the particle. If the particle is at its maximum position $x = A$ at $t = 0$, then $\phi = 0$ and the curve of x versus t is as shown in Figure 13.3b. If the particle is at some other position at $t = 0$, the constants ϕ and A tell us what the position was at time $t = 0$. The quantity $(\omega t + \phi)$ is called the **phase** of the motion and is useful in comparing the motions of two oscillators.

Note from Equation 13.3 that the trigonometric function x is *periodic* and repeats itself every time ωt increases by 2π rad. **The period T of the motion is the time it takes for the particle to go through one full cycle.** We say that the particle has made *one oscillation.* This definition of T tells us that the value of x at time t equals the value of x at time $t + T$. We can show that $T = 2\pi/\omega$ by using the preceding observation that the phase $(\omega t + \phi)$ increases by 2π rad in a time T:

$$\omega t + \phi + 2\pi = \omega(t + T) + \phi$$

Hence, $\omega T = 2\pi$, or

$$T = \frac{2\pi}{\omega} \tag{13.4}$$

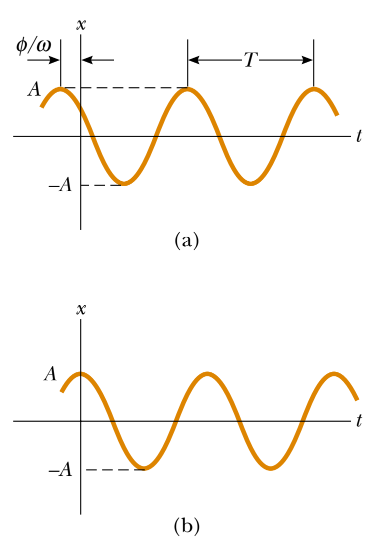


Figure 13.3 (a) An x–t curve for a particle undergoing simple harmonic motion. The amplitude of the motion is A, the period is T, and the phase constant is ϕ. (b) The x–t curve in the special case in which $x = A$ at $t = 0$ and hence $\phi = 0$.

The inverse of the period is called the **frequency** f of the motion. **The frequency represents the number of oscillations that the particle makes per unit time:**

Frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \textbf{(13.5)}$$

The units of f are cycles per second $= s^{-1}$, or **hertz** (Hz).

Rearranging Equation 13.5, we obtain the angular frequency:

Angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \textbf{(13.6)}$$

Quick Quiz 13.1

What would the phase constant ϕ have to be in Equation 13.3 if we were describing an oscillating object that happened to be at the origin at $t = 0$?

Quick Quiz 13.2

An object undergoes simple harmonic motion of amplitude A. Through what total distance does the object move during one complete cycle of its motion? (a) $A/2$. (b) A. (c) $2A$. (d) $4A$.

We can obtain the linear velocity of a particle undergoing simple harmonic motion by differentiating Equation 13.3 with respect to time:

Velocity in simple harmonic motion

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \quad \textbf{(13.7)}$$

The acceleration of the particle is

Acceleration in simple harmonic motion

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) \quad \textbf{(13.8)}$$

Because $x = A\cos(\omega t + \phi)$, we can express Equation 13.8 in the form

$$a = -\omega^2 x \quad \textbf{(13.9)}$$

From Equation 13.7 we see that, because the sine function oscillates between ± 1, the extreme values of v are $\pm \omega A$. Because the cosine function also oscillates between ± 1, Equation 13.8 tells us that the extreme values of a are $\pm \omega^2 A$. Therefore, the maximum speed and the magnitude of the maximum acceleration of a particle moving in simple harmonic motion are

Maximum values of speed and acceleration in simple harmonic motion

$$v_{max} = \omega A \quad \textbf{(13.10)}$$

$$a_{max} = \omega^2 A \quad \textbf{(13.11)}$$

Figure 13.4a represents the displacement versus time for an arbitrary value of the phase constant. The velocity and acceleration curves are illustrated in Figure 13.4b and c. These curves show that the phase of the velocity differs from the phase of the displacement by $\pi/2$ rad, or 90°. That is, when x is a maximum or a minimum, the velocity is zero. Likewise, when x is zero, the speed is a maximum.

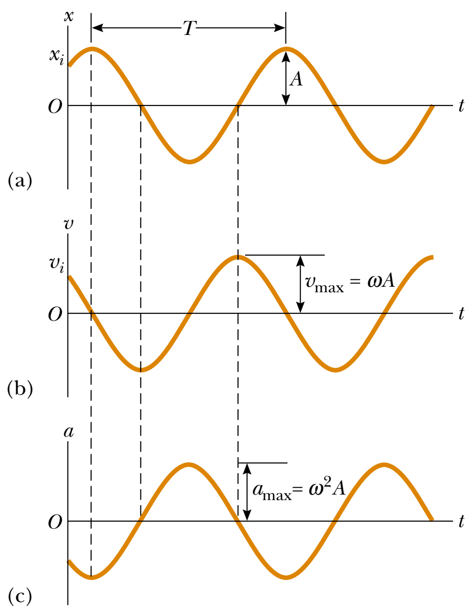


Figure 13.4 Graphical representation of simple harmonic motion. (a) Displacement versus time. (b) Velocity versus time. (c) Acceleration versus time. Note that at any specified time the velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement.

Furthermore, note that the phase of the acceleration differs from the phase of the displacement by π rad, or 180°. That is, when x is a maximum, a is a maximum in the opposite direction.

The phase constant ϕ is important when we compare the motion of two or more oscillating objects. Imagine two identical pendulum bobs swinging side by side in simple harmonic motion, with one having been released later than the other. The pendulum bobs have different phase constants. Let us show how the phase constant and the amplitude of any particle moving in simple harmonic motion can be determined if we know the particle's initial speed and position and the angular frequency of its motion.

Suppose that at $t = 0$ the initial position of a single oscillator is $x = x_i$ and its initial speed is $v = v_i$. Under these conditions, Equations 13.3 and 13.7 give

$$x_i = A \cos \phi \tag{13.12}$$

$$v_i = -\omega A \sin \phi \tag{13.13}$$

Dividing Equation 13.13 by Equation 13.12 eliminates A, giving $v_i/x_i = -\omega \tan \phi$, or

$$\tan \phi = -\frac{v_i}{\omega x_i} \tag{13.14}$$

Furthermore, if we square Equations 13.12 and 13.13, divide the velocity equation by ω^2, and then add terms, we obtain

$$x_i^2 + \left(\frac{v_i}{\omega}\right)^2 = A^2 \cos^2 \phi + A^2 \sin^2 \phi$$

Using the identity $\sin^2 \phi + \cos^2 \phi = 1$, we can solve for A:

$$A = \sqrt{x_i^2 + \left(\frac{v_i}{\omega}\right)^2} \tag{13.15}$$

The following properties of a particle moving in simple harmonic motion are important:

Properties of simple harmonic motion

- The acceleration of the particle is proportional to the displacement but is in the opposite direction. This is the *necessary and sufficient condition for simple harmonic motion,* as opposed to all other kinds of vibration.
- The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase, as shown in Figure 13.4.
- The frequency and the period of the motion are independent of the amplitude. (We show this explicitly in the next section.)

Quick Quiz 13.3

Can we use Equations 2.8, 2.10, 2.11, and 2.12 (see pages 35 and 36) to describe the motion of a simple harmonic oscillator?

EXAMPLE 13.1 An Oscillating Object

An object oscillates with simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation

$$x = (4.00\text{ m})\cos\left(\pi t + \frac{\pi}{4}\right)$$

where t is in seconds and the angles in the parentheses are in radians. (a) Determine the amplitude, frequency, and period of the motion.

Solution By comparing this equation with Equation 13.3, the general equation for simple harmonic motion—$x = A\cos(\omega t + \phi)$—we see that $A = 4.00$ m and $\omega = \pi$ rad/s. Therefore, $f = \omega/2\pi = \pi/2\pi = 0.500$ Hz and $T = 1/f = 2.00$ s.

(b) Calculate the velocity and acceleration of the object at any time t.

Solution

$$v = \frac{dx}{dt} = -(4.00\text{ m})\sin\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$= -(4.00\pi\text{ m/s})\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$a = \frac{dv}{dt} = -(4.00\pi\text{ m/s})\cos\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$= -(4.00\pi^2\text{ m/s}^2)\cos\left(\pi t + \frac{\pi}{4}\right)$$

(c) Using the results of part (b), determine the position, velocity, and acceleration of the object at $t = 1.00$ s.

Solution Noting that the angles in the trigonometric functions are in radians, we obtain, at $t = 1.00$ s,

$$x = (4.00\text{ m})\cos\left(\pi + \frac{\pi}{4}\right) = (4.00\text{ m})\cos\left(\frac{5\pi}{4}\right)$$

$$= (4.00\text{ m})(-0.707) = -2.83\text{ m}$$

$$v = -(4.00\pi\text{ m/s})\sin\left(\frac{5\pi}{4}\right) = -(4.00\pi\text{ m/s})(-0.707)$$

$$= 8.89\text{ m/s}$$

$$a = -(4.00\pi^2\text{ m/s}^2)\cos\left(\frac{5\pi}{4}\right)$$

$$= -(4.00\pi^2\text{ m/s}^2)(-0.707) = 27.9\text{ m/s}^2$$

(d) Determine the maximum speed and maximum acceleration of the object.

Solution In the general expressions for v and a found in part (b), we use the fact that the maximum values of the sine and cosine functions are unity. Therefore, v varies between $\pm 4.00\pi$ m/s, and a varies between $\pm 4.00\pi^2$ m/s^2. Thus,

$$v_{max} = 4.00\pi\text{ m/s} = 12.6\text{ m/s}$$

$$a_{max} = 4.00\pi^2\text{ m/s}^2 = 39.5\text{ m/s}^2$$

We obtain the same results using $v_{max} = \omega A$ and $a_{max} = \omega^2 A$, where $A = 4.00$ m and $\omega = \pi$ rad/s.

(e) Find the displacement of the object between $t = 0$ and $t = 1.00$ s.

Solution The x coordinate at $t = 0$ is

$$x_i = (4.00 \text{ m}) \cos\left(0 + \frac{\pi}{4}\right) = (4.00 \text{ m})(0.707) = 2.83 \text{ m}$$

In part (c), we found that the x coordinate at $t = 1.00$ s is -2.83 m; therefore, the displacement between $t = 0$ and $t = 1.00$ s is

$$\Delta x = x_f - x_i = -2.83 \text{ m} - 2.83 \text{ m} = -5.66 \text{ m}$$

Because the object's velocity changes sign during the first second, the magnitude of Δx is not the same as the distance traveled in the first second. (By the time the first second is over, the object has been through the point $x = -2.83$ m once, traveled to $x = -4.00$ m, and come back to $x = -2.83$ m.)

Exercise What is the phase of the motion at $t = 2.00$ s?

Answer $9\pi/4$ rad.

13.2 THE BLOCK–SPRING SYSTEM REVISITED

Let us return to the block–spring system (Fig. 13.5). Again we assume that the surface is frictionless; hence, when the block is displaced from equilibrium, the only force acting on it is the restoring force of the spring. As we saw in Equation 13.2, when the block is displaced a distance x from equilibrium, it experiences an acceleration $a = -(k/m)x$. If the block is displaced a maximum distance $x = A$ at some initial time and then released from rest, its initial acceleration at that instant is $-kA/m$ (its extreme negative value). When the block passes through the equilibrium position $x = 0$, its acceleration is zero. At this instant, its speed is a maximum. The block then continues to travel to the left of equilibrium and finally reaches $x = -A$, at which time its acceleration is kA/m (maximum positive) and its speed is again zero. Thus, we see that the block oscillates between the turning points $x = \pm A$.

Let us now describe the oscillating motion in a quantitative fashion. Recall that $a = dv/dt = d^2x/dt^2$, and so we can express Equation 13.2 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{13.16}$$

If we denote the ratio k/m with the symbol ω^2, this equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{13.17}$$

Now we require a solution to Equation 13.17—that is, a function $x(t)$ that satisfies this second-order differential equation. Because Equations 13.17 and 13.9 are equivalent, each solution must be that of simple harmonic motion:

$$x = A\cos(\omega t + \phi)$$

To see this explicitly, assume that $x = A\cos(\omega t + \phi)$. Then

$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 A\cos(\omega t + \phi)$$

Comparing the expressions for x and d^2x/dt^2, we see that $d^2x/dt^2 = -\omega^2 x$, and Equation 13.17 is satisfied. We conclude that **whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium**

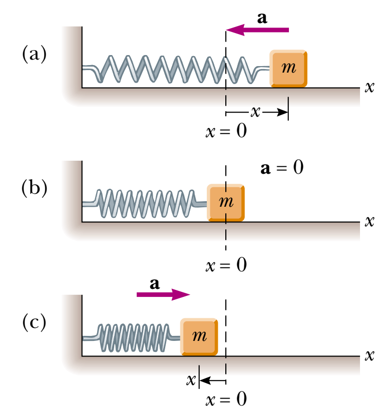


Figure 13.5 A block of mass m attached to a spring on a frictionless surface undergoes simple harmonic motion. (a) When the block is displaced to the right of equilibrium, the displacement is positive and the acceleration is negative. (b) At the equilibrium position, $x = 0$, the acceleration is zero and the speed is a maximum. (c) When the block is displaced to the left of equilibrium, the displacement is negative and the acceleration is positive.

position and in the opposite direction ($F = -kx$), the particle moves in simple harmonic motion.

Recall that the period of any simple harmonic oscillator is $T = 2\pi/\omega$ (Eq. 13.4) and that the frequency is the inverse of the period. We know from Equations 13.16 and 13.17 that $\omega = \sqrt{k/m}$, so we can express the period and frequency of the block–spring system as

Period and frequency for a block–spring system

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad (13.18)$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (13.19)$$

That is, **the frequency and period depend only on the mass of the block and on the force constant of the spring.** Furthermore, the frequency and period are independent of the amplitude of the motion. As we might expect, the frequency is greater for a stiffer spring (the stiffer the spring, the greater the value of k) and decreases with increasing mass.

QuickLab

Hang an object from a rubber band and start it oscillating. Measure T. Now tie four identical rubber bands together, end to end. How should k for this longer band compare with k for the single band? Again, time the oscillations with the same object. Can you verify Equation 13.19?

Special Case 1. Let us consider a special case to better understand the physical significance of Equation 13.3, the defining expression for simple harmonic motion. We shall use this equation to describe the motion of an oscillating block–spring system. Suppose we pull the block a distance A from equilibrium and then release it from rest at this stretched position, as shown in Figure 13.6. Our solution for x must obey the initial conditions that $x_i = A$ and $v_i = 0$ at $t = 0$. It does if we choose $\phi = 0$, which gives $x = A\cos\omega t$ as the solution. To check this solution, we note that it satisfies the condition that $x_i = A$ at $t = 0$ because $\cos 0 = 1$. Thus, we see that A and ϕ contain the information on initial conditions.

Now let us investigate the behavior of the velocity and acceleration for this special case. Because $x = A\cos\omega t$,

$$v = \frac{dx}{dt} = -\omega A \sin\omega t$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos\omega t$$

From the velocity expression we see that, because $\sin 0 = 0$, $v_i = 0$ at $t = 0$, as we require. The expression for the acceleration tells us that $a = -\omega^2 A$ at $t = 0$. Physically, this negative acceleration makes sense because the force acting on the block is directed to the left when the displacement is positive. In fact, at the extreme po-

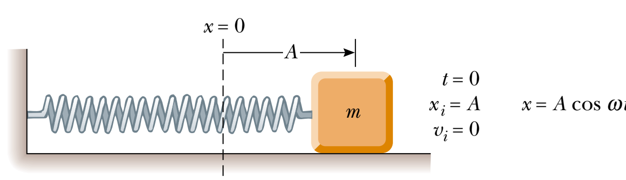


Figure 13.6 A block–spring system that starts from rest at $x_i = A$. In this case, $\phi = 0$ and thus $x = A\cos\omega t$.

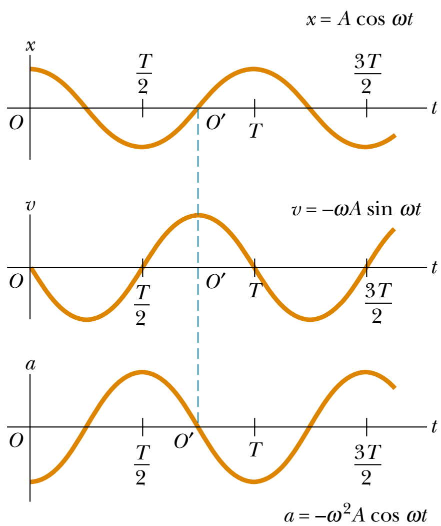


Figure 13.7 Displacement, velocity, and acceleration versus time for a block–spring system like the one shown in Figure 13.6, undergoing simple harmonic motion under the initial conditions that at $t = 0$, $x_i = A$ and $v_i = 0$ (Special Case 1). The origins at O' correspond to Special Case 2, the block–spring system under the initial conditions shown in Figure 13.8.

sition shown in Figure 13.6, $F_s = -kA$ (to the left) and the initial acceleration is $-\omega^2 A = -kA/m$.

Another approach to showing that $x = A \cos \omega t$ is the correct solution involves using the relationship $\tan \phi = -v_i/\omega x_i$ (Eq. 13.14). Because $v_i = 0$ at $t = 0$, $\tan \phi = 0$ and thus $\phi = 0$. (The tangent of π also equals zero, but $\phi = \pi$ gives the wrong value for x_i.)

Figure 13.7 is a plot of displacement, velocity, and acceleration versus time for this special case. Note that the acceleration reaches extreme values of $\pm\omega^2 A$ while the displacement has extreme values of $\pm A$ because the force is maximal at those positions. Furthermore, the velocity has extreme values of $\pm\omega A$, which both occur at $x = 0$. Hence, the quantitative solution agrees with our qualitative description of this system.

Special Case 2. Now suppose that the block is given an initial velocity $\mathbf{v}_i$ to the right at the instant it is at the equilibrium position, so that $x_i = 0$ and $v = v_i$ at $t = 0$ (Fig. 13.8). The expression for x must now satisfy these initial conditions. Because the block is moving in the positive x direction at $t = 0$ and because $x_i = 0$ at $t = 0$, the expression for x must have the form $x = A \sin \omega t$.

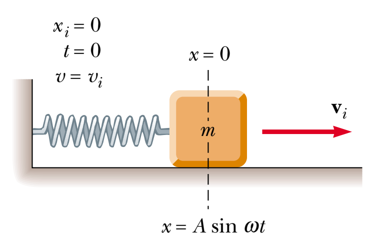


Figure 13.8 The block–spring system starts its motion at the equilibrium position at $t = 0$. If its initial velocity is v_i to the right, the block's x coordinate varies as $x = (v_i/\omega) \sin \omega t$.

Applying Equation 13.14 and the initial condition that $x_i = 0$ at $t = 0$, we find that $\tan \phi = -\infty$ and $\phi = -\pi/2$. Hence, Equation 13.3 becomes $x = A \cos(\omega t - \pi/2)$, which can be written $x = A \sin \omega t$. Furthermore, from Equation 13.15 we see that $A = v_i/\omega$; therefore, we can express x as

$$x = \frac{v_i}{\omega} \sin \omega t$$

The velocity and acceleration in this case are

$$v = \frac{dx}{dt} = v_i \cos \omega t$$

$$a = \frac{dv}{dt} = -\omega v_i \sin \omega t$$

These results are consistent with the facts that (1) the block always has a maximum speed at $x = 0$ and (2) the force and acceleration are zero at this position. The graphs of these functions versus time in Figure 13.7 correspond to the origin at O'.

Quick Quiz 13.4

What is the solution for x if the block is initially moving to the left in Figure 13.8?

EXAMPLE 13.2 Watch Out for Potholes!

A car with a mass of 1 300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20 000 N/m. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

Solution We assume that the mass is evenly distributed. Thus, each spring supports one fourth of the load. The total mass is 1 460 kg, and therefore each spring supports 365 kg. Hence, the frequency of vibration is, from Equation 13.19,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20\ 000\ \text{N/m}}{365\ \text{kg}}} = \boxed{1.18\ \text{Hz}}$$

Exercise How long does it take the car to execute two complete vibrations?

Answer 1.70 s.

EXAMPLE 13.3 A Block–Spring System

A block with a mass of 200 g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from rest, as shown in Figure 13.6. (a) Find the period of its motion.

Solution From Equations 13.16 and 13.17, we know that the angular frequency of any block–spring system is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00\ \text{N/m}}{200 \times 10^{-3}\ \text{kg}}} = 5.00\ \text{rad/s}$$

and the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00\ \text{rad/s}} = \boxed{1.26\ \text{s}}$$

(b) Determine the maximum speed of the block.

Solution We use Equation 13.10:

$$v_{\text{max}} = \omega A = (5.00\ \text{rad/s})(5.00 \times 10^{-2}\ \text{m}) = \boxed{0.250\ \text{m/s}}$$

(c) What is the maximum acceleration of the block?

Solution We use Equation 13.11:

$$a_{\text{max}} = \omega^2 A = (5.00\ \text{rad/s})^2(5.00 \times 10^{-2}\ \text{m}) = \boxed{1.25\ \text{m/s}^2}$$

(d) Express the displacement, speed, and acceleration as functions of time.

Solution This situation corresponds to Special Case 1, where our solution is $x = A\cos\omega t$. Using this expression and the results from (a), (b), and (c), we find that

$$x = A\cos\omega t = \boxed{(0.050\ \text{m})\cos 5.00t}$$

$$v = \omega A\sin\omega t = \boxed{-(0.250\ \text{m/s})\sin 5.00t}$$

$$a = \omega^2 A\cos\omega t = \boxed{-(1.25\ \text{m/s}^2)\cos 5.00t}$$

13.3 ENERGY OF THE SIMPLE HARMONIC OSCILLATOR

Let us examine the mechanical energy of the block–spring system illustrated in Figure 13.6. Because the surface is frictionless, we expect the total mechanical energy to be constant, as was shown in Chapter 8. We can use Equation 13.7 to express the kinetic energy as

$$K = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) \quad (13.20)$$

Kinetic energy of a simple harmonic oscillator

The elastic potential energy stored in the spring for any elongation x is given by $\frac{1}{2}kx^2$ (see Eq. 8.4). Using Equation 13.3, we obtain

$$U = \tfrac{1}{2} kx^2 = \tfrac{1}{2} kA^2 \cos^2(\omega t + \phi) \quad (13.21)$$

Potential energy of a simple harmonic oscillator

We see that K and U are *always* positive quantities. Because $\omega^2 = k/m$, we can express the total mechanical energy of the simple harmonic oscillator as

$$E = K + U = \tfrac{1}{2} kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

From the identity $\sin^2\theta + \cos^2\theta = 1$, we see that the quantity in square brackets is unity. Therefore, this equation reduces to

$$E = \tfrac{1}{2} kA^2 \quad (13.22)$$

Total energy of a simple harmonic oscillator

That is, **the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.** Note that U is small when K is large, and vice versa, because the sum must be constant. In fact, the total mechanical energy is equal to the maximum potential energy stored in the spring when $x = \pm A$ because $v = 0$ at these points and thus there is no kinetic energy. At the equilibrium position, where $U = 0$ because $x = 0$, the total energy, all in the form of kinetic energy, is again $\frac{1}{2}kA^2$. That is,

$$E = \tfrac{1}{2} mv_{\text{max}}^2 = \tfrac{1}{2} m\omega^2 A^2 = \tfrac{1}{2} m\frac{k}{m}A^2 = \tfrac{1}{2} kA^2 \qquad (\text{at } x = 0)$$

Plots of the kinetic and potential energies versus time appear in Figure 13.9a, where we have taken $\phi = 0$. As already mentioned, both K and U are always positive, and at all times their sum is a constant equal to $\frac{1}{2}kA^2$, the total energy of the system. The variations of K and U with the displacement x of the block are plotted

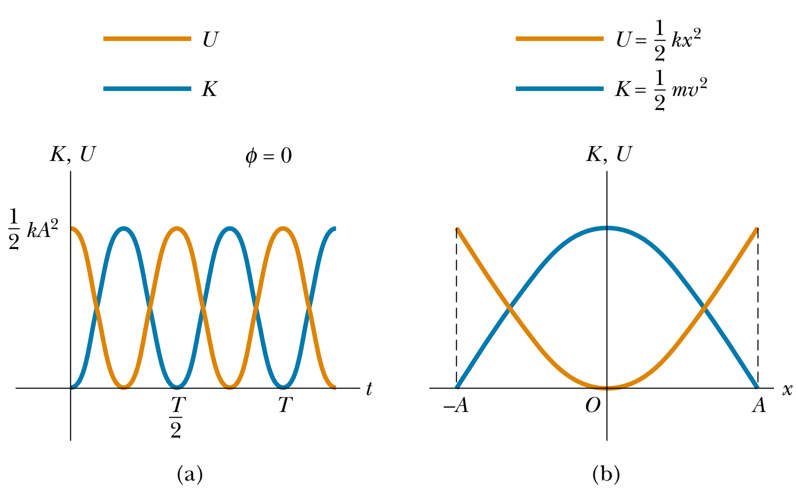


Figure 13.9 (a) Kinetic energy and potential energy versus time for a simple harmonic oscillator with $\phi = 0$. (b) Kinetic energy and potential energy versus displacement for a simple harmonic oscillator. In either plot, note that $K + U =$ constant.

in Figure 13.9b. Energy is continuously being transformed between potential energy stored in the spring and kinetic energy of the block.

Figure 13.10 illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the block–spring system for one full period of the motion. Most of the ideas discussed so far are incorporated in this important figure. Study it carefully.

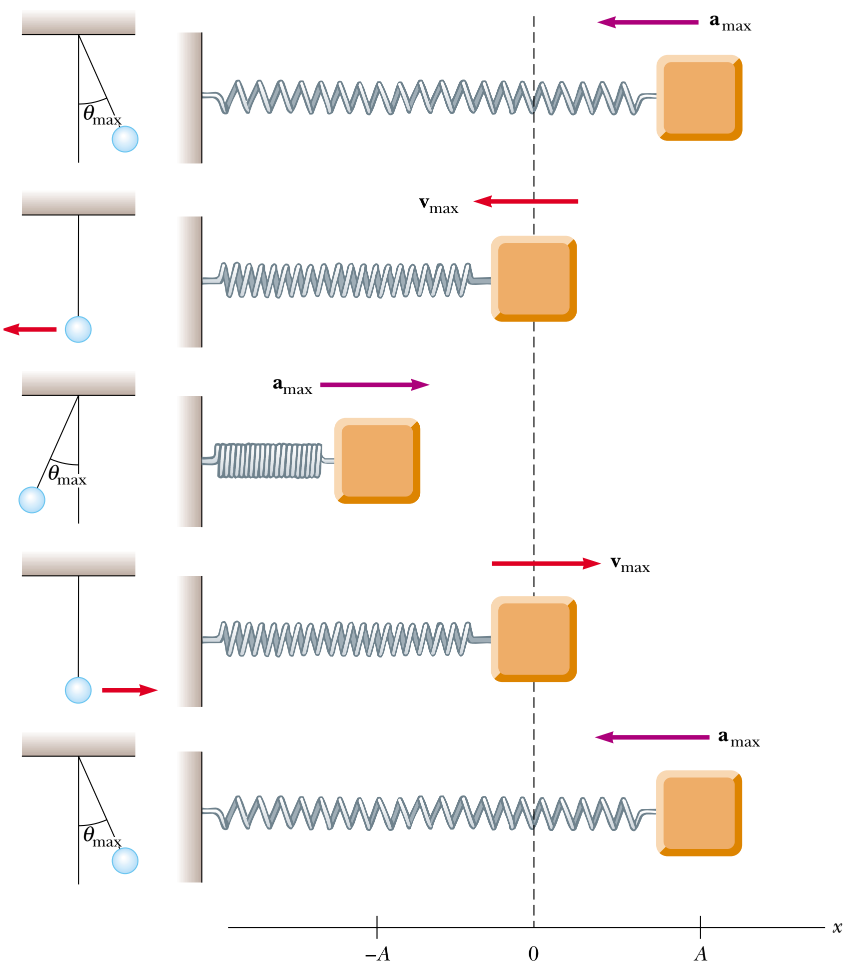
Finally, we can use the principle of conservation of energy to obtain the velocity for an arbitrary displacement by expressing the total energy at some arbitrary position x as

$$E = K + U = \tfrac{1}{2} mv^2 + \tfrac{1}{2} kx^2 = \tfrac{1}{2} kA^2$$

Velocity as a function of position for a simple harmonic oscillator

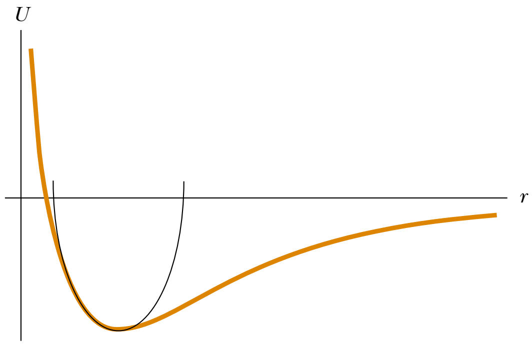
$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\,\omega\sqrt{A^2 - x^2} \qquad \textbf{(13.23)}$$

When we check Equation 13.23 to see whether it agrees with known cases, we find that it substantiates the fact that the speed is a maximum at $x = 0$ and is zero at the turning points $x = \pm A$.



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Figure 13.10 Simple harmonic motion for a block–spring system and its relationship to the motion of a simple pendulum. The parameters in the table refer to the block–spring system, assuming that $x = A$ at $t = 0$; thus, $x = A\cos\omega t$ (see Special Case 1).



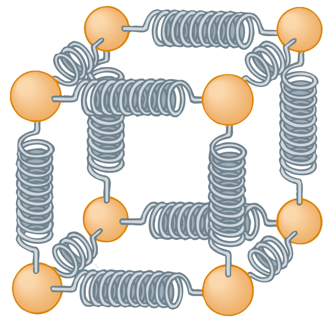


Figure 13.11 (a) If the atoms in a molecule do not move too far from their equilibrium positions, a graph of potential energy versus separation distance between atoms is similar to the graph of potential energy versus position for a simple harmonic oscillator. (b) Tiny springs approximate the forces holding atoms together.

You may wonder why we are spending so much time studying simple harmonic oscillators. We do so because they are good models of a wide variety of physical phenomena. For example, recall the Lennard–Jones potential discussed in Example 8.11. This complicated function describes the forces holding atoms together. Figure 13.11a shows that, for small displacements from the equilibrium position, the potential energy curve for this function approximates a parabola, which represents the potential energy function for a simple harmonic oscillator. Thus, we can approximate the complex atomic binding forces as tiny springs, as depicted in Figure 13.11b.

The ideas presented in this chapter apply not only to block–spring systems and atoms, but also to a wide range of situations that include bungee jumping, tuning in a television station, and viewing the light emitted by a laser. You will see more examples of simple harmonic oscillators as you work through this book.

EXAMPLE 13.4 Oscillations on a Horizontal Surface

A 0.500-kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. (a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

Solution Using Equation 13.22, we obtain

$$E = K + U = \tfrac{1}{2} kA^2 = \tfrac{1}{2}(20.0\ \text{N/m})\ (3.00 \times 10^{-2}\ \text{m})^2$$

$$= \boxed{9.00 \times 10^{-3}\ \text{J}}$$

When the cube is at $x = 0$, we know that $U = 0$ and $E = \frac{1}{2} mv_{\text{max}}^2$; therefore,

$$\tfrac{1}{2} mv_{\text{max}}^2 = 9.00 \times 10^{-3}\ \text{J}$$

$$v_{\text{max}} = \sqrt{\frac{18.0 \times 10^{-3}\ \text{J}}{0.500\ \text{kg}}} = \boxed{0.190\ \text{m/s}}$$

(b) What is the velocity of the cube when the displacement is 2.00 cm?

Solution We can apply Equation 13.23 directly:

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

$$= \pm\sqrt{\frac{20.0\ \text{N/m}}{0.500\ \text{kg}}[(0.030\,0\ \text{m})^2 - (0.020\,0\ \text{m})^2]}$$

$$= \boxed{\pm 0.141\ \text{m/s}}$$

The positive and negative signs indicate that the cube could be moving to either the right or the left at this instant.

(c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Solution Using the result of (b), we find that

$$K = \frac{1}{2} mv^2 = \frac{1}{2}(0.500 \text{ kg})(0.141 \text{ m/s})^2 = \boxed{5.00 \times 10^{-3} \text{ J}}$$

$$U = \frac{1}{2} kx^2 = \frac{1}{2}(20.0 \text{ N/m})(0.020\,0 \text{ m})^2 = \boxed{4.00 \times 10^{-3} \text{ J}}$$

Note that $K + U = E$.

Exercise For what values of x is the speed of the cube 0.100 m/s?

Answer ±2.55 cm.

13.4 THE PENDULUM

8.11 & 8.12

The **simple pendulum** is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end, as shown in Figure 13.12. The motion occurs in the vertical plane and is driven by the force of gravity. We shall show that, provided the angle θ is small (less than about 10°), the motion is that of a simple harmonic oscillator.

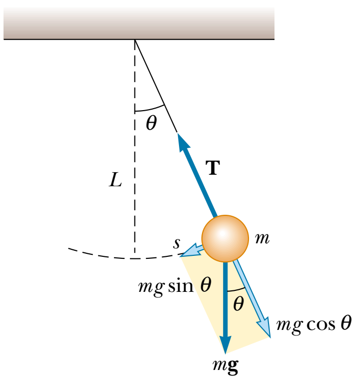


Figure 13.12 When θ is small, a simple pendulum oscillates in simple harmonic motion about the equilibrium position $\theta = 0$. The restoring force is $mg \sin\theta$, the component of the gravitational force tangent to the arc.

The forces acting on the bob are the force **T** exerted by the string and the gravitational force $m\mathbf{g}$. The tangential component of the gravitational force, $mg \sin\theta$, always acts toward $\theta = 0$, opposite the displacement. Therefore, the tangential force is a restoring force, and we can apply Newton's second law for motion in the tangential direction:

$$\sum F_t = -mg \sin\theta = m\frac{d^2s}{dt^2}$$

where s is the bob's displacement measured along the arc and the minus sign indicates that the tangential force acts toward the equilibrium (vertical) position. Because $s = L\theta$ (Eq. 10.1a) and L is constant, this equation reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

The right side is proportional to $\sin\theta$ rather than to θ; hence, with $\sin\theta$ present, we would not expect simple harmonic motion because this expression is not of the form of Equation 13.17. However, if we assume that θ is small, we can use the approximation $\sin\theta \approx \theta$; thus the equation of motion for the simple pen-

The motion of a simple pendulum, captured with multiflash photography. Is the oscillating motion simple harmonic in this case?

dulum becomes

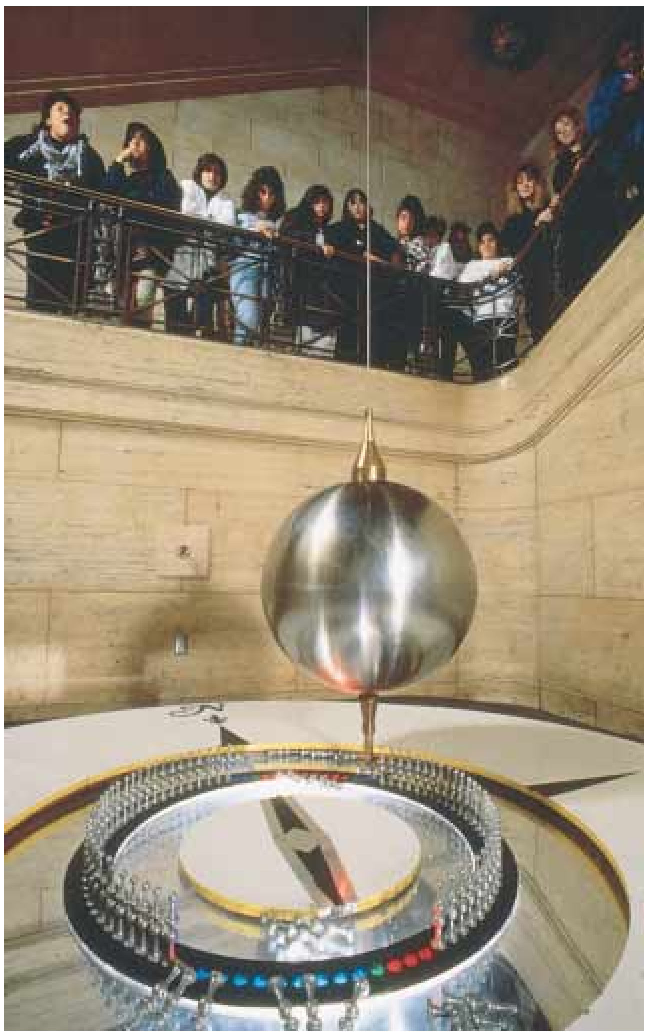
$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\,\theta \tag{13.24}$$

Equation of motion for a simple pendulum (small θ)

Now we have an expression of the same form as Equation 13.17, and we conclude that the motion for small amplitudes of oscillation is simple harmonic motion. Therefore, θ can be written as $\theta = \theta_{\max}\cos(\omega t + \phi)$, where $\theta_{\max}$ is the *maximum angular displacement* and the angular frequency ω is

$$\omega = \sqrt{\frac{g}{L}} \tag{13.25}$$

Angular frequency of motion for a simple pendulum



The Foucault pendulum at the Franklin Institute in Philadelphia. This type of pendulum was first used by the French physicist Jean Foucault to verify the Earth's rotation experimentally. As the pendulum swings, the vertical plane in which it oscillates appears to rotate as the bob successively knocks over the indicators arranged in a circle on the floor. In reality, the plane of oscillation is fixed in space, and the Earth rotating beneath the swinging pendulum moves the indicators into position to be knocked down, one after the other.

The period of the motion is

Period of motion for a simple pendulum

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{13.26}$$

In other words, **the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.** Because the period is independent of the mass, we conclude that all simple pendulums that are of equal length and are at the same location (so that g is constant) oscillate with the same period. The analogy between the motion of a simple pendulum and that of a block–spring system is illustrated in Figure 13.10.

The simple pendulum can be used as a timekeeper because its period depends only on its length and the local value of g. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of g can provide information on the location of oil and of other valuable underground resources.

Quick Quiz 13.5

A block of mass m is first allowed to hang from a spring in static equilibrium. It stretches the spring a distance L beyond the spring's unstressed length. The block and spring are then set into oscillation. Is the period of this system less than, equal to, or greater than the period of a simple pendulum having a length L and a bob mass m?

EXAMPLE 13.5 A Connection Between Length and Time

Christian Huygens (1629–1695), the greatest clockmaker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1 s. How much shorter would our length unit be had his suggestion been followed?

Solution Solving Equation 13.26 for the length gives

$$L = \frac{T^2 g}{4\pi^2} = \frac{(1\ \text{s})^2(9.80\ \text{m/s}^2)}{4\pi^2} = \boxed{0.248\ \text{m}}$$

Thus, the meter's length would be slightly less than one-fourth its current length. Note that the number of significant digits depends only on how precisely we know g because the time has been defined to be exactly 1 s.

Physical Pendulum

QuickLab

Firmly hold a ruler so that about half of it is over the edge of your desk. With your other hand, pull down and then release the free end, watching how it vibrates. Now slide the ruler so that only about a quarter of it is free to vibrate. This time when you release it, how does the vibrational period compare with its earlier value? Why?

Suppose you balance a wire coat hanger so that the hook is supported by your extended index finger. When you give the hanger a small displacement (with your other hand) and then release it, it oscillates. If a hanging object oscillates about a fixed axis that does not pass through its center of mass and the object cannot be approximated as a point mass, we cannot treat the system as a simple pendulum. In this case the system is called a **physical pendulum.**

Consider a rigid body pivoted at a point O that is a distance d from the center of mass (Fig. 13.13). The force of gravity provides a torque about an axis through O, and the magnitude of that torque is $mgd \sin\theta$, where θ is as shown in Figure 13.13. Using the law of motion $\Sigma\tau = I\alpha$, where I is the moment of inertia about the axis through O, we obtain

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

The minus sign indicates that the torque about O tends to decrease θ. That is, the force of gravity produces a restoring torque. Because this equation gives us the angular acceleration $d^2\theta/dt^2$ of the pivoted body, we can consider it the equation of motion for the system. If we again assume that θ is small, the approximation $\sin\theta \approx \theta$ is valid, and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta \qquad \textbf{(13.27)}$$

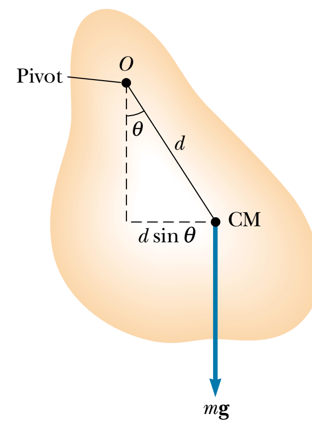


Figure 13.13 A physical pendulum.

Because this equation is of the same form as Equation 13.17, the motion is simple harmonic motion. That is, the solution of Equation 13.27 is $\theta = \theta_{max}\cos(\omega t + \phi)$, where θ_{max} is the maximum angular displacement and

$$\omega = \sqrt{\frac{mgd}{I}}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} \qquad \textbf{(13.28)}$$

Period of motion for a physical pendulum

One can use this result to measure the moment of inertia of a flat rigid body. If the location of the center of mass—and hence the value of d—are known, the moment of inertia can be obtained by measuring the period. Finally, note that Equation 13.28 reduces to the period of a simple pendulum (Eq. 13.26) when $I = md^2$—that is, when all the mass is concentrated at the center of mass.

EXAMPLE 13.6 A Swinging Rod

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane (Fig. 13.14). Find the period of oscillation if the amplitude of the motion is small.

Solution In Chapter 10 we found that the moment of inertia of a uniform rod about an axis through one end is $\frac{1}{3}ML^2$. The distance d from the pivot to the center of mass is $L/2$. Substituting these quantities into Equation 13.28 gives

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Comment In one of the Moon landings, an astronaut walking on the Moon's surface had a belt hanging from his space suit, and the belt oscillated as a physical pendulum. A scientist on the Earth observed this motion on television and used it to estimate the free-fall acceleration on the Moon. How did the scientist make this calculation?

Exercise Calculate the period of a meter stick that is pivoted about one end and is oscillating in a vertical plane.

Answer 1.64 s.

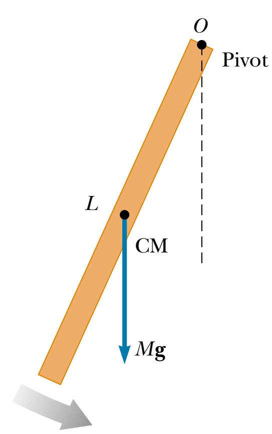


Figure 13.14 A rigid rod oscillating about a pivot through one end is a physical pendulum with $d = L/2$ and, from Table 10.2, $I = \frac{1}{3}ML^2$.

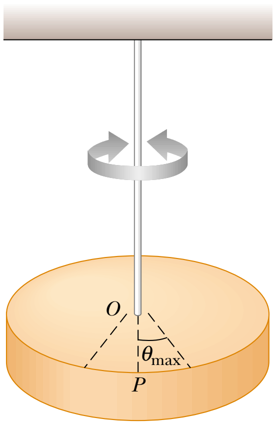


Figure 13.15 A torsional pendulum consists of a rigid body suspended by a wire attached to a rigid support. The body oscillates about the line *OP* with an amplitude θ_{max}.

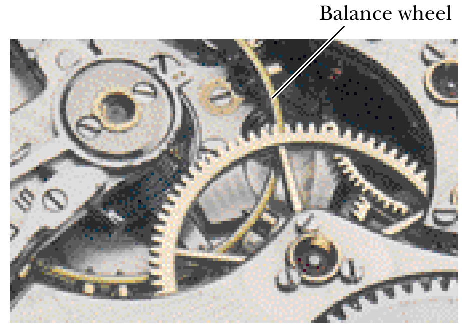


Figure 13.16 The balance wheel of this antique pocket watch is a torsional pendulum and regulates the time-keeping mechanism.

Torsional Pendulum

Figure 13.15 shows a rigid body suspended by a wire attached at the top to a fixed support. When the body is twisted through some small angle θ, the twisted wire exerts on the body a restoring torque that is proportional to the angular displacement. That is,

$$\tau = -\kappa\theta$$

where κ (kappa) is called the *torsion constant* of the support wire. The value of κ can be obtained by applying a known torque to twist the wire through a measurable angle θ. Applying Newton's second law for rotational motion, we find

$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \qquad \textbf{(13.29)}$$

Again, this is the equation of motion for a simple harmonic oscillator, with $\omega = \sqrt{\kappa/I}$ and a period

Period of motion for a torsional pendulum

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \qquad \textbf{(13.30)}$$

This system is called a *torsional pendulum.* There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded. Figure 13.16 shows the balance wheel of a watch oscillating as a torsional pendulum, energized by the mainspring.

13.5 COMPARING SIMPLE HARMONIC MOTION WITH UNIFORM CIRCULAR MOTION

8.8

We can better understand and visualize many aspects of simple harmonic motion by studying its relationship to uniform circular motion. Figure 13.17 is an overhead view of an experimental arrangement that shows this relationship. A ball is attached to the rim of a turntable of radius *A*, which is illuminated from the side by a lamp. The ball casts a shadow on a screen. We find that as the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.

Consider a particle located at point P on the circumference of a circle of radius A, as shown in Figure 13.18a, with the line OP making an angle ϕ with the x axis at $t = 0$. We call this circle a *reference circle* for comparing simple harmonic motion and uniform circular motion, and we take the position of P at $t = 0$ as our reference position. If the particle moves along the circle with constant angular speed ω until OP makes an angle θ with the x axis, as illustrated in Figure 13.18b, then at some time $t > 0$, the angle between OP and the x axis is $\theta = \omega t + \phi$. As the particle moves along the circle, the projection of P on the x axis, labeled point Q, moves back and forth along the x axis, between the limits $x = \pm A$.

Note that points P and Q always have the same x coordinate. From the right triangle OPQ, we see that this x coordinate is

$$x = A\cos(\omega t + \phi) \tag{13.31}$$

This expression shows that the point Q moves with simple harmonic motion along the x axis. Therefore, we conclude that

simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

We can make a similar argument by noting from Figure 13.18b that the projection of P along the y axis also exhibits simple harmonic motion. Therefore, **uniform circular motion can be considered a combination of two simple harmonic motions,** one along the x axis and one along the y axis, with the two differing in phase by 90°.

This geometric interpretation shows that the time for one complete revolution of the point P on the reference circle is equal to the period of motion T for simple harmonic motion between $x = \pm A$. That is, the angular speed ω of P is the same as the angular frequency ω of simple harmonic motion along the x axis (this is why we use the same symbol). The phase constant ϕ for simple harmonic motion corresponds to the initial angle that OP makes with the x axis. The radius A of the reference circle equals the amplitude of the simple harmonic motion.

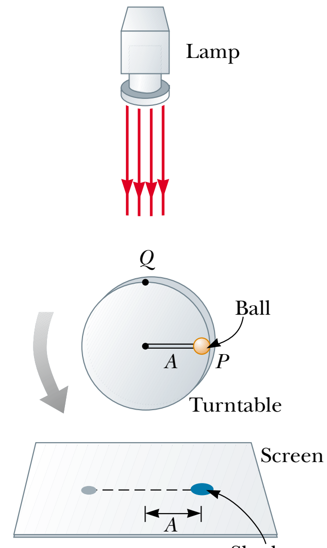


Figure 13.17 An experimental setup for demonstrating the connection between simple harmonic motion and uniform circular motion. As the ball rotates on the turntable with constant angular speed, its shadow on the screen moves back and forth in simple harmonic motion.

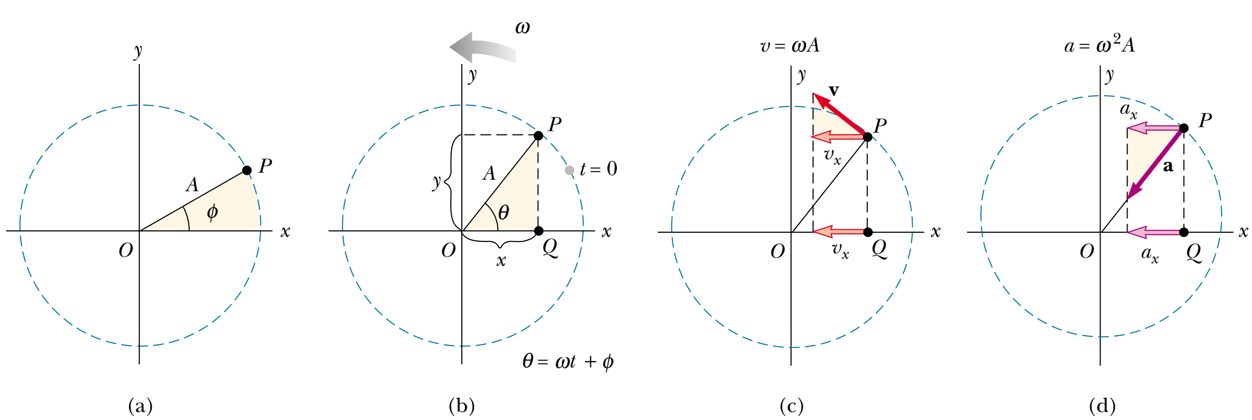


Figure 13.18 Relationship between the uniform circular motion of a point P and the simple harmonic motion of a point Q. A particle at P moves in a circle of radius A with constant angular speed ω. (a) A reference circle showing the position of P at $t = 0$. (b) The x coordinates of points P and Q are equal and vary in time as $x = A\cos(\omega t + \phi)$. (c) The x component of the velocity of P equals the velocity of Q. (d) The x component of the acceleration of P equals the acceleration of Q.

Because the relationship between linear and angular speed for circular motion is $v = r\omega$ (see Eq. 10.10), the particle moving on the reference circle of radius A has a velocity of magnitude ωA. From the geometry in Figure 13.18c, we see that the x component of this velocity is $-\omega A \sin(\omega t + \phi)$. By definition, the point Q has a velocity given by dx/dt. Differentiating Equation 13.31 with respect to time, we find that the velocity of Q is the same as the x component of the velocity of P.

The acceleration of P on the reference circle is directed radially inward toward O and has a magnitude $v^2/A = \omega^2 A$. From the geometry in Figure 13.18d, we see that the x component of this acceleration is $-\omega^2 A \cos(\omega t + \phi)$. This value is also the acceleration of the projected point Q along the x axis, as you can verify by taking the second derivative of Equation 13.31.

EXAMPLE 13.7 Circular Motion with Constant Angular Speed

A particle rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. At $t = 0$, the particle has an x coordinate of 2.00 m and is moving to the right. (a) Determine the x coordinate as a function of time.

Solution Because the amplitude of the particle's motion equals the radius of the circle and $\omega = 8.00$ rad/s, we have

$$x = A\cos(\omega t + \phi) = (3.00 \text{ m})\cos(8.00t + \phi)$$

We can evaluate ϕ by using the initial condition that $x = 2.00$ m at $t = 0$:

$$2.00 \text{ m} = (3.00 \text{ m})\cos(0 + \phi)$$

$$\phi = \cos^{-1}\left(\frac{2.00 \text{ m}}{3.00 \text{ m}}\right)$$

If we were to take our answer as $\phi = 48.2°$, then the coordinate $x = (3.00 \text{ m}) \cos(8.00t + 48.2°)$ would be decreasing at time $t = 0$ (that is, moving to the left). Because our particle is first moving to the right, we must choose $\phi = -48.2° = -0.841$ rad. The x coordinate as a function of time is then

$$x = (3.00 \text{ m}) \cos(8.00t - 0.841)$$

Note that ϕ in the cosine function must be in radians.

(b) Find the x components of the particle's velocity and acceleration at any time t.

Solution

$$v_x = \frac{dx}{dt} = (-3.00 \text{ m})(8.00 \text{ rad/s})\sin(8.00t - 0.841)$$

$$= -(24.0 \text{ m/s})\sin(8.00t - 0.841)$$

$$a_x = \frac{dv_x}{dt} = (-24.0 \text{ m/s})(8.00 \text{ rad/s})\cos(8.00t - 0.841)$$

$$= -(192 \text{ m/s}^2)\cos(8.00t - 0.841)$$

From these results, we conclude that $v_{max} = 24.0$ m/s and that $a_{max} = 192$ m/s^2. Note that these values also equal the tangential speed ωA and the centripetal acceleration $\omega^2 A$.

Optional Section

13.6 DAMPED OSCILLATIONS

The oscillatory motions we have considered so far have been for ideal systems—that is, systems that oscillate indefinitely under the action of a linear restoring force. In many real systems, dissipative forces, such as friction, retard the motion. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be *damped.*

One common type of retarding force is the one discussed in Section 6.4, where the force is proportional to the speed of the moving object and acts in the direction opposite the motion. This retarding force is often observed when an object moves through air, for instance. Because the retarding force can be expressed as $\mathbf{R} = -b\mathbf{v}$ (where b is a constant called the *damping coefficient*) and the restoring force of the system is $-kx$, we can write Newton's second law as

$$\sum F_x = -kx - bv = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad \textbf{(13.32)}$$

The solution of this equation requires mathematics that may not be familiar to you yet; we simply state it here without proof. When the retarding force is small compared with the maximum restoring force—that is, when b is small—the solution to Equation 13.32 is

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi) \qquad \textbf{(13.33)}$$

where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \qquad \textbf{(13.34)}$$

This result can be verified by substituting Equation 13.33 into Equation 13.32.

Figure 13.19a shows the displacement as a function of time for an object oscillating in the presence of a retarding force, and Figure 13.19b depicts one such system: a block attached to a spring and submersed in a viscous liquid. We see that **when the retarding force is much smaller than the restoring force, the oscillatory character of the motion is preserved but the amplitude decreases in time, with the result that the motion ultimately ceases.** Any system that behaves in this way is known as a **damped oscillator.** The dashed blue lines in Figure 13.19a, which define the *envelope* of the oscillatory curve, represent the exponential factor in Equation 13.33. This envelope shows that **the amplitude decays exponentially with time.** For motion with a given spring constant and block mass, the oscillations dampen more rapidly as the maximum value of the retarding force approaches the maximum value of the restoring force.

It is convenient to express the angular frequency of a damped oscillator in the form

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

where $\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency** of the system. When the magnitude of the maximum retarding force $R_{max} = bv_{max} < kA$, the system is said to be **underdamped.** As the value of R approaches kA, the amplitudes of the oscillations decrease more and more rapidly. This motion is represented by the blue curve in Figure 13.20. When b reaches a critical value b_c such that $b_c/2m = \omega_0$, the system does not oscillate and is said to be **critically damped.** In this case the system, once released from rest at some nonequilibrium position, returns to equilibrium and then stays there. The graph of displacement versus time for this case is the red curve in Figure 13.20.

If the medium is so viscous that the retarding force is greater than the restoring force—that is, if $R_{max} = bv_{max} > kA$ and $b/2m > \omega_0$—the system is **overdamped.** Again, the displaced system, when free to move, does not oscillate but simply returns to its equilibrium position. As the damping increases, the time it takes the system to approach equilibrium also increases, as indicated by the black curve in Figure 13.20.

In any case in which friction is present, whether the system is overdamped or underdamped, the energy of the oscillator eventually falls to zero. The lost mechanical energy dissipates into internal energy in the retarding medium.

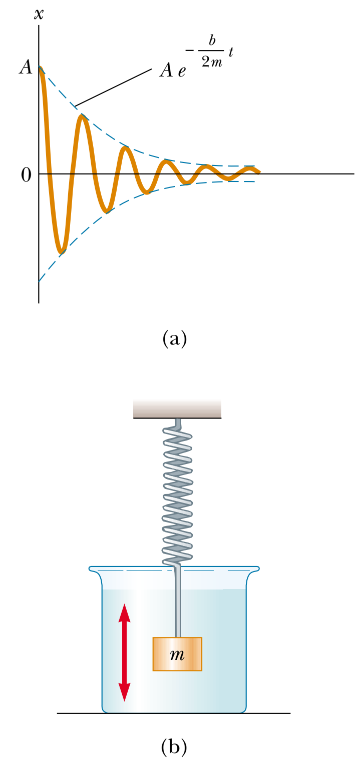


Figure 13.19 (a) Graph of displacement versus time for a damped oscillator. Note the decrease in amplitude with time. (b) One example of a damped oscillator is a mass attached to a spring and submersed in a viscous liquid.

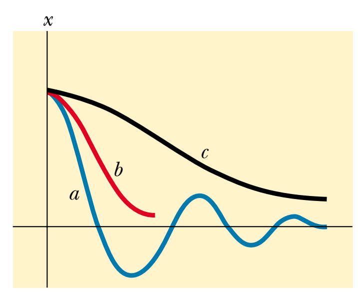


Figure 13.20 Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

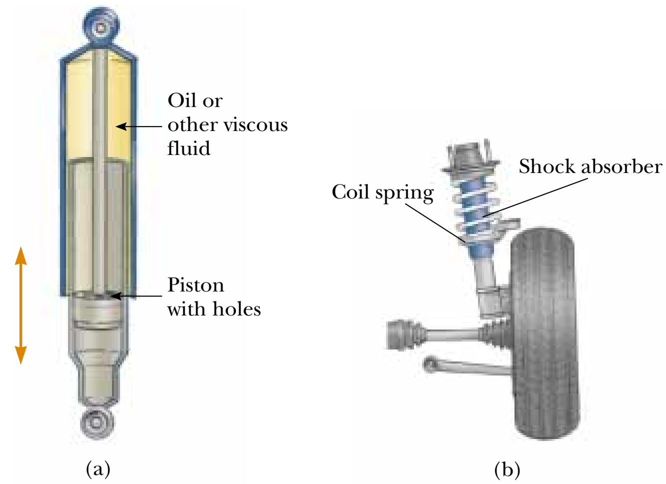


Figure 13.21 (a) A shock absorber consists of a piston oscillating in a chamber filled with oil. As the piston oscillates, the oil is squeezed through holes between the piston and the chamber, causing a damping of the piston's oscillations. (b) One type of automotive suspension system, in which a shock absorber is placed inside a coil spring at each wheel.

web

To learn more about shock absorbers, visit **http://www.hdridecontrol.com**

Quick Quiz 13.6

An automotive suspension system consists of a combination of springs and shock absorbers, as shown in Figure 13.21. If you were an automotive engineer, would you design a suspension system that was underdamped, critically damped, or overdamped? Discuss each case.

Optional Section

13.7 FORCED OSCILLATIONS

It is possible to compensate for energy loss in a damped system by applying an external force that does positive work on the system. At any instant, energy can be put into the system by an applied force that acts in the direction of motion of the oscillator. For example, a child on a swing can be kept in motion by appropriately timed pushes. The amplitude of motion remains constant if the energy input per cycle exactly equals the energy lost as a result of damping. Any motion of this type is called **forced oscillation.**

A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as $F = F_{\text{ext}} \cos \omega t$, where ω is the angular frequency of the periodic force and F_{ext} is a constant. Adding this driving force to the left side of Equation 13.32 gives

$$F_{\text{ext}} \cos \omega t - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad \textbf{(13.35)}$$

(As earlier, we present the solution of this equation without proof.) After a sufficiently long period of time, when the energy input per cycle equals the energy lost per cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude. At this time, when the system is in a steady state, the solution of Equation 13.35 is

$$x = A\cos(\omega t + \phi) \qquad \textbf{(13.36)}$$

where

$$A = \frac{F_{\text{ext}}/m}{\sqrt{(\omega^2 - \omega_0{}^2)^2 + \left(\frac{b\omega}{m}\right)^2}} \quad \textbf{(13.37)}$$

and where $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped oscillator ($b = 0$). One could argue that in steady state the oscillator must physically have the same frequency as the driving force, and thus the solution given by Equation 13.36 is expected. In fact, when this solution is substituted into Equation 13.35, one finds that it is indeed a solution, provided the amplitude is given by Equation 13.37.

Equation 13.37 shows that, because an external force is driving it, the motion of the forced oscillator is not damped. The external agent provides the necessary energy to overcome the losses due to the retarding force. Note that the system oscillates at the angular frequency ω of the driving force. For small damping, the amplitude becomes very large when the frequency of the driving force is near the natural frequency of oscillation. The dramatic increase in amplitude near the natural frequency ω_0 is called **resonance,** and for this reason ω_0 is sometimes called the **resonance frequency** of the system.

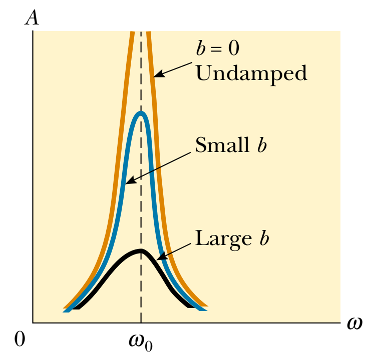


Figure 13.22 Graph of amplitude versus frequency for a damped oscillator when a periodic driving force is present. When the frequency of the driving force equals the natural frequency ω_0, resonance occurs. Note that the shape of the resonance curve depends on the size of the damping coefficient b.

The reason for large-amplitude oscillations at the resonance frequency is that energy is being transferred to the system under the most favorable conditions. We can better understand this by taking the first time derivative of x in Equation 13.36, which gives an expression for the velocity of the oscillator. We find that v is proportional to $\sin(\omega t + \phi)$. When the applied force $\mathbf{F}$ is in phase with the velocity, the rate at which work is done on the oscillator by $\mathbf{F}$ equals the dot product $\mathbf{F} \cdot \mathbf{v}$. Remember that "rate at which work is done" is the definition of power. Because the product $\mathbf{F} \cdot \mathbf{v}$ is a maximum when $\mathbf{F}$ and $\mathbf{v}$ are in phase, we conclude that **at resonance the applied force is in phase with the velocity and that the power transferred to the oscillator is a maximum.**

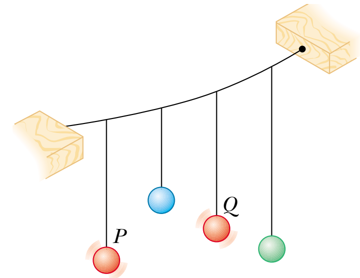
Figure 13.22 is a graph of amplitude as a function of frequency for a forced oscillator with and without damping. Note that the amplitude increases with decreasing damping ($b \rightarrow 0$) and that the resonance curve broadens as the damping increases. Under steady-state conditions and at any driving frequency, the energy transferred into the system equals the energy lost because of the damping force; hence, the average total energy of the oscillator remains constant. In the absence of a damping force ($b = 0$), we see from Equation 13.37 that the steady-state amplitude approaches infinity as $\omega \rightarrow \omega_0$. In other words, if there are no losses in the system and if we continue to drive an initially motionless oscillator with a periodic force that is in phase with the velocity, the amplitude of motion builds without limit (see the red curve in Fig. 13.22). This limitless building does not occur in practice because some damping is always present.

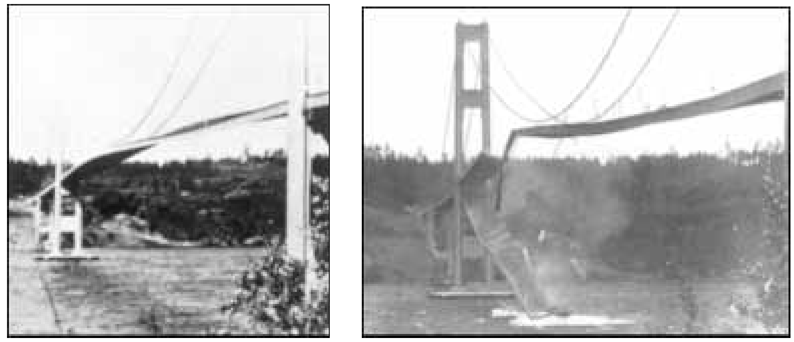
The behavior of a driven oscillating system after the driving force is removed depends on b and on how close ω was to ω_0. This behavior is sometimes quantified by a parameter called the *quality factor* Q. The closer a system is to being undamped, the greater its Q. The amplitude of oscillation drops by a factor of e (= 2.718 . . .) in Q/π cycles.

Later in this book we shall see that resonance appears in other areas of physics. For example, certain electrical circuits have natural frequencies. A bridge has natural frequencies that can be set into resonance by an appropriate driving force. A dramatic example of such resonance occurred in 1940, when the Tacoma Narrows Bridge in the state of Washington was destroyed by resonant vibrations. Although the winds were not particularly strong on that occasion, the bridge ultimately collapsed (Fig. 13.23) because the bridge design had no built-in safety features.

QuickLab

Tie several objects to strings and suspend them from a horizontal string, as illustrated in the figure. Make two of the hanging strings approximately the same length. If one of this pair, such as P, is set into sideways motion, all the others begin to oscillate. But Q, whose length is the same as that of P, oscillates with the greatest amplitude. Must all the masses have the same value?





(a) (b)

Figure 13.23 (a) In 1940 turbulent winds set up torsional vibrations in the Tacoma Narrows Bridge, causing it to oscillate at a frequency near one of the natural frequencies of the bridge structure. (b) Once established, this resonance condition led to the bridge's collapse.

Many other examples of resonant vibrations can be cited. A resonant vibration that you may have experienced is the "singing" of telephone wires in the wind. Machines often break if one vibrating part is at resonance with some other moving part. Soldiers marching in cadence across a bridge have been known to set up resonant vibrations in the structure and thereby cause it to collapse. Whenever any real physical system is driven near its resonance frequency, you can expect oscillations of very large amplitudes.

SUMMARY

When the acceleration of an object is proportional to its displacement from some equilibrium position and is in the direction opposite the displacement, the object moves with simple harmonic motion. The position x of a simple harmonic oscillator varies periodically in time according to the expression

$$x = A\cos(\omega t + \phi) \tag{13.3}$$

where A is the **amplitude** of the motion, ω is the **angular frequency,** and ϕ is the **phase constant.** The value of ϕ depends on the initial position and initial velocity of the oscillator. You should be able to use this formula to describe the motion of an object undergoing simple harmonic motion.

The time T needed for one complete oscillation is defined as the **period** of the motion:

$$T = \frac{2\pi}{\omega} \tag{13.4}$$

The inverse of the period is the **frequency** of the motion, which equals the number of oscillations per second.

The velocity and acceleration of a simple harmonic oscillator are

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \tag{13.7}$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \tag{13.8}$$

$$v = \pm\omega\sqrt{A^2 - x^2} \tag{13.23}$$

Thus, the maximum speed is ωA, and the maximum acceleration is $\omega^2 A$. The speed is zero when the oscillator is at its turning points, $x = \pm A$, and is a maximum when the oscillator is at the equilibrium position $x = 0$. The magnitude of the acceleration is a maximum at the turning points and zero at the equilibrium position. You should be able to find the velocity and acceleration of an oscillating object at any time if you know the amplitude, angular frequency, and phase constant.

A block–spring system moves in simple harmonic motion on a frictionless surface, with a period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \textbf{(13.18)}$$

The kinetic energy and potential energy for a simple harmonic oscillator vary with time and are given by

$$K = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) \quad \textbf{(13.20)}$$

$$U = \tfrac{1}{2} kx^2 = \tfrac{1}{2} kA^2 \cos^2(\omega t + \phi) \quad \textbf{(13.21)}$$

These three formulas allow you to analyze a wide variety of situations involving oscillations. Be sure you recognize how the mass of the block and the spring constant of the spring enter into the calculations.

The total energy of a simple harmonic oscillator is a constant of the motion and is given by

$$E = \tfrac{1}{2} kA^2 \quad \textbf{(13.22)}$$

The potential energy of the oscillator is a maximum when the oscillator is at its turning points and is zero when the oscillator is at the equilibrium position. The kinetic energy is zero at the turning points and a maximum at the equilibrium position. You should be able to determine the division of energy between potential and kinetic forms at any time t.

A **simple pendulum** of length L moves in simple harmonic motion. For small angular displacements from the vertical, its period is

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \textbf{(13.26)}$$

For small angular displacements from the vertical, a **physical pendulum** moves in simple harmonic motion about a pivot that does not go through the center of mass. The period of this motion is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad \textbf{(13.28)}$$

where I is the moment of inertia about an axis through the pivot and d is the distance from the pivot to the center of mass. You should be able to distinguish when to use the simple-pendulum formula and when the system must be considered a physical pendulum.

Uniform circular motion can be considered a combination of two simple harmonic motions, one along the x axis and the other along the y axis, with the two differing in phase by 90°.

QUESTIONS

1. Is a bouncing ball an example of simple harmonic motion? Is the daily movement of a student from home to school and back simple harmonic motion? Why or why not?

2. If the coordinate of a particle varies as $x = -A\cos\omega t$, what is the phase constant in Equation 13.3? At what position does the particle begin its motion?

3. Does the displacement of an oscillating particle between $t = 0$ and a later time t necessarily equal the position of the particle at time t? Explain.
4. Determine whether the following quantities can be in the same direction for a simple harmonic oscillator: (a) displacement and velocity, (b) velocity and acceleration, (c) displacement and acceleration.
5. Can the amplitude A and the phase constant ϕ be determined for an oscillator if only the position is specified at $t = 0$? Explain.
6. Describe qualitatively the motion of a mass–spring system when the mass of the spring is not neglected.
7. Make a graph showing the potential energy of a stationary block hanging from a spring, $U = \frac{1}{2}ky^2 + mgy$. Why is the lowest part of the graph offset from the origin?
8. A block–spring system undergoes simple harmonic motion with an amplitude A. Does the total energy change if the mass is doubled but the amplitude is not changed? Do the kinetic and potential energies depend on the mass? Explain.
9. What happens to the period of a simple pendulum if the pendulum's length is doubled? What happens to the period if the mass of the suspended bob is doubled?
10. A simple pendulum is suspended from the ceiling of a stationary elevator, and the period is determined. Describe the changes, if any, in the period when the elevator (a) accelerates upward, (b) accelerates downward, and (c) moves with constant velocity.
11. A simple pendulum undergoes simple harmonic motion when θ is small. Is the motion periodic when θ is large? How does the period of motion change as θ increases?
12. Will damped oscillations occur for any values of b and k? Explain.
13. As it possible to have damped oscillations when a system is at resonance? Explain.
14. At resonance, what does the phase constant ϕ equal in Equation 13.36? (*Hint:* Compare this equation with the expression for the driving force, which must be in phase with the velocity at resonance.)
15. Some parachutes have holes in them to allow air to move smoothly through them. Without such holes, sometimes the air that has gathered beneath the chute as a parachutist falls is released from under its edges alternately and periodically, at one side and then at the other. Why might this periodic release of air cause a problem?
16. If a grandfather clock were running slowly, how could we adjust the length of the pendulum to correct the time?
17. A pendulum bob is made from a sphere filled with water. What would happen to the frequency of vibration of this pendulum if the sphere had a hole in it that allowed the water to leak out slowly?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Section 13.1 Simple Harmonic Motion

1. The displacement of a particle at $t = 0.250$ s is given by the expression $x = (4.00 \text{ m}) \cos(3.00\pi t + \pi)$, where x is in meters and t is in seconds. Determine (a) the frequency and period of the motion, (b) the amplitude of the motion, (c) the phase constant, and (d) the displacement of the particle at $t = 0.250$ s.
2. A ball dropped from a height of 4.00 m makes a perfectly elastic collision with the ground. Assuming that no energy is lost due to air resistance, (a) show that the motion is periodic and (b) determine the period of the motion. (c) Is the motion simple harmonic? Explain.
3. A particle moves in simple harmonic motion with a frequency of 3.00 oscillations/s and an amplitude of 5.00 cm. (a) Through what total distance does the particle move during one cycle of its motion? (b) What is its maximum speed? Where does this occur? (c) Find the maximum acceleration of the particle. Where in the motion does the maximum acceleration occur?
4. In an engine, a piston oscillates with simple harmonic motion so that its displacement varies according to the expression

$$x = (5.00 \text{ cm}) \cos(2t + \pi/6)$$

where x is in centimeters and t is in seconds. At $t = 0$, find (a) the displacement of the particle, (b) its velocity, and (c) its acceleration. (d) Find the period and amplitude of the motion.

WEB 5. A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is 2.00 cm, and the frequency is 1.50 Hz. (a) Show that the displacement of the particle is given by $x = (2.00 \text{ cm}) \sin(3.00\pi t)$. Determine (b) the maximum speed and the earliest time ($t > 0$) at which the particle has this speed, (c) the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration, and (d) the total distance traveled between $t = 0$ and $t = 1.00$ s.

6. The initial position and initial velocity of an object moving in simple harmonic motion are x_i and v_i; the angular frequency of oscillation is ω. (a) Show that the position and velocity of the object for all time can be written as

$$x(t) = x_i \cos \omega t + \left(\frac{v_i}{\omega}\right) \sin \omega t$$

$$v(t) = -x_i\omega \sin \omega t + v_i \cos \omega t$$

(b) If the amplitude of the motion is A, show that

$$v^2 - ax = v_i^2 - a_i x_i = \omega^2 A^2$$

Section 13.2 The Block–Spring System Revisited

Note: Neglect the mass of the spring in all problems in this section.

7. A spring stretches by 3.90 cm when a 10.0-g mass is hung from it. If a 25.0-g mass attached to this spring oscillates in simple harmonic motion, calculate the period of the motion.

8. A simple harmonic oscillator takes 12.0 s to undergo five complete vibrations. Find (a) the period of its motion, (b) the frequency in hertz, and (c) the angular frequency in radians per second.

9. A 0.500-kg mass attached to a spring with a force constant of 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate (a) the maximum value of its speed and acceleration, (b) the speed and acceleration when the mass is 6.00 cm from the equilibrium position, and (c) the time it takes the mass to move from $x = 0$ to $x = 8.00$ cm.

10. A 1.00-kg mass attached to a spring with a force constant of 25.0 N/m oscillates on a horizontal, frictionless track. At $t = 0$, the mass is released from rest at $x = -3.00$ cm. (That is, the spring is compressed by 3.00 cm.) Find (a) the period of its motion; (b) the maximum values of its speed and acceleration; and (c) the displacement, velocity, and acceleration as functions of time.

11. A 7.00-kg mass is hung from the bottom end of a vertical spring fastened to an overhead beam. The mass is set into vertical oscillations with a period of 2.60 s. Find the force constant of the spring.

12. A block of unknown mass is attached to a spring with a spring constant of 6.50 N/m and undergoes simple harmonic motion with an amplitude of 10.0 cm. When the mass is halfway between its equilibrium position and the end point, its speed is measured to be + 30.0 cm/s. Calculate (a) the mass of the block, (b) the period of the motion, and (c) the maximum acceleration of the block.

13. A particle that hangs from a spring oscillates with an angular frequency of 2.00 rad/s. The spring–particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed of 1.50 m/s. The car then stops suddenly. (a) With what amplitude does the particle oscillate? (b) What is the equation of motion for the particle? (Choose upward as the positive direction.)

14. A particle that hangs from a spring oscillates with an angular frequency ω. The spring–particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed v. The car then stops suddenly. (a) With what amplitude does the particle oscillate? (b) What is the equation of motion for the particle? (Choose upward as the positive direction.)

15. A 1.00-kg mass is attached to a horizontal spring. The spring is initially stretched by 0.100 m, and the mass is released from rest there. It proceeds to move without friction. After 0.500 s, the speed of the mass is zero. What is the maximum speed of the mass?

Section 13.3 Energy of the Simple Harmonic Oscillator

Note: Neglect the mass of the spring in all problems in this section.

16. A 200-g mass is attached to a spring and undergoes simple harmonic motion with a period of 0.250 s. If the total energy of the system is 2.00 J, find (a) the force constant of the spring and (b) the amplitude of the motion.

WEB **17.** An automobile having a mass of 1 000 kg is driven into a brick wall in a safety test. The bumper behaves as a spring of constant 5.00×10^6 N/m and compresses 3.16 cm as the car is brought to rest. What was the speed of the car before impact, assuming that no energy is lost during impact with the wall?

18. A mass–spring system oscillates with an amplitude of 3.50 cm. If the spring constant is 250 N/m and the mass is 0.500 kg, determine (a) the mechanical energy of the system, (b) the maximum speed of the mass, and (c) the maximum acceleration.

19. A 50.0-g mass connected to a spring with a force constant of 35.0 N/m oscillates on a horizontal, frictionless surface with an amplitude of 4.00 cm. Find (a) the total energy of the system and (b) the speed of the mass when the displacement is 1.00 cm. Find (c) the kinetic energy and (d) the potential energy when the displacement is 3.00 cm.

20. A 2.00-kg mass is attached to a spring and placed on a horizontal, smooth surface. A horizontal force of 20.0 N is required to hold the mass at rest when it is pulled 0.200 m from its equilibrium position (the origin of the x axis). The mass is now released from rest with an initial displacement of $x_i = 0.200$ m, and it subsequently undergoes simple harmonic oscillations. Find (a) the force constant of the spring, (b) the frequency of the oscillations, and (c) the maximum speed of the mass. Where does this maximum speed occur? (d) Find the maximum acceleration of the mass. Where does it occur? (e) Find the total energy of the oscillating system. Find (f) the speed and (g) the acceleration when the displacement equals one third of the maximum value.

21. A 1.50-kg block at rest on a tabletop is attached to a horizontal spring having force constant of 19.6 N/m. The spring is initially unstretched. A constant 20.0-N horizontal force is applied to the object, causing the spring to stretch. (a) Determine the speed of the block after it has moved 0.300 m from equilibrium, assuming that the surface between the block and the tabletop is frictionless. (b) Answer part (a) for a coefficient of kinetic friction of 0.200 between the block and the tabletop.

22. The amplitude of a system moving in simple harmonic motion is doubled. Determine the change in (a) the total energy, (b) the maximum speed, (c) the maximum acceleration, and (d) the period.

23. A particle executes simple harmonic motion with an amplitude of 3.00 cm. At what displacement from the midpoint of its motion does its speed equal one half of its maximum speed?

24. A mass on a spring with a constant of 3.24 N/m vibrates, with its position given by the equation $x = (5.00 \text{ cm}) \cos(3.60t \text{ rad/s})$. (a) During the first cycle, for $0 < t < 1.75$ s, when is the potential energy of the system changing most rapidly into kinetic energy? (b) What is the maximum rate of energy transformation?

Section 13.4 The Pendulum

25. A man enters a tall tower, needing to know its height. He notes that a long pendulum extends from the ceiling almost to the floor and that its period is 12.0 s. (a) How tall is the tower? (b) If this pendulum is taken to the Moon, where the free-fall acceleration is 1.67 m/s^2, what is its period there?

26. A "seconds" pendulum is one that moves through its equilibrium position once each second. (The period of the pendulum is 2.000 s.) The length of a seconds pendulum is 0.992 7 m at Tokyo and 0.994 2 m at Cambridge, England. What is the ratio of the free-fall accelerations at these two locations?

27. A rigid steel frame above a street intersection supports standard traffic lights, each of which is hinged to hang immediately below the frame. A gust of wind sets a light swinging in a vertical plane. Find the order of magnitude of its period. State the quantities you take as data and their values.

28. The angular displacement of a pendulum is represented by the equation $\theta = (0.320 \text{ rad})\cos \omega t$, where θ is in radians and $\omega = 4.43$ rad/s. Determine the period and length of the pendulum.

WEB 29. A simple pendulum has a mass of 0.250 kg and a length of 1.00 m. It is displaced through an angle of 15.0° and then released. What are (a) the maximum speed, (b) the maximum angular acceleration, and (c) the maximum restoring force?

30. A simple pendulum is 5.00 m long. (a) What is the period of simple harmonic motion for this pendulum if it is hanging in an elevator that is accelerating upward at 5.00 m/s^2? (b) What is its period if the elevator is accelerating downward at 5.00 m/s^2? (c) What is the period of simple harmonic motion for this pendulum if it is placed in a truck that is accelerating horizontally at 5.00 m/s^2?

31. A particle of mass m slides without friction inside a hemispherical bowl of radius R. Show that, if it starts from rest with a small displacement from equilibrium, the particle moves in simple harmonic motion with an angular frequency equal to that of a simple pendulum of length R. That is, $\omega = \sqrt{g/R}$.

32. A mass is attached to the end of a string to form a simple pendulum. The period of its harmonic motion is measured for small angular displacements and three lengths; in each case, the motion is clocked with a stopwatch for 50 oscillations. For lengths of 1.000 m, 0.750 m, and 0.500 m, total times of 99.8 s, 86.6 s, and 71.1 s, respectively, are measured for the 50 oscillations. (a) Determine the period of motion for each length. (b) Determine the mean value of g obtained from these three independent measurements, and compare it with the accepted value. (c) Plot T^2 versus L, and obtain a value for g from the slope of your best-fit straight-line graph. Compare this value with that obtained in part (b).

33. A physical pendulum in the form of a planar body moves in simple harmonic motion with a frequency of 0.450 Hz. If the pendulum has a mass of 2.20 kg and the pivot is located 0.350 m from the center of mass, determine the moment of inertia of the pendulum.

34. A very light, rigid rod with a length of 0.500 m extends straight out from one end of a meter stick. The stick is suspended from a pivot at the far end of the rod and is set into oscillation. (a) Determine the period of oscillation. (b) By what percentage does this differ from a 1.00-m-long simple pendulum?

35. Consider the physical pendulum of Figure 13.13. (a) If I_{CM} is its moment of inertia about an axis that passes through its center of mass and is parallel to the axis that passes through its pivot point, show that its period is

$$T = 2\pi\sqrt{\frac{I_{CM} + md^2}{mgd}}$$

where d is the distance between the pivot point and the center of mass. (b) Show that the period has a minimum value when d satisfies $md^2 = I_{CM}$.

36. A torsional pendulum is formed by attaching a wire to the center of a meter stick with a mass of 2.00 kg. If the resulting period is 3.00 min, what is the torsion constant for the wire?

37. A clock balance wheel has a period of oscillation of 0.250 s. The wheel is constructed so that 20.0 g of mass is concentrated around a rim of radius 0.500 cm. What are (a) the wheel's moment of inertia and (b) the torsion constant of the attached spring?

Section 13.5 Comparing Simple Harmonic Motion with Uniform Circular Motion

38. While riding behind a car that is traveling at 3.00 m/s, you notice that one of the car's tires has a small hemispherical boss on its rim, as shown in Figure P13.38. (a) Explain why the boss, from your viewpoint behind the car, executes simple harmonic motion. (b) If the radius of the car's tires is 0.300 m, what is the boss's period of oscillation?

39. Consider the simplified single-piston engine shown in Figure P13.39. If the wheel rotates with constant angular speed, explain why the piston rod oscillates in simple harmonic motion.

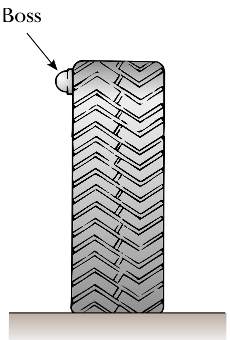


Figure P13.38

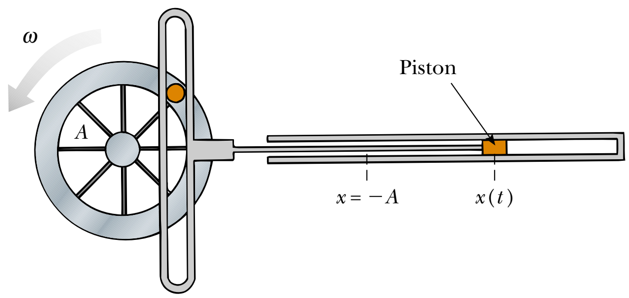


Figure P13.39

(Optional)
Section 13.6 Damped Oscillations

40. Show that the time rate of change of mechanical energy for a damped, undriven oscillator is given by $dE/dt = -bv^2$ and hence is always negative. (*Hint:* Differentiate the expression for the mechanical energy of an oscillator, $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, and use Eq. 13.32.)

41. A pendulum with a length of 1.00 m is released from an initial angle of 15.0°. After 1 000 s, its amplitude is reduced by friction to 5.50°. What is the value of $b/2m$?

42. Show that Equation 13.33 is a solution of Equation 13.32 provided that $b^2 < 4mk$.

(Optional)
Section 13.7 Forced Oscillations

43. A baby rejoices in the day by crowing and jumping up and down in her crib. Her mass is 12.5 kg, and the crib mattress can be modeled as a light spring with a force constant of 4.30 kN/m. (a) The baby soon learns to bounce with maximum amplitude and minimum effort by bending her knees at what frequency? (b) She learns to use the mattress as a trampoline—losing contact with it for part of each cycle—when her amplitude exceeds what value?

44. A 2.00-kg mass attached to a spring is driven by an external force $F = (3.00\text{ N})\cos(2\pi t)$. If the force constant of the spring is 20.0 N/m, determine (a) the period and (b) the amplitude of the motion. (*Hint:* Assume that there is no damping—that is, that $b = 0$—and use Eq. 13.37.)

45. Considering an *undamped,* forced oscillator ($b = 0$), show that Equation 13.36 is a solution of Equation 13.35, with an amplitude given by Equation 13.37.

46. A weight of 40.0 N is suspended from a spring that has a force constant of 200 N/m. The system is undamped and is subjected to a harmonic force with a frequency of 10.0 Hz, which results in a forced-motion amplitude of 2.00 cm. Determine the maximum value of the force.

47. Damping is negligible for a 0.150-kg mass hanging from a light 6.30-N/m spring. The system is driven by a force oscillating with an amplitude of 1.70 N. At what frequency will the force make the mass vibrate with an amplitude of 0.440 m?

48. You are a research biologist. Before dining at a fine restaurant, you set your pager to vibrate instead of beep, and you place it in the side pocket of your suit coat. The arm of your chair presses the light cloth against your body at one spot. Fabric with a length of 8.21 cm hangs freely below that spot, with the pager at the bottom. A co-worker telephones you. The motion of the vibrating pager makes the hanging part of your coat swing back and forth with remarkably large amplitude. The waiter, maître d', wine steward, and nearby diners notice immediately and fall silent. Your daughter pipes up and says, "Daddy, look! Your cockroaches must have gotten out again!" Find the frequency at which your pager vibrates.

ADDITIONAL PROBLEMS

49. A car with bad shock absorbers bounces up and down with a period of 1.50 s after hitting a bump. The car has a mass of 1 500 kg and is supported by four springs of equal force constant k. Determine the value of k.

50. A large passenger with a mass of 150 kg sits in the middle of the car described in Problem 49. What is the new period of oscillation?

51. A compact mass M is attached to the end of a uniform rod, of equal mass M and length L, that is pivoted at the top (Fig. P13.51). (a) Determine the tensions in the rod

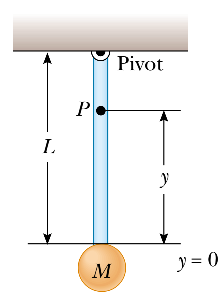


Figure P13.51

at the pivot and at the point P when the system is stationary. (b) Calculate the period of oscillation for small displacements from equilibrium, and determine this period for $L = 2.00$ m. (*Hint:* Assume that the mass at the end of the rod is a point mass, and use Eq. 13.28.)

52. A mass, $m_1 = 9.00$ kg, is in equilibrium while connected to a light spring of constant $k = 100$ N/m that is fastened to a wall, as shown in Figure P13.52a. A second mass, $m_2 = 7.00$ kg, is slowly pushed up against mass m_1, compressing the spring by the amount $A = 0.200$ m (see Fig. P13.52b). The system is then released, and both masses start moving to the right on the frictionless surface. (a) When m_1 reaches the equilibrium point, m_2 loses contact with m_1 (see Fig. P13.52c) and moves to the right with speed v. Determine the value of v. (b) How far apart are the masses when the spring is fully stretched for the first time (D in Fig. P13.52d)? (*Hint:* First determine the period of oscillation and the amplitude of the m_1–spring system after m_2 loses contact with m_1.)

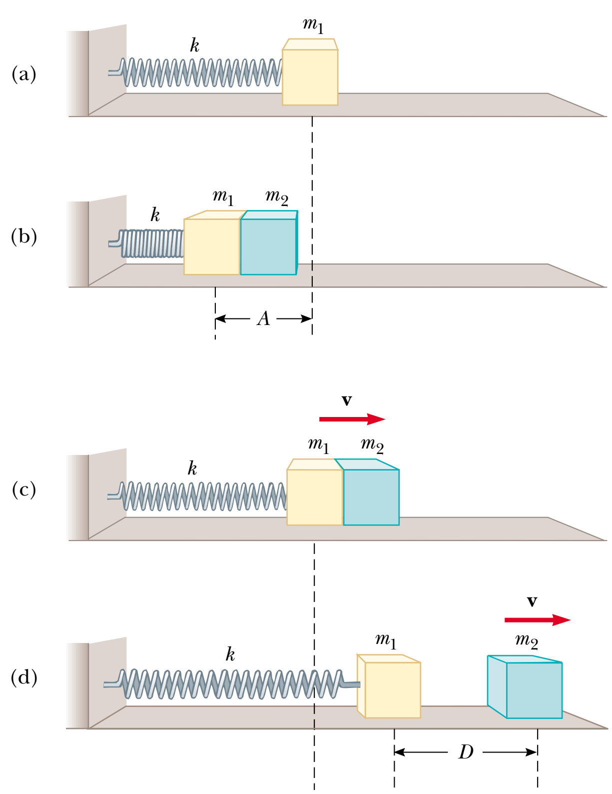


Figure P13.52

WEB **53.** A large block P executes horizontal simple harmonic motion as it slides across a frictionless surface with a frequency of $f = 1.50$ Hz. Block B rests on it, as shown in Figure P13.53, and the coefficient of static friction between the two is $\mu_s = 0.600$. What maximum amplitude of oscillation can the system have if block B is not to slip?

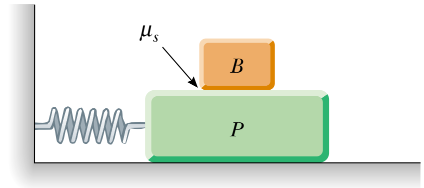


Figure P13.53 Problems 53 and 54.

54. A large block P executes horizontal simple harmonic motion as it slides across a frictionless surface with a frequency f. Block B rests on it, as shown in Figure P13.53, and the coefficient of static friction between the two is μ_s. What maximum amplitude of oscillation can the system have if the upper block is not to slip?

55. The mass of the deuterium molecule (D_2) is twice that of the hydrogen molecule (H_2). If the vibrational frequency of H_2 is 1.30×10^{14} Hz, what is the vibrational frequency of D_2? Assume that the "spring constant" of attracting forces is the same for the two molecules.

56. A solid sphere (radius $= R$) rolls without slipping in a cylindrical trough (radius $= 5R$), as shown in Figure P13.56. Show that, for small displacements from equilibrium perpendicular to the length of the trough, the sphere executes simple harmonic motion with a period $T = 2\pi\sqrt{28R/5g}$.

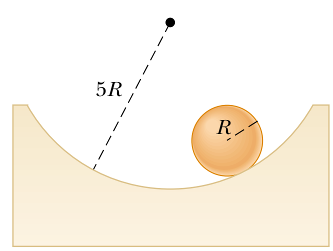


Figure P13.56

57. A light cubical container of volume a^3 is initially filled with a liquid of mass density ρ. The container is initially supported by a light string to form a pendulum of length L_i, measured from the center of mass of the filled container. The liquid is allowed to flow from the bottom of the container at a constant rate (dM/dt). At any time t, the level of the liquid in the container is h

and the length of the pendulum is L (measured relative to the instantaneous center of mass). (a) Sketch the apparatus and label the dimensions a, h, L_i, and L. (b) Find the time rate of change of the period as a function of time t. (c) Find the period as a function of time.

58. After a thrilling plunge, bungee-jumpers bounce freely on the bungee cords through many cycles. Your little brother can make a pest of himself by figuring out the mass of each person, using a proportion he set up by solving this problem: A mass m is oscillating freely on a vertical spring with a period T (Fig. P13.58a). An unknown mass m' on the same spring oscillates with a period T'. Determine (a) the spring constant k and (b) the unknown mass m'.

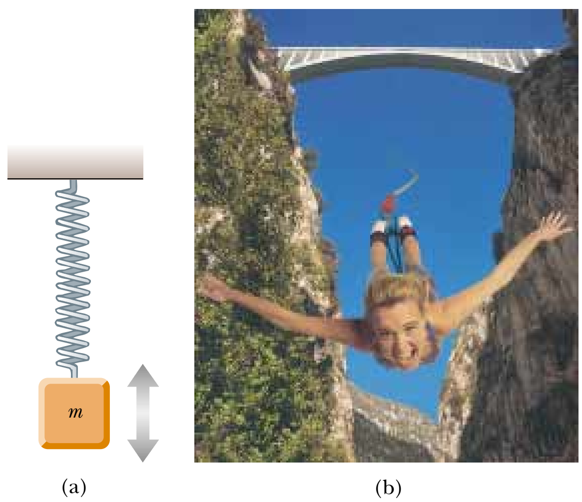


Figure P13.58 (a) Mass–spring system for Problems 58 and 68. (b) Bungee-jumping from a bridge. *(Telegraph Colour Library/ FPG International)*

59. A pendulum of length L and mass M has a spring of force constant k connected to it at a distance h below its point of suspension (Fig. P13.59). Find the frequency of vibration of the system for small values of the amplitude (small θ). (Assume that the vertical suspension of length L is rigid, but neglect its mass.)

60. A horizontal plank of mass m and length L is pivoted at one end. The plank's other end is supported by a spring of force constant k (Fig. P13.60). The moment of inertia of the plank about the pivot is $\frac{1}{3}\,mL^2$. (a) Show that the plank, after being displaced a small angle θ from its horizontal equilibrium position and released, moves with simple harmonic motion of angular frequency $\omega = \sqrt{3k/m}$. (b) Evaluate the frequency if the mass is 5.00 kg and the spring has a force constant of 100 N/m.

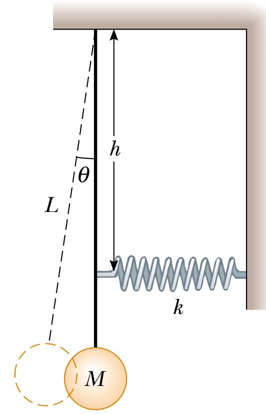


Figure P13.59

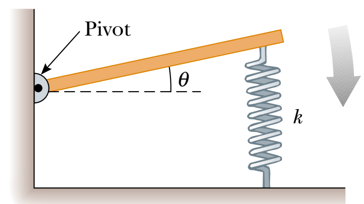


Figure P13.60

61. One end of a light spring with a force constant of 100 N/m is attached to a vertical wall. A light string is tied to the other end of the horizontal spring. The string changes from horizontal to vertical as it passes over a 4.00-cm-diameter solid pulley that is free to turn on a fixed smooth axle. The vertical section of the string supports a 200-g mass. The string does not slip at its contact with the pulley. Find the frequency of oscillation of the mass if the mass of the pulley is (a) negligible, (b) 250 g, and (c) 750 g.

62. A 2.00-kg block hangs without vibrating at the end of a spring ($k = 500$ N/m) that is attached to the ceiling of an elevator car. The car is rising with an upward acceleration of $g/3$ when the acceleration suddenly ceases (at $t = 0$). (a) What is the angular frequency of oscillation of the block after the acceleration ceases? (b) By what amount is the spring stretched during the acceleration of the elevator car? (c) What are the amplitude of the oscillation and the initial phase angle observed by a rider in the car? Take the upward direction to be positive.

63. A simple pendulum with a length of 2.23 m and a mass of 6.74 kg is given an initial speed of 2.06 m/s at its equilibrium position. Assume that it undergoes simple harmonic motion, and determine its (a) period, (b) total energy, and (c) maximum angular displacement.

64. People who ride motorcycles and bicycles learn to look out for bumps in the road and especially for *washboarding,* which is a condition of many equally spaced ridges worn into the road. What is so bad about washboarding? A motorcycle has several springs and shock absorbers in its suspension, but you can model it as a single spring supporting a mass. You can estimate the spring constant by thinking about how far the spring compresses when a big biker sits down on the seat. A motorcyclist traveling at highway speed must be particularly careful of washboard bumps that are a certain distance apart. What is the order of magnitude of their separation distance? State the quantities you take as data and the values you estimate or measure for them.

65. A wire is bent into the shape of one cycle of a cosine curve. It is held in a vertical plane so that the height y of the wire at any horizontal distance x from the center is given by $y = 20.0\text{ cm}[1 - \cos(0.160x\text{ rad/m})]$. A bead can slide without friction on the stationary wire. Show that if its excursion away from $x = 0$ is never large, the bead moves with simple harmonic motion. Determine its angular frequency. (*Hint:* $\cos\theta \cong 1 - \theta^2/2$ for small θ.)

66. A block of mass M is connected to a spring of mass m and oscillates in simple harmonic motion on a horizontal, frictionless track (Fig. P13.66). The force constant of the spring is k, and the equilibrium length is ℓ. Find (a) the kinetic energy of the system when the block has a speed v, and (b) the period of oscillation. (*Hint:* Assume that all portions of the spring oscillate in phase and that the velocity of a segment dx is proportional to the distance x from the fixed end; that is, $v_x = [x/\ell]v$. Also, note that the mass of a segment of the spring is $dm = [m/\ell]\,dx$.)

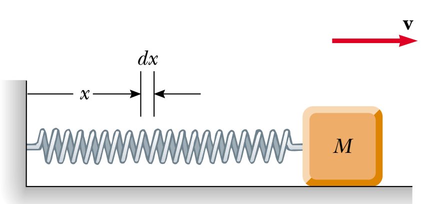


Figure P13.66

WEB **67.** A ball of mass m is connected to two rubber bands of length L, each under tension T, as in Figure P13.67. The ball is displaced by a small distance y perpendicular to the length of the rubber bands. Assuming that the tension does not change, show that (a) the restoring force is $-(2T/L)y$ and (b) the system exhibits simple harmonic motion with an angular frequency $\omega = \sqrt{2T/mL}$.

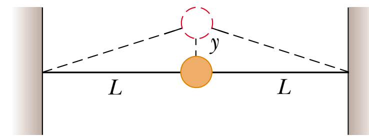


Figure P13.67

68. When a mass M, connected to the end of a spring of mass $m_s = 7.40$ g and force constant k, is set into simple harmonic motion, the period of its motion is

$$T = 2\pi\sqrt{\frac{M + (m_s/3)}{k}}$$

A two-part experiment is conducted with the use of various masses suspended vertically from the spring, as shown in Figure P13.58a. (a) Static extensions of 17.0, 29.3, 35.3, 41.3, 47.1, and 49.3 cm are measured for M values of 20.0, 40.0, 50.0, 60.0, 70.0, and 80.0 g, respectively. Construct a graph of Mg versus x, and perform a linear least-squares fit to the data. From the slope of your graph, determine a value for k for this spring. (b) The system is now set into simple harmonic motion, and periods are measured with a stopwatch. With $M =$ 80.0 g, the total time for 10 oscillations is measured to be 13.41 s. The experiment is repeated with M values of 70.0, 60.0, 50.0, 40.0, and 20.0 g, with corresponding times for 10 oscillations of 12.52, 11.67, 10.67, 9.62, and 7.03 s. Compute the experimental value for T for each of these measurements. Plot a graph of T^2 versus M, and determine a value for k from the slope of the linear least-squares fit through the data points. Compare this value of k with that obtained in part (a). (c) Obtain a value for m_s from your graph, and compare it with the given value of 7.40 g.

69. A small, thin disk of radius r and mass m is attached rigidly to the face of a second thin disk of radius R and mass M, as shown in Figure P13.69. The center of the small disk is located at the edge of the large disk. The large disk is mounted at its center on a frictionless axle. The assembly is rotated through a small angle θ from its equilibrium position and released. (a) Show that the

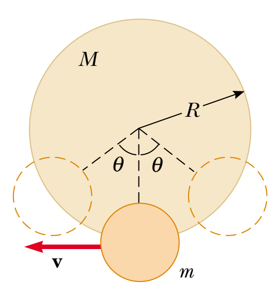


Figure P13.69

speed of the center of the small disk as it passes through the equilibrium position is

$$v = 2\left[\frac{Rg(1 - \cos\theta)}{(M/m) + (r/R)^2 + 2}\right]^{1/2}$$

(b) Show that the period of the motion is

$$T = 2\pi\left[\frac{(M + 2m)R^2 + mr^2}{2mgR}\right]^{1/2}$$

70. Consider the damped oscillator illustrated in Figure 13.19. Assume that the mass is 375 g, the spring constant is 100 N/m, and $b = 0.100$ kg/s. (a) How long does it takes for the amplitude to drop to half its initial value? (b) How long does it take for the mechanical energy to drop to half its initial value? (c) Show that, in general, the fractional rate at which the amplitude decreases in a damped harmonic oscillator is one-half the fractional rate at which the mechanical energy decreases.

71. A mass m is connected to two springs of force constants k_1 and k_2, as shown in Figure P13.71a and b. In each case, the mass moves on a frictionless table and is displaced from equilibrium and then released. Show that in the two cases the mass exhibits simple harmonic motion with periods

(a) $$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

(b) $$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

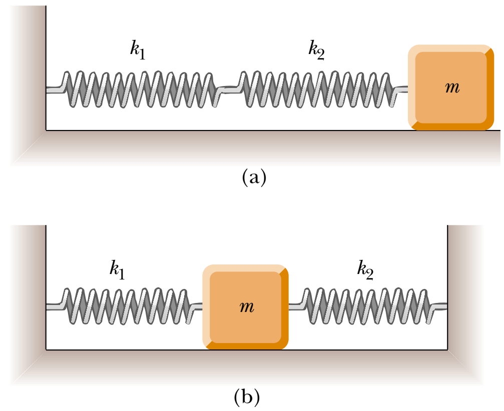


Figure P13.71

72. Consider a simple pendulum of length $L = 1.20$ m that is displaced from the vertical by an angle θ_{max} and then released. You are to predict the subsequent angular displacements when θ_{max} is small and also when it is large. Set up and carry out a numerical method to integrate the equation of motion for the simple pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Take the initial conditions to be $\theta = \theta_{max}$ and $d\theta/dt = 0$ at $t = 0$. On one trial choose $\theta_{max} = 5.00°$, and on another trial take $\theta_{max} = 100°$. In each case, find the displacement θ as a function of time. Using the same values for θ_{max}, compare your results for θ with those obtained from $\theta_{max}\cos\omega t$. How does the period for the large value of θ_{max} compare with that for the small value of θ_{max}? *Note:* Using the Euler method to solve this differential equation, you may find that the amplitude tends to increase with time. The fourth-order Runge–Kutta method would be a better choice to solve the differential equation. However, if you choose Δt small enough, the solution that you obtain using Euler's method can still be good.

Answers to Quick Quizzes

13.1 Because A can never be zero, ϕ must be any value that results in the cosine function's being zero at $t = 0$. In other words, $\phi = \cos^{-1}(0)$. This is true at $\phi = \pi/2$, $3\pi/2$ or, more generally, $\phi = \pm n\pi/2$, where n is any nonzero odd integer. If we want to restrict our choices of ϕ to values between 0 and 2π, we need to know whether the object was moving to the right or to the left at $t = 0$. If it was moving with a positive velocity, then $\phi = 3\pi/2$. If $v_i < 0$, then $\phi = \pi/2$.

13.2 (d) $4A$. From its maximum positive position to the equilibrium position, it travels a distance A, by definition of *amplitude*. It then goes an equal distance past the equilibrium position to its maximum negative position. It then repeats these two motions in the reverse direction to return to its original position and complete one cycle.

13.3 No, because in simple harmonic motion, the acceleration is not constant.

13.4 $x = -A\sin\omega t$, where $A = v_i/\omega$.

13.5 From Hooke's law, the spring constant must be $k = mg/L$. If we substitute this value for k into Equation 13.18, we find that

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

This is the same as Equation 13.26, which gives the period of a simple pendulum. Thus, when an object stretches a vertically hung spring, the period of the system is the same as that of a simple pendulum having a length equal to the amount of static extension of the spring.

13.6 If your goal is simply to stop the bounce from an absorbed shock as rapidly as possible, you should critically damp the suspension. Unfortunately, the stiffness of this design makes for an uncomfortable ride. If you underdamp the suspension, the ride is more comfortable but the car bounces. If you overdamp the suspension, the wheel is displaced from its equilibrium position longer than it should be. (For example, after hitting a bump, the spring stays compressed for a short time and the wheel does not quickly drop back down into contact with the road after the wheel is past the bump—a dangerous situation.) Because of all these considerations, automotive engineers usually design suspensions to be slightly underdamped. This allows the suspension to absorb a shock rapidly (minimizing the roughness of the ride) and then return to equilibrium after only one or two noticeable oscillations.

PUZZLER

More than 300 years ago, Isaac Newton realized that the same gravitational force that causes apples to fall to the Earth also holds the Moon in its orbit. In recent years, scientists have used the Hubble Space Telescope to collect evidence of the gravitational force acting even farther away, such as at this protoplanetary disk in the constellation Taurus. What properties of an object such as a protoplanet or the Moon determine the strength of its gravitational attraction to another object? *(Left, Larry West/FPG International; right, Courtesy of NASA)*

web

For more information about the Hubble, visit the Space Telescope Science Institute at **http://www.stsci.edu/**

chapter

14

The Law of Gravity

Chapter Outline

Before 1687, a large amount of data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces causing these motions was not available. In that year, Isaac Newton provided the key that unlocked the secrets of the heavens. He knew, from his first law, that a net force had to be acting on the Moon because without such a force the Moon would move in a straight-line path rather than in its almost circular orbit. Newton reasoned that this force was the gravitational attraction exerted by the Earth on the Moon. He realized that the forces involved in the Earth–Moon attraction and in the Sun–planet attraction were not something special to those systems, but rather were particular cases of a general and universal attraction between objects. In other words, Newton saw that the same force of attraction that causes the Moon to follow its path around the Earth also causes an apple to fall from a tree. As he put it, "I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centers about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the Earth; and found them answer pretty nearly."

In this chapter we study the law of gravity. We place emphasis on describing the motion of the planets because astronomical data provide an important test of the validity of the law of gravity. We show that the laws of planetary motion developed by Johannes Kepler follow from the law of gravity and the concept of conservation of angular momentum. We then derive a general expression for gravitational potential energy and examine the energetics of planetary and satellite motion. We close by showing how the law of gravity is also used to determine the force between a particle and an extended object.

14.1 NEWTON'S LAW OF UNIVERSAL GRAVITATION

You may have heard the legend that Newton was struck on the head by a falling apple while napping under a tree. This alleged accident supposedly prompted him to imagine that perhaps all bodies in the Universe were attracted to each other in the same way the apple was attracted to the Earth. Newton analyzed astronomical data on the motion of the Moon around the Earth. From that analysis, he made the bold assertion that the force law governing the motion of planets was the *same* as the force law that attracted a falling apple to the Earth. This was the first time that "earthly" and "heavenly" motions were unified. We shall look at the mathematical details of Newton's analysis in Section 14.5.

In 1687 Newton published his work on the law of gravity in his treatise *Mathematical Principles of Natural Philosophy.* **Newton's law of universal gravitation** states that

> every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

If the particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of this gravitational force is

The law of gravity

$$F_g = G\frac{m_1 m_2}{r^2} \tag{14.1}$$

where G is a constant, called the *universal gravitational constant,* that has been measured experimentally. As noted in Example 6.6, its value in SI units is

$$G = 6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2} \qquad \textbf{(14.2)}$$

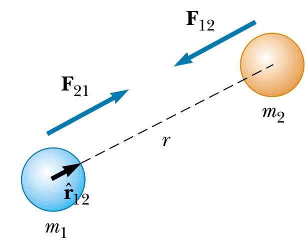


Figure 14.1 The gravitational force between two particles is attractive. The unit vector $\hat{\mathbf{r}}_{12}$ is directed from particle 1 to particle 2. Note that $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

The form of the force law given by Equation 14.1 is often referred to as an **inverse-square law** because the magnitude of the force varies as the inverse square of the separation of the particles.[1] We shall see other examples of this type of force law in subsequent chapters. We can express this force in vector form by defining a unit vector $\hat{\mathbf{r}}_{12}$ (Fig. 14.1). Because this unit vector is directed from particle 1 to particle 2, the force exerted by particle 1 on particle 2 is

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r^2}\,\hat{\mathbf{r}}_{12} \qquad \textbf{(14.3)}$$

where the minus sign indicates that particle 2 is attracted to particle 1, and hence the force must be directed toward particle 1. By Newton's third law, the force exerted by particle 2 on particle 1, designated $\mathbf{F}_{21}$, is equal in magnitude to $\mathbf{F}_{12}$ and in the opposite direction. That is, these forces form an action–reaction pair, and $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Properties of the gravitational force

Several features of Equation 14.3 deserve mention. The gravitational force is a field force that always exists between two particles, regardless of the medium that separates them. Because the force varies as the inverse square of the distance between the particles, it decreases rapidly with increasing separation. We can relate this fact to the geometry of the situation by noting that the intensity of light emanating from a point source drops off in the same $1/r^2$ manner, as shown in Figure 14.2.

Another important point about Equation 14.3 is that **the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.** For example, the force exerted by the

QuickLab

Inflate a balloon just enough to form a small sphere. Measure its diameter. Use a marker to color in a 1-cm square on its surface. Now continue inflating the balloon until it reaches twice the original diameter. Measure the size of the square you have drawn. Also note how the color of the marked area has changed. Have you verified what is shown in Figure 14.2?

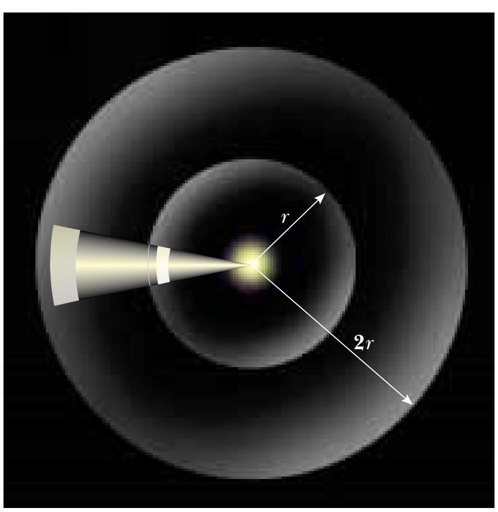


Figure 14.2 Light radiating from a point source drops off as $1/r^2$, a relationship that matches the way the gravitational force depends on distance. When the distance from the light source is doubled, the light has to cover four times the area and thus is one fourth as bright.

[1] An inverse relationship between two quantities x and y is one in which $y = k/x$, where k is a constant. A direct proportion between x and y exists when $y = kx$.

Earth on a particle of mass m near the Earth's surface has the magnitude

$$F_g = G\frac{M_E m}{{R_E}^2} \tag{14.4}$$

where M_E is the Earth's mass and R_E its radius. This force is directed toward the center of the Earth.

We have evidence of the fact that the gravitational force acting on an object is directly proportional to its mass from our observations of falling objects, discussed in Chapter 2. All objects, regardless of mass, fall in the absence of air resistance at the same acceleration g near the surface of the Earth. According to Newton's second law, this acceleration is given by $g = F_g/m$, where m is the mass of the falling object. If this ratio is to be the same for all falling objects, then F_g must be directly proportional to m, so that the mass cancels in the ratio. If we consider the more general situation of a gravitational force between any two objects with mass, such as two planets, this same argument can be applied to show that the gravitational force is proportional to one of the masses. We can choose *either* of the masses in the argument, however; thus, the gravitational force must be directly proportional to *both* masses, as can be seen in Equation 14.3.

14.2 MEASURING THE GRAVITATIONAL CONSTANT

The universal gravitational constant G was measured in an important experiment by Henry Cavendish (1731–1810) in 1798. The Cavendish apparatus consists of two small spheres, each of mass m, fixed to the ends of a light horizontal rod suspended by a fine fiber or thin metal wire, as illustrated in Figure 14.3. When two large spheres, each of mass M, are placed near the smaller ones, the attractive force between smaller and larger spheres causes the rod to rotate and twist the wire suspension to a new equilibrium orientation. The angle of rotation is measured by the deflection of a light beam reflected from a mirror attached to the vertical suspension. The deflection of the light is an effective technique for amplifying the motion. The experiment is carefully repeated with different masses at various separations. In addition to providing a value for G, the results show experimentally that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.

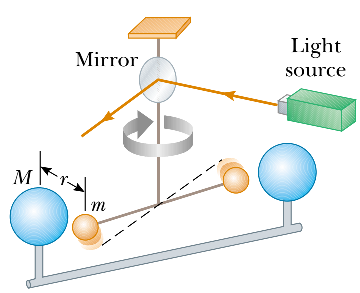


Figure 14.3 Schematic diagram of the Cavendish apparatus for measuring G. As the small spheres of mass m are attracted to the large spheres of mass M, the rod between the two small spheres rotates through a small angle. A light beam reflected from a mirror on the rotating apparatus measures the angle of rotation. The dashed line represents the original position of the rod.

EXAMPLE 14.1 Billiards, Anyone?

Three 0.300-kg billiard balls are placed on a table at the corners of a right triangle, as shown in Figure 14.4. Calculate the gravitational force on the cue ball (designated m_1) resulting from the other two balls.

Solution First we calculate separately the individual forces on the cue ball due to the other two balls, and then we find the vector sum to get the resultant force. We can see graphically that this force should point upward and toward the right. We locate our coordinate axes as shown in Figure 14.4, placing our origin at the position of the cue ball.

The force exerted by m_2 on the cue ball is directed upward and is given by

$$\mathbf{F}_{21} = G\frac{m_2 m_1}{{r_{21}}^2}\mathbf{j}$$

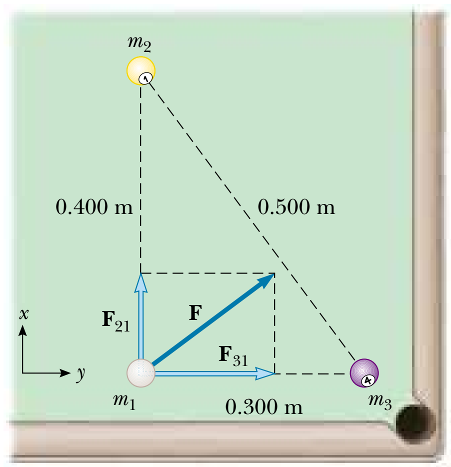


Figure 14.4 The resultant gravitational force acting on the cue ball is the vector sum $\mathbf{F}_{21} + \mathbf{F}_{31}$.

$$= \left(6.67 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\right)\frac{(0.300\ \mathrm{kg})(0.300\ \mathrm{kg})}{(0.400\ \mathrm{m})^2}\mathbf{j}$$
$$= 3.75 \times 10^{-11}\mathbf{j}\ \mathrm{N}$$

This result shows that the gravitational forces between everyday objects have extremely small magnitudes. The force exerted by m_3 on the cue ball is directed to the right:

$$\mathbf{F}_{31} = G\frac{m_3 m_1}{{r_{31}}^2}\mathbf{i}$$
$$= \left(6.67 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\right)\frac{(0.300\ \mathrm{kg})(0.300\ \mathrm{kg})}{(0.300\ \mathrm{m})^2}\mathbf{i}$$
$$= 6.67 \times 10^{-11}\,\mathbf{i}\ \mathrm{N}$$

Therefore, the resultant force on the cue ball is

$$\mathbf{F} = \mathbf{F}_{21} + \mathbf{F}_{31} = (3.75\mathbf{j} + 6.67\mathbf{i}) \times 10^{-11}\ \mathrm{N}$$

and the magnitude of this force is

$$F = \sqrt{{F_{21}}^2 + {F_{31}}^2} = \sqrt{(3.75)^2 + (6.67)^2} \times 10^{-11}$$
$$= 7.65 \times 10^{-11}\ \mathrm{N}$$

Exercise Find the direction of **F**.

Answer 29.3° counterclockwise from the positive x axis.

14.3 FREE-FALL ACCELERATION AND THE GRAVITATIONAL FORCE

In Chapter 5, when defining mg as the weight of an object of mass m, we referred to g as the magnitude of the free-fall acceleration. Now we are in a position to obtain a more fundamental description of g. Because the force acting on a freely falling object of mass m near the Earth's surface is given by Equation 14.4, we can equate mg to this force to obtain

$$mg = G\frac{M_E m}{{R_E}^2}$$

$$g = G\frac{M_E}{{R_E}^2} \qquad \textbf{(14.5)}$$

Free-fall acceleration near the Earth's surface

Now consider an object of mass m located a distance h above the Earth's surface or a distance r from the Earth's center, where $r = R_E + h$. The magnitude of the gravitational force acting on this object is

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

The gravitational force acting on the object at this position is also $F_g = mg'$, where g' is the value of the free-fall acceleration at the altitude h. Substituting this expression for F_g into the last equation shows that g' is

Variation of g with altitude

$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \qquad (14.6)$$

Thus, it follows that g' *decreases* with *increasing altitude*. Because the weight of a body is mg', we see that as $r \to \infty$, its weight approaches zero.

EXAMPLE 14.2 Variation of g with Altitude h

The International Space Station is designed to operate at an altitude of 350 km. When completed, it will have a weight (measured at the Earth's surface) of 4.22×10^6 N. What is its weight when in orbit?

Solution Because the station is above the surface of the Earth, we expect its weight in orbit to be less than its weight on Earth, 4.22×10^6 N. Using Equation 14.6 with $h = 350$ km, we obtain

$$g' = \frac{GM_E}{(R_E + h)^2}$$

$$= \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{(6.37 \times 10^6\ \text{m} + 0.350 \times 10^6\ \text{m})^2}$$

$$= 8.83\ \text{m/s}^2$$

Because $g'/g = 8.83/9.80 = 0.901$, we conclude that the weight of the station at an altitude of 350 km is 90.1% of the value at the Earth's surface. So the station's weight in orbit is

$$(0.901)(4.22 \times 10^6\ \text{N}) = \boxed{3.80 \times 10^6\ \text{N}}$$

Values of g' at other altitudes are listed in Table 14.1.

TABLE 14.1 Free-Fall Acceleration g' at Various Altitudes Above the Earth's Surface

Altitude h (km)	g' (m/s^2)
1 000	7.33
2 000	5.68
3 000	4.53
4 000	3.70
5 000	3.08
6 000	2.60
7 000	2.23
8 000	1.93
9 000	1.69
10 000	1.49
50 000	0.13
∞	0

web

The official web site for the International Space Station is **www.station.nasa.gov**

EXAMPLE 14.3 The Density of the Earth

Using the fact that $g = 9.80\ \text{m/s}^2$ at the Earth's surface, find the average density of the Earth.

Solution Using $g = 9.80\ \text{m/s}^2$ and $R_E = 6.37 \times 10^6$ m, we find from Equation 14.5 that $M_E = 5.96 \times 10^{24}$ kg. From this result, and using the definition of density from Chapter 1, we obtain

$$\rho_E = \frac{M_E}{V_E} = \frac{M_E}{\frac{4}{3}\pi R_E^3} = \frac{5.96 \times 10^{24}\ \text{kg}}{\frac{4}{3}\pi(6.37 \times 10^6\ \text{m})^3}$$

$$= \boxed{5.50 \times 10^3\ \text{kg/m}^3}$$

Because this value is about twice the density of most rocks at the Earth's surface, we conclude that the inner core of the Earth has a density much higher than the average value. It is most amazing that the Cavendish experiment, which determines G (and can be done on a tabletop), combined with simple free-fall measurements of g, provides information about the core of the Earth.

Astronauts F. Story Musgrave and Jeffrey A. Hoffman, along with the Hubble Space Telescope and the space shuttle *Endeavor,* are all falling around the Earth.

14.4 KEPLER'S LAWS

People have observed the movements of the planets, stars, and other celestial bodies for thousands of years. In early history, scientists regarded the Earth as the center of the Universe. This so-called geocentric model was elaborated and formalized by the Greek astronomer Claudius Ptolemy (c. 100–c. 170) in the second century A.D. and was accepted for the next 1 400 years. In 1543 the Polish astronomer Nicolaus Copernicus (1473–1543) suggested that the Earth and the other planets revolved in circular orbits around the Sun (the heliocentric model).

The Danish astronomer Tycho Brahe (1546–1601) wanted to determine how the heavens were constructed, and thus he developed a program to determine the positions of both stars and planets. It is interesting to note that those observations of the planets and 777 stars visible to the naked eye were carried out with only a large sextant and a compass. (The telescope had not yet been invented.)

The German astronomer Johannes Kepler was Brahe's assistant for a short while before Brahe's death, whereupon he acquired his mentor's astronomical data and spent 16 years trying to deduce a mathematical model for the motion of the planets. Such data are difficult to sort out because the Earth is also in motion around the Sun. After many laborious calculations, Kepler found that Brahe's data on the revolution of Mars around the Sun provided the answer.

Johannes Kepler German astronomer (1571–1630) The German astronomer Johannes Kepler is best known for developing the laws of planetary motion based on the careful observations of Tycho Brahe. *(Art Resource)*

For more information about Johannes Kepler, visit our Web site at **www.saunderscollege.com/physics/**

Kepler's analysis first showed that the concept of circular orbits around the Sun had to be abandoned. He eventually discovered that the orbit of Mars could be accurately described by an **ellipse.** Figure 14.5 shows the geometric description of an ellipse. The longest dimension is called the major axis and is of length $2a$, where a is the **semimajor axis.** The shortest dimension is the minor axis, of length $2b$, where b is the **semiminor axis.** On either side of the center is a **focal point,** a distance c from the center, where $a^2 = b^2 + c^2$. The Sun is located at one of the focal points of Mars's orbit. Kepler generalized his analysis to include the motions of all planets. The complete analysis is summarized in three statements known as **Kepler's laws:**

Kepler's laws

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

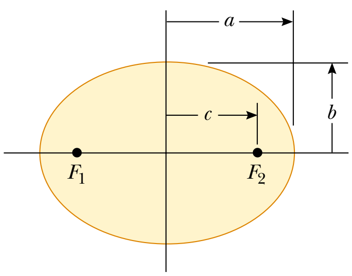


Figure 14.5 Plot of an ellipse. The semimajor axis has a length a, and the semiminor axis has a length b. The focal points are located at a distance c from the center, where $a^2 = b^2 + c^2$.

Most of the planetary orbits are close to circular in shape; for example, the semimajor and semiminor axes of the orbit of Mars differ by only 0.4%. Mercury and Pluto have the most elliptical orbits of the nine planets. In addition to the planets, there are many asteroids and comets orbiting the Sun that obey Kepler's laws. Comet Halley is such an object; it becomes visible when it is close to the Sun every 76 years. Its orbit is very elliptical, with a semiminor axis 76% smaller than its semimajor axis.

Although we do not prove it here, Kepler's first law is a direct consequence of the fact that the gravitational force varies as $1/r^2$. That is, under an inverse-square gravitational-force law, the orbit of a planet can be shown mathematically to be an ellipse with the Sun at one focal point. Indeed, half a century after Kepler developed his laws, Newton demonstrated that these laws are a consequence of the gravitational force that exists between any two masses. Newton's law of universal gravitation, together with his development of the laws of motion, provides the basis for a full mathematical solution to the motion of planets and satellites.

14.5 THE LAW OF GRAVITY AND THE MOTION OF PLANETS

In formulating his law of gravity, Newton used the following reasoning, which supports the assumption that the gravitational force is proportional to the inverse square of the separation between the two interacting bodies. He compared the acceleration of the Moon in its orbit with the acceleration of an object falling near the Earth's surface, such as the legendary apple (Fig. 14.6). Assuming that both accelerations had the same cause—namely, the gravitational attraction of the Earth—Newton used the inverse-square law to reason that the acceleration of the Moon toward the Earth (centripetal acceleration) should be proportional to $1/r_M^2$, where r_M is the distance between the centers of the Earth and the Moon. Furthermore, the acceleration of the apple toward the Earth should be proportional to $1/R_E^2$, where R_E is the radius of the Earth, or the distance between the centers of the Earth and the apple. Using the values $r_M = 3.84 \times 10^8$ m and

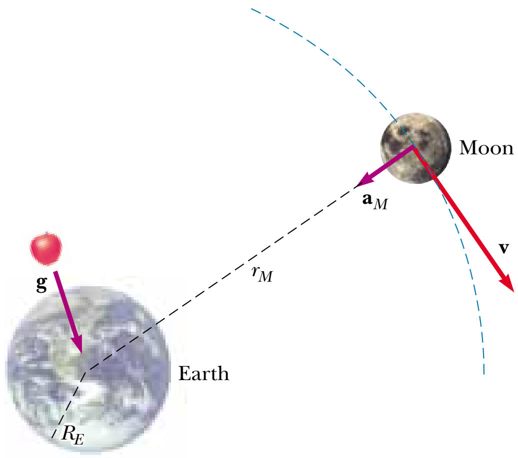


Figure 14.6 As it revolves around the Earth, the Moon experiences a centripetal acceleration $\mathbf{a}_M$ directed toward the Earth. An object near the Earth's surface, such as the apple shown here, experiences an acceleration $\mathbf{g}$. (Dimensions are not to scale.)

$R_E = 6.37 \times 10^6$ m, Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}}\right)^2 = 2.75 \times 10^{-4}$$

Therefore, the centripetal acceleration of the Moon is

Acceleration of the Moon

$$a_M = (2.75 \times 10^{-4})(9.80 \text{ m/s}^2) = 2.70 \times 10^{-3} \text{ m/s}^2$$

Newton also calculated the centripetal acceleration of the Moon from a knowledge of its mean distance from the Earth and its orbital period, $T = 27.32$ days $= 2.36 \times 10^6$ s. In a time T, the Moon travels a distance $2\pi r_M$, which equals the circumference of its orbit. Therefore, its orbital speed is $2\pi r_M/T$ and its centripetal acceleration is

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2(3.84 \times 10^8 \text{ m})}{(2.36 \times 10^6 \text{ s})^2}$$

$$= 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80 \text{ m/s}^2}{60^2}$$

In other words, because the Moon is roughly 60 Earth radii away, the gravitational acceleration at that distance should be about $1/60^2$ of its value at the Earth's surface. This is just the acceleration needed to account for the circular motion of the Moon around the Earth. The nearly perfect agreement between this value and the value Newton obtained using g provides strong evidence of the inverse-square nature of the gravitational force law.

Although these results must have been very encouraging to Newton, he was deeply troubled by an assumption he made in the analysis. To evaluate the acceleration of an object at the Earth's surface, Newton treated the Earth as if its mass were all concentrated at its center. That is, he assumed that the Earth acted as a particle as far as its influence on an exterior object was concerned. Several years later, in 1687, on the basis of his pioneering work in the development of calculus, Newton proved that this assumption was valid and was a natural consequence of the law of universal gravitation.

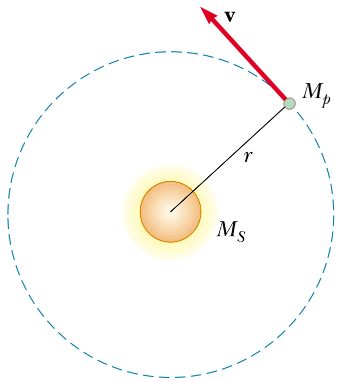


Figure 14.7 A planet of mass M_p moving in a circular orbit around the Sun. The orbits of all planets except Mercury and Pluto are nearly circular.

Kepler's third law

Kepler's Third Law

It is informative to show that Kepler's third law can be predicted from the inverse-square law for circular orbits.[2] Consider a planet of mass M_p moving around the Sun of mass M_S in a circular orbit, as shown in Figure 14.7. Because the gravitational force exerted by the Sun on the planet is a radially directed force that keeps the planet moving in a circle, we can apply Newton's second law ($\Sigma F = ma$) to the planet:

$$\frac{GM_SM_p}{r^2} = \frac{M_pv^2}{r}$$

Because the orbital speed v of the planet is simply $2\pi r/T$, where T is its period of revolution, the preceding expression becomes

$$\frac{GM_S}{r^2} = \frac{(2\pi r/T)^2}{r}$$

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = K_S r^3 \quad \textbf{(14.7)}$$

where K_S is a constant given by

$$K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19}\ \text{s}^2/\text{m}^3$$

Equation 14.7 is Kepler's third law. It can be shown that the law is also valid for elliptical orbits if we replace r with the length of the semimajor axis a. Note that the constant of proportionality K_S is independent of the mass of the planet. Therefore, Equation 14.7 is valid for *any* planet.[3] Table 14.2 contains a collection of useful planetary data. The last column verifies that T^2/r^3 is a constant. The small variations in the values in this column reflect uncertainties in the measured values of the periods and semimajor axes of the planets.

If we were to consider the orbit around the Earth of a satellite such as the Moon, then the proportionality constant would have a different value, with the Sun's mass replaced by the Earth's mass.

EXAMPLE 14.4 The Mass of the Sun

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.156×10^7 s and its distance from the Sun is 1.496×10^{11} m.

Solution Using Equation 14.7, we find that

$$M_S = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.496 \times 10^{11}\ \text{m})^3}{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(3.156 \times 10^7\ \text{s})^2}$$

$$= 1.99 \times 10^{30}\ \text{kg}$$

In Example 14.3, an understanding of gravitational forces enabled us to find out something about the density of the Earth's core, and now we have used this understanding to determine the mass of the Sun.

[2] The orbits of all planets except Mercury and Pluto are very close to being circular; hence, we do not introduce much error with this assumption. For example, the ratio of the semiminor axis to the semimajor axis for the Earth's orbit is $b/a = 0.999\ 86$.

[3] Equation 14.7 is indeed a proportion because the ratio of the two quantities T^2 and r^3 is a constant. The variables in a proportion are not required to be limited to the first power only.

TABLE 14.2 Useful Planetary Data

Body	Mass (kg)	Mean Radius (m)	Period of Revolution (s)	Mean Distance from Sun (m)	$\frac{T^2}{r^3}$ (s^2/m^3)
Mercury	3.18×10^{23}	2.43×10^{6}	7.60×10^{6}	5.79×10^{10}	2.97×10^{-19}
Venus	4.88×10^{24}	6.06×10^{6}	1.94×10^{7}	1.08×10^{11}	2.99×10^{-19}
Earth	5.98×10^{24}	6.37×10^{6}	3.156×10^{7}	1.496×10^{11}	2.97×10^{-19}
Mars	6.42×10^{23}	3.37×10^{6}	5.94×10^{7}	2.28×10^{11}	2.98×10^{-19}
Jupiter	1.90×10^{27}	6.99×10^{7}	3.74×10^{8}	7.78×10^{11}	2.97×10^{-19}
Saturn	5.68×10^{26}	5.85×10^{7}	9.35×10^{8}	1.43×10^{12}	2.99×10^{-19}
Uranus	8.68×10^{25}	2.33×10^{7}	2.64×10^{9}	2.87×10^{12}	2.95×10^{-19}
Neptune	1.03×10^{26}	2.21×10^{7}	5.22×10^{9}	4.50×10^{12}	2.99×10^{-19}
Pluto	$\approx 1.4 \times 10^{22}$	$\approx 1.5 \times 10^{6}$	7.82×10^{9}	5.91×10^{12}	2.96×10^{-19}
Moon	7.36×10^{22}	1.74×10^{6}	—	—	—
Sun	1.991×10^{30}	6.96×10^{8}	—	—	—

Kepler's Second Law and Conservation of Angular Momentum

Consider a planet of mass M_p moving around the Sun in an elliptical orbit (Fig. 14.8). The gravitational force acting on the planet is always along the radius vector, directed toward the Sun, as shown in Figure 14.9a. When a force is directed toward or away from a fixed point and is a function of r only, it is called a **central force.** The torque acting on the planet due to this force is clearly zero; that is, because $\mathbf{F}$ is parallel to $\mathbf{r}$,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F\hat{\mathbf{r}} = 0$$

(You may want to revisit Section 11.2 to refresh your memory on the vector product.) Recall from Equation 11.19, however, that torque equals the time rate of change of angular momentum: $\boldsymbol{\tau} = d\mathbf{L}/dt$. Therefore, **because the gravitational**

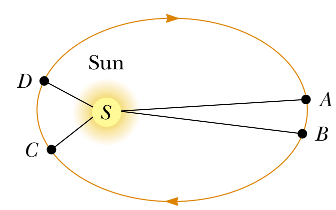
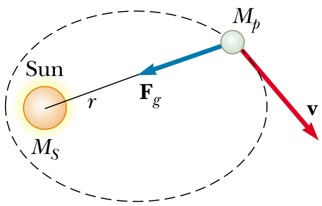
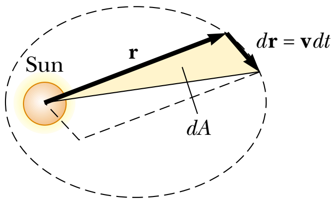


Figure 14.8 Kepler's second law is called the law of equal areas. When the time interval required for a planet to travel from A to B is equal to the time interval required for it to go from C to D, the two areas swept out by the planet's radius vector are equal. Note that in order for this to be true, the planet must be moving faster between C and D than between A and B.

Separate views of Jupiter and of Periodic Comet Shoemaker–Levy 9—both taken with the Hubble Space Telescope about two months before Jupiter and the comet collided in July 1994—were put together with the use of a computer. Their relative sizes and distances were altered. The black spot on Jupiter is the shadow of its moon Io.



(a)



(b)

Figure 14.9 (a) The gravitational force acting on a planet is directed toward the Sun, along the radius vector. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time dt is equal to one-half the area of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r} = \mathbf{v}dt$.

force exerted by the Sun on a planet results in no torque on the planet, the angular momentum L of the planet is constant:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times M_p\mathbf{v} = M_p\mathbf{r} \times \mathbf{v} = \text{constant} \qquad \textbf{(14.8)}$$

Because $\mathbf{L}$ remains constant, the planet's motion at any instant is restricted to the plane formed by $\mathbf{r}$ and $\mathbf{v}$.

We can relate this result to the following geometric consideration. The radius vector $\mathbf{r}$ in Figure 14.9b sweeps out an area dA in a time dt. This area equals one-half the area $|\mathbf{r} \times d\mathbf{r}|$ of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r}$ (see Section 11.2). Because the displacement of the planet in a time dt is $d\mathbf{r} = \mathbf{v}dt$, we can say that

$$dA = \tfrac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}\, dt| = \frac{L}{2M_p}\, dt$$

$$\frac{dA}{dt} = \frac{L}{2M_p} = \text{constant} \qquad \textbf{(14.9)}$$

where L and M_p are both constants. Thus, we conclude that

the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

It is important to recognize that this result, which is Kepler's second law, is a consequence of the fact that the force of gravity is a central force, which in turn implies that angular momentum is constant. Therefore, Kepler's second law applies to *any* situation involving a central force, whether inverse-square or not.

EXAMPLE 14.5 Motion in an Elliptical Orbit

A satellite of mass m moves in an elliptical orbit around the Earth (Fig. 14.10). The minimum distance of the satellite from the Earth is called the *perigee* (indicated by p in Fig. 14.10), and the maximum distance is called the *apogee* (indicated by a). If the speed of the satellite at p is v_p, what is its speed at a?

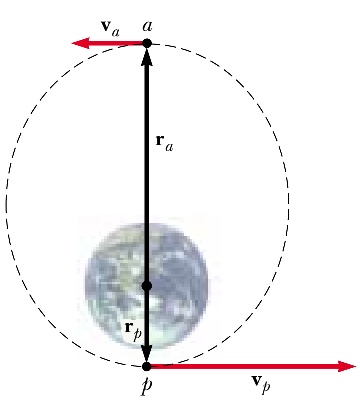


Figure 14.10 As a satellite moves around the Earth in an elliptical orbit, its angular momentum is constant. Therefore, $mv_ar_a = mv_pr_p$, where the subscripts a and p represent apogee and perigee, respectively.

Solution As the satellite moves from perigee toward apogee, it is moving farther from the Earth. Thus, a component of the gravitational force exerted by the Earth on the satellite is opposite the velocity vector. Negative work is done on the satellite, which causes it to slow down, according to the work–kinetic energy theorem. As a result, we expect the speed at apogee to be lower than the speed at perigee.

The angular momentum of the satellite relative to the Earth is $\mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times \mathbf{v}$. At the points a and p, $\mathbf{v}$ is perpendicular to $\mathbf{r}$. Therefore, the magnitude of the angular momentum at these positions is $L_a = mv_ar_a$ and $L_p = mv_pr_p$. Because angular momentum is constant, we see that

$$mv_ar_a = mv_pr_p$$

$$v_a = \frac{r_p}{r_a}v_p$$

Quick Quiz 14.1

How would you explain the fact that Saturn and Jupiter have periods much greater than one year?

14.6 THE GRAVITATIONAL FIELD

When Newton published his theory of universal gravitation, it was considered a success because it satisfactorily explained the motion of the planets. Since 1687 the same theory has been used to account for the motions of comets, the deflection of a Cavendish balance, the orbits of binary stars, and the rotation of galaxies. Nevertheless, both Newton's contemporaries and his successors found it difficult to accept the concept of a force that acts through a distance, as mentioned in Section 5.1. They asked how it was possible for two objects to interact when they were not in contact with each other. Newton himself could not answer that question.

An approach to describing interactions between objects that are not in contact came well after Newton's death, and it enables us to look at the gravitational interaction in a different way. As described in Section 5.1, this alternative approach uses the concept of a **gravitational field** that exists at every point in space. When a particle of mass m is placed at a point where the gravitational field is $\mathbf{g}$, the particle experiences a force $\mathbf{F}_g = m\mathbf{g}$. In other words, the field exerts a force on the particle. Hence, the gravitational field $\mathbf{g}$ is defined as

Gravitational field

$$\mathbf{g} \equiv \frac{\mathbf{F}_g}{m} \qquad (14.10)$$

That is, the gravitational field at a point in space equals the gravitational force experienced by a *test particle* placed at that point divided by the mass of the test particle. Notice that the presence of the test particle is not necessary for the field to exist—the Earth creates the gravitational field. We call the object creating the field the *source particle* (although the Earth is clearly not a particle; we shall discuss shortly the fact that we can approximate the Earth as a particle for the purpose of finding the gravitational field that it creates). We can detect the presence of the field and measure its strength by placing a test particle in the field and noting the force exerted on it.

Although the gravitational force is inherently an interaction between two objects, the concept of a gravitational field allows us to "factor out" the mass of one of the objects. In essence, we are describing the "effect" that any object (in this case, the Earth) has on the empty space around itself in terms of the force that *would* be present *if* a second object were somewhere in that space.[4]

As an example of how the field concept works, consider an object of mass m near the Earth's surface. Because the gravitational force acting on the object has a magnitude GM_Em/r^2 (see Eq. 14.4), the field $\mathbf{g}$ at a distance r from the center of the Earth is

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\frac{GM_E}{r^2}\,\hat{\mathbf{r}} \qquad (14.11)$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the Earth and the minus

[4] We shall return to this idea of mass affecting the space around it when we discuss Einstein's theory of gravitation in Chapter 39.

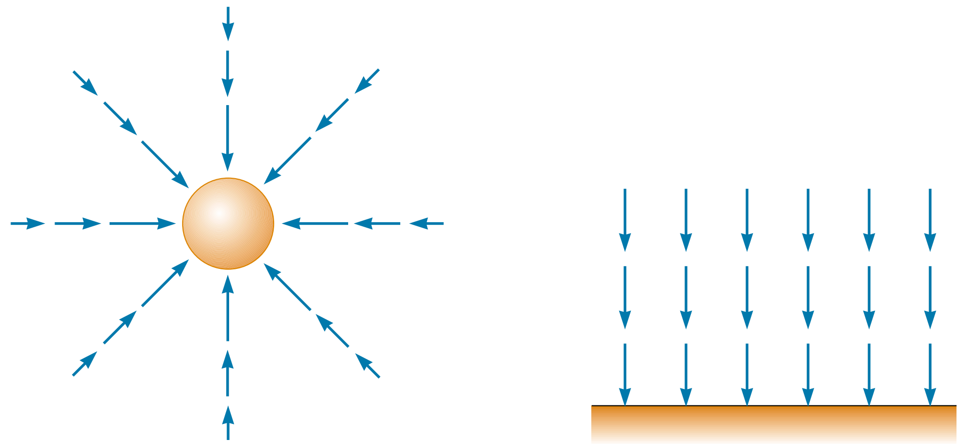


Figure 14.11 (a) The gravitational field vectors in the vicinity of a uniform spherical mass such as the Earth vary in both direction and magnitude. The vectors point in the direction of the acceleration a particle would experience if it were placed in the field. The magnitude of the field vector at any location is the magnitude of the free-fall acceleration at that location. (b) The gravitational field vectors in a small region near the Earth's surface are uniform in both direction and magnitude.

sign indicates that the field points toward the center of the Earth, as illustrated in Figure 14.11a. Note that the field vectors at different points surrounding the Earth vary in both direction and magnitude. In a small region near the Earth's surface, the downward field **g** is approximately constant and uniform, as indicated in Figure 14.11b. Equation 14.11 is valid at all points *outside* the Earth's surface, assuming that the Earth is spherical. At the Earth's surface, where $r = R_E$, **g** has a magnitude of 9.80 N/kg.

14.7 GRAVITATIONAL POTENTIAL ENERGY

In Chapter 8 we introduced the concept of gravitational potential energy, which is the energy associated with the position of a particle. We emphasized that the gravitational potential energy function $U = mgy$ is valid only when the particle is near the Earth's surface, where the gravitational force is constant. Because the gravitational force between two particles varies as $1/r^2$, we expect that a more general potential energy function—one that is valid without the restriction of having to be near the Earth's surface—will be significantly different from $U = mgy$.

Before we calculate this general form for the gravitational potential energy function, let us first verify that *the gravitational force is conservative.* (Recall from Section 8.2 that a force is conservative if the work it does on an object moving between any two points is independent of the path taken by the object.) To do this, we first note that the gravitational force is a central force. By definition, a central force is any force that is directed along a radial line to a fixed center and has a magnitude that depends only on the radial coordinate r. Hence, a central force can be represented by $F(r)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector directed from the origin to the particle, as shown in Figure 14.12.

Consider a central force acting on a particle moving along the general path P to Q in Figure 14.12. The path from P to Q can be approximated by a series of steps according to the following procedure. In Figure 14.12, we draw several thin wedges, which are shown as dashed lines. The outer boundary of our set of wedges is a path consisting of short radial line segments and arcs (gray in the figure). We select the length of the radial dimension of each wedge such that the short arc at the wedge's wide end intersects the actual path of the particle. Then we can approximate the actual path with a series of zigzag movements that alternate between moving along an arc and moving along a radial line.

By definition, a central force is always directed along one of the radial segments; therefore, the work done by **F** along any radial segment is

$$dW = \mathbf{F} \cdot d\mathbf{r} = F(r)\ dr$$

You should recall that, by definition, the work done by a force that is perpendicular to the displacement is zero. Hence, the work done in moving along any arc is zero because **F** is perpendicular to the displacement along these segments. Therefore, the total work done by **F** is the sum of the contributions along the radial segments:

Work done by a central force

$$W = \int_{r_i}^{r_f} F(r)\ dr$$

where the subscripts i and f refer to the initial and final positions. Because the integrand is a function only of the radial position, this integral depends only on the initial and final values of r. Thus, the work done is the same over *any* path from P to Q. Because the work done is independent of the path and depends only on the end points, we conclude that *any central force is conservative*. We are now assured that a potential energy function can be obtained once the form of the central force is specified.

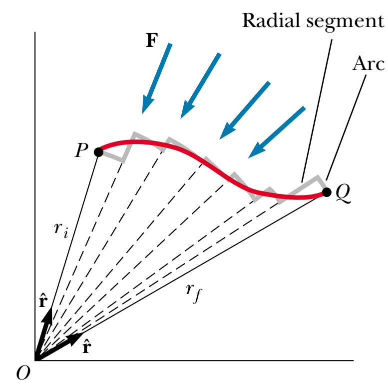


Figure 14.12 A particle moves from P to Q while acted on by a central force **F**, which is directed radially. The path is broken into a series of radial segments and arcs. Because the work done along the arcs is zero, the work done is independent of the path and depends only on r_f and r_i.

Recall from Equation 8.2 that the change in the gravitational potential energy associated with a given displacement is defined as the negative of the work done by the gravitational force during that displacement:

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)\ dr \qquad \textbf{(14.12)}$$

We can use this result to evaluate the gravitational potential energy function. Consider a particle of mass m moving between two points P and Q above the Earth's surface (Fig. 14.13). The particle is subject to the gravitational force given by Equation 14.1. We can express this force as

$$F(r) = -\frac{GM_E m}{r^2}$$

where the negative sign indicates that the force is attractive. Substituting this expression for $F(r)$ into Equation 14.12, we can compute the change in the gravita-

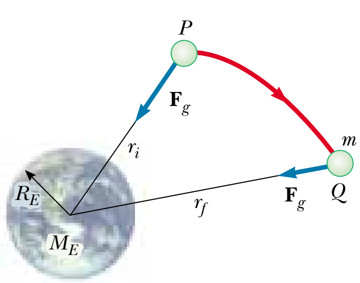


Figure 14.13 As a particle of mass m moves from P to Q above the Earth's surface, the gravitational potential energy changes according to Equation 14.12.

tional potential energy function:

$$U_f - U_i = GM_E m \int_{r_i}^{r_f} \frac{dr}{r^2} = GM_E m \left[-\frac{1}{r} \right]_{r_i}^{r_f}$$

Change in gravitational potential energy

$$U_f - U_i = -GM_E m \left(\frac{1}{r_f} - \frac{1}{r_i} \right) \quad \textbf{(14.13)}$$

As always, the choice of a reference point for the potential energy is completely arbitrary. It is customary to choose the reference point where the force is zero. Taking $U_i = 0$ at $r_i = \infty$, we obtain the important result

Gravitational potential energy of the Earth–particle system for $r \geq R_E$

$$U = -\frac{GM_E m}{r} \quad \textbf{(14.14)}$$

This expression applies to the Earth–particle system where the two masses are separated by a distance r, provided that $r \geq R_E$. The result is not valid for particles inside the Earth, where $r < R_E$. (The situation in which $r < R_E$ is treated in Section 14.10.) Because of our choice of U_i, the function U is always negative (Fig. 14.14).

Although Equation 14.14 was derived for the particle–Earth system, it can be applied to any two particles. That is, the gravitational potential energy associated with any pair of particles of masses m_1 and m_2 separated by a distance r is

$$U = -\frac{Gm_1 m_2}{r} \quad \textbf{(14.15)}$$

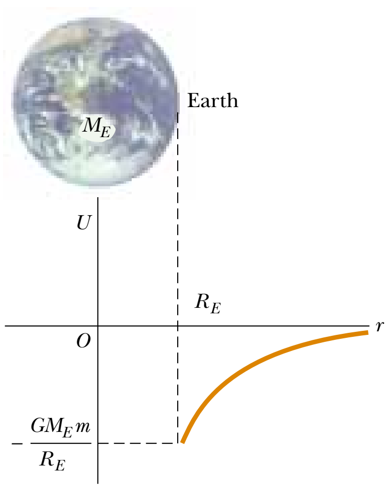


Figure 14.14 Graph of the gravitational potential energy U versus r for a particle above the Earth's surface. The potential energy goes to zero as r approaches infinity.

This expression shows that the gravitational potential energy for any pair of particles varies as $1/r$, whereas the force between them varies as $1/r^2$. Furthermore, the potential energy is negative because the force is attractive and we have taken the potential energy as zero when the particle separation is infinite. Because the force between the particles is attractive, we know that an external agent must do positive work to increase the separation between them. The work done by the external agent produces an increase in the potential energy as the two particles are separated. That is, U becomes less negative as r increases.

When two particles are at rest and separated by a distance r, an external agent has to supply an energy at least equal to $+Gm_1 m_2/r$ in order to separate the particles to an infinite distance. It is therefore convenient to think of the absolute value of the potential energy as the *binding energy* of the system. If the external agent supplies an energy greater than the binding energy, the excess energy of the system will be in the form of kinetic energy when the particles are at an infinite separation.

We can extend this concept to three or more particles. In this case, the total potential energy of the system is the sum over all pairs of particles.[5] Each pair contributes a term of the form given by Equation 14.15. For example, if the system contains three particles, as in Figure 14.15, we find that

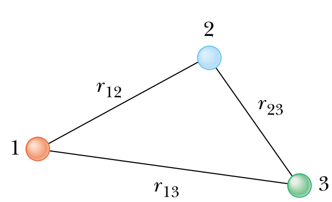


Figure 14.15 Three interacting particles.

$$U_{\text{total}} = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right) \quad \textbf{(14.16)}$$

The absolute value of U_{total} represents the work needed to separate the particles by an infinite distance.

[5] The fact that potential energy terms can be added for all pairs of particles stems from the experimental fact that gravitational forces obey the superposition principle.

EXAMPLE 14.6 The Change in Potential Energy

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Show that in this situation the general expression for the change in gravitational potential energy given by Equation 14.13 reduces to the familiar relationship $\Delta U = mg\,\Delta y$.

Solution We can express Equation 14.13 in the form

$$\Delta U = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = GM_E m\left(\frac{r_f - r_i}{r_i r_f}\right)$$

If both the initial and final positions of the particle are close to the Earth's surface, then $r_f - r_i = \Delta y$ and $r_i r_f \approx R_E^2$. (Recall that r is measured from the center of the Earth.) Therefore, the change in potential energy becomes

$$\Delta U \approx \frac{GM_E m}{R_E^2}\,\Delta y = mg\,\Delta y$$

where we have used the fact that $g = GM_E/R_E^2$ (Eq. 14.5). Keep in mind that the reference point is arbitrary because it is the *change* in potential energy that is meaningful.

14.8 ENERGY CONSIDERATIONS IN PLANETARY AND SATELLITE MOTION

Consider a body of mass m moving with a speed v in the vicinity of a massive body of mass M, where $M \gg m$. The system might be a planet moving around the Sun, a satellite in orbit around the Earth, or a comet making a one-time flyby of the Sun. If we assume that the body of mass M is at rest in an inertial reference frame, then the total mechanical energy E of the two-body system when the bodies are separated by a distance r is the sum of the kinetic energy of the body of mass m and the potential energy of the system, given by Equation 14.15:[6]

$$E = K + U$$

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad \textbf{(14.17)}$$

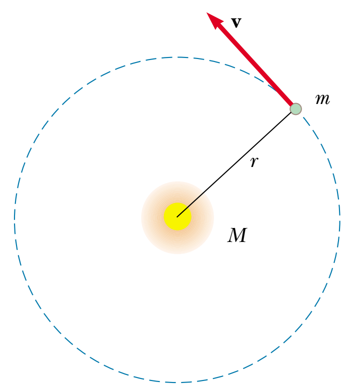


Figure 14.16 A body of mass m moving in a circular orbit about a much larger body of mass M.

This equation shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system,[7] such as the Earth–Sun system, E is necessarily *less than zero* because we have chosen the convention that $U \rightarrow 0$ as $r \rightarrow \infty$.

We can easily establish that $E < 0$ for the system consisting of a body of mass m moving in a circular orbit about a body of mass $M \gg m$ (Fig. 14.16). Newton's second law applied to the body of mass m gives

$$\frac{GMm}{r^2} = ma = \frac{mv^2}{r}$$

[6] You might recognize that we have ignored the acceleration and kinetic energy of the larger body. To see that this simplification is reasonable, consider an object of mass m falling toward the Earth. Because the center of mass of the object–Earth system is effectively stationary, it follows that $mv = M_E v_E$. Thus, the Earth acquires a kinetic energy equal to

$$\tfrac{1}{2}M_E v_E^2 = \tfrac{1}{2}\frac{m^2}{M_E}v^2 = \frac{m}{M_E}K$$

where K is the kinetic energy of the object. Because $M_E \gg m$, this result shows that the kinetic energy of the Earth is negligible.

[7] Of the three examples provided at the beginning of this section, the planet moving around the Sun and a satellite in orbit around the Earth are bound systems—the Earth will always stay near the Sun, and the satellite will always stay near the Earth. The one-time comet flyby represents an unbound system—the comet interacts once with the Sun but is not bound to it. Thus, in theory the comet can move infinitely far away from the Sun.

Multiplying both sides by r and dividing by 2 gives

$$\tfrac{1}{2}mv^2 = \frac{GMm}{2r} \quad (14.18)$$

Substituting this into Equation 14.17, we obtain

$$E = \frac{GMm}{2r} - \frac{GMm}{r}$$

Total energy for circular orbits

$$E = -\frac{GMm}{2r} \quad (14.19)$$

This result clearly shows that **the total mechanical energy is negative in the case of circular orbits.** Note that **the kinetic energy is positive and equal to one-half the absolute value of the potential energy.** The absolute value of E is also equal to the binding energy of the system, because this amount of energy must be provided to the system to move the two masses infinitely far apart.

The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for elliptical orbits is the same as Equation 14.19 with r replaced by the semimajor axis length a. Furthermore, the total energy is constant if we assume that the system is isolated. Therefore, as the body of mass m moves from P to Q in Figure 14.13, the total energy remains constant and Equation 14.17 gives

$$E = \tfrac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \tfrac{1}{2}mv_f^2 - \frac{GMm}{r_f} \quad (14.20)$$

Combining this statement of energy conservation with our earlier discussion of conservation of angular momentum, we see that **both the total energy and the total angular momentum of a gravitationally bound, two-body system are constants of the motion.**

EXAMPLE 14.7 Changing the Orbit of a Satellite

The space shuttle releases a 470-kg communications satellite while in an orbit that is 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth. How much energy did the engine have to provide?

Solution First we must determine the radius of a geosynchronous orbit. Then we can calculate the change in energy needed to boost the satellite into orbit.

The period of the orbit T must be one day (86 400 s), so that the satellite travels once around the Earth in the same time that the Earth spins once on its axis. Knowing the period, we can then apply Kepler's third law (Eq. 14.7) to find the radius, once we replace K_S with $K_E = 4\pi^2/GM_E = 9.89 \times 10^{-14}\ \text{s}^2/\text{m}^3$:

$$T^2 = K_E r^3$$

$$r = \sqrt[3]{\frac{T^2}{K_E}} = \sqrt[3]{\frac{(86\,400\ \text{s})^2}{9.89 \times 10^{-14}\ \text{s}^2/\text{m}^3}} = 4.23 \times 10^7\ \text{m} = R_f$$

This is a little more than 26 000 mi above the Earth's surface.

We must also determine the initial radius (not the altitude above the Earth's surface) of the satellite's orbit when it was still in the shuttle's cargo bay. This is simply

$$R_E + 280\ \text{km} = 6.65 \times 10^6\ \text{m} = R_i$$

Now, applying Equation 14.19, we obtain, for the total initial and final energies,

$$E_i = -\frac{GM_E m}{2R_i} \qquad E_f = -\frac{GM_E m}{2R_f}$$

The energy required from the engine to boost the satellite is

$$E_{\text{engine}} = E_f - E_i = -\frac{GM_E m}{2}\left(\frac{1}{R_f} - \frac{1}{R_i}\right)$$

$$= -\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(470\ \text{kg})}{2}$$

$$\times \left(\frac{1}{4.23 \times 10^7\ \text{m}} - \frac{1}{6.65 \times 10^6\ \text{m}}\right)$$

$$= 1.19 \times 10^{10}\ \text{J}$$

This is the energy equivalent of 89 gal of gasoline. NASA engineers must account for the changing mass of the spacecraft as it ejects burned fuel, something we have not done here. Would you expect the calculation that includes the effect of this changing mass to yield a greater or lesser amount of energy required from the engine?

If we wish to determine how the energy is distributed after the engine is fired, we find from Equation 14.18 that the change in kinetic energy is $\Delta K = (GM_E m/2)(1/R_f - 1/R_i) = -1.19 \times 10^{10}$ J (a decrease), and the corresponding change in potential energy is $\Delta U = -GM_E m(1/R_f - 1/R_i) = 2.38 \times 10^{10}$ J (an increase). Thus, the change in mechanical energy of the system is $\Delta E = \Delta K + \Delta U = 1.19 \times 10^{10}$ J, as we already calculated. The firing of the engine results in an increase in the total mechanical energy of the system. Because an increase in potential energy is accompanied by a decrease in kinetic energy, we conclude that the speed of an orbiting satellite decreases as its altitude increases.

Escape Speed

Suppose an object of mass m is projected vertically upward from the Earth's surface with an initial speed v_i, as illustrated in Figure 14.17. We can use energy considerations to find the minimum value of the initial speed needed to allow the object to escape the Earth's gravitational field. Equation 14.17 gives the total energy of the object at any point. At the surface of the Earth, $v = v_i$ and $r = r_i = R_E$. When the object reaches its maximum altitude, $v = v_f = 0$ and $r = r_f = r_{\text{max}}$. Because the total energy of the system is constant, substituting these conditions into Equation 14.20 gives

$$\tfrac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{\text{max}}}$$

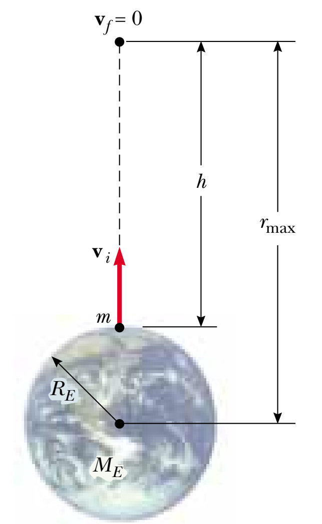


Figure 14.17 An object of mass m projected upward from the Earth's surface with an initial speed v_i reaches a maximum altitude h.

Solving for v_i^2 gives

$$v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{\text{max}}}\right) \tag{14.21}$$

Therefore, if the initial speed is known, this expression can be used to calculate the maximum altitude h because we know that

$$h = r_{\text{max}} - R_E$$

We are now in a position to calculate **escape speed,** which is the minimum speed the object must have at the Earth's surface in order to escape from the influence of the Earth's gravitational field. Traveling at this minimum speed, the object continues to move farther and farther away from the Earth as its speed asymptotically approaches zero. Letting $r_{\text{max}} \rightarrow \infty$ in Equation 14.21 and taking $v_i = v_{\text{esc}}$, we obtain

Escape speed

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}} \tag{14.22}$$

Note that this expression for v_{esc} is independent of the mass of the object. In other words, a spacecraft has the same escape speed as a molecule. Furthermore, the result is independent of the direction of the velocity and ignores air resistance.

If the object is given an initial speed equal to v_{esc}, its total energy is equal to zero. This can be seen by noting that when $r \rightarrow \infty$, the object's kinetic energy and its potential energy are both zero. If v_i is greater than v_{esc}, the total energy is greater than zero and the object has some residual kinetic energy as $r \rightarrow \infty$.

EXAMPLE 14.8 Escape Speed of a Rocket

Calculate the escape speed from the Earth for a 5 000-kg spacecraft, and determine the kinetic energy it must have at the Earth's surface in order to escape the Earth's gravitational field.

Solution Using Equation 14.22 gives

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{6.37 \times 10^6\ \text{m}}}$$

$$= \boxed{1.12 \times 10^4\ \text{m/s}}$$

This corresponds to about 25 000 mi/h.

The kinetic energy of the spacecraft is

$$K = \tfrac{1}{2}mv_{\text{esc}}^2 = \tfrac{1}{2}(5.00 \times 10^3\ \text{kg})(1.12 \times 10^4\ \text{m/s})^2$$

$$= \boxed{3.14 \times 10^{11}\ \text{J}}$$

This is equivalent to about 2 300 gal of gasoline.

Equations 14.21 and 14.22 can be applied to objects projected from any planet. That is, in general, the escape speed from the surface of any planet of mass M and radius R is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Escape speeds for the planets, the Moon, and the Sun are provided in Table 14.3. Note that the values vary from 1.1 km/s for Pluto to about 618 km/s for the Sun. These results, together with some ideas from the kinetic theory of gases (see Chapter 21), explain why some planets have atmospheres and others do not. As we shall see later, a gas molecule has an average kinetic energy that depends on the temperature of the gas. Hence, lighter molecules, such as hydrogen and helium, have a higher average speed than heavier species at the same temperature. When the average speed of the lighter molecules is not much less than the escape speed of a planet, a significant fraction of them have a chance to escape from the planet.

This mechanism also explains why the Earth does not retain hydrogen molecules and helium atoms in its atmosphere but does retain heavier molecules, such as oxygen and nitrogen. On the other hand, the very large escape speed for Jupiter enables that planet to retain hydrogen, the primary constituent of its atmosphere.

TABLE 14.3
Escape Speeds from the Surfaces of the Planets, Moon, and Sun

Body	v_{esc} (km/s)
Mercury	4.3
Venus	10.3
Earth	11.2
Moon	2.3
Mars	5.0
Jupiter	60
Saturn	36
Uranus	22
Neptune	24
Pluto	1.1
Sun	618

Quick Quiz 14.2

If you were a space prospector and discovered gold on an asteroid, it probably would not be a good idea to jump up and down in excitement over your find. Why?

Quick Quiz 14.3

Figure 14.18 is a drawing by Newton showing the path of a stone thrown from a mountaintop. He shows the stone landing farther and farther away when thrown at higher and higher speeds (at points D, E, F, and G), until finally it is thrown all the way around the Earth. Why didn't Newton show the stone landing at B and A before it was going fast enough to complete an orbit?

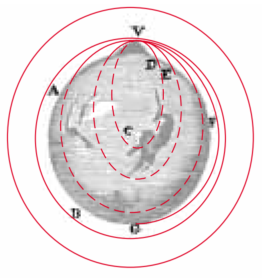


Figure 14.18 "The greater the velocity . . . with which [a stone] is projected, the farther it goes before it falls to the Earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at last, exceeding the limits of the Earth, it should pass into space without touching." Sir Isaac Newton, *System of the World.*

Optional Section

14.9 THE GRAVITATIONAL FORCE BETWEEN AN EXTENDED OBJECT AND A PARTICLE

We have emphasized that the law of universal gravitation given by Equation 14.3 is valid only if the interacting objects are treated as particles. In view of this, how can we calculate the force between a particle and an object having finite dimensions? This is accomplished by treating the extended object as a collection of particles and making use of integral calculus. We first evaluate the potential energy function, and then calculate the gravitational force from that function.

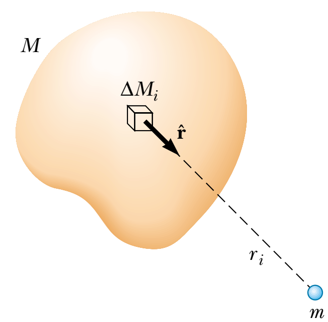


Figure 14.19 A particle of mass m interacting with an extended object of mass M. The total gravitational force exerted by the object on the particle can be obtained by dividing the object into numerous elements, each having a mass ΔM_i, and then taking a vector sum over the forces exerted by all elements.

We obtain the potential energy associated with a system consisting of a particle of mass m and an extended object of mass M by dividing the object into many elements, each having a mass ΔM_i (Fig. 14.19). The potential energy associated with the system consisting of any one element and the particle is $U = -Gm\,\Delta M_i/r_i$, where r_i is the distance from the particle to the element ΔM_i. The total potential energy of the overall system is obtained by taking the sum over all elements as $\Delta M_i \rightarrow 0$. In this limit, we can express U in integral form as

$$U = -Gm \int \frac{dM}{r} \qquad \textbf{(14.23)}$$

Once U has been evaluated, we obtain the force exerted by the extended object on the particle by taking the negative derivative of this scalar function (see Section 8.6). If the extended object has spherical symmetry, the function U depends only on r, and the force is given by $-dU/dr$. We treat this situation in Section 14.10. In principle, one can evaluate U for any geometry; however, the integration can be cumbersome.

An alternative approach to evaluating the gravitational force between a particle and an extended object is to perform a vector sum over all mass elements of the object. Using the procedure outlined in evaluating U and the law of universal gravitation in the form shown in Equation 14.3, we obtain, for the total force exerted on the particle

Total force exerted on a particle by an extended object

$$\mathbf{F}_g = -Gm \int \frac{dM}{r^2}\,\hat{\mathbf{r}} \qquad \textbf{(14.24)}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the element dM toward the particle (see Fig. 14.19) and the minus sign indicates that the direction of the force is opposite that of $\hat{\mathbf{r}}$. This procedure is not always recommended because working with a vector function is more difficult than working with the scalar potential energy function. However, if the geometry is simple, as in the following example, the evaluation of $\mathbf{F}$ can be straightforward.

EXAMPLE 14.9 Gravitational Force Between a Particle and a Bar

The left end of a homogeneous bar of length L and mass M is at a distance h from a particle of mass m (Fig. 14.20). Calculate the total gravitational force exerted by the bar on the particle.

Solution The arbitrary segment of the bar of length dx has a mass dM. Because the mass per unit length is constant, it follows that the ratio of masses dM/M is equal to the ratio of lengths dx/L, and so $dM = (M/L)\,dx$. In this problem, the variable r in Equation 14.24 is the distance x shown in Figure 14.20, the unit vector $\hat{\mathbf{r}}$ is $\hat{\mathbf{r}} = -\mathbf{i}$, and the force acting on the particle is to the right; therefore, Equation 14.24 gives us

$$\mathbf{F}_g = -Gm\int_h^{h+L}\frac{M\,dx}{L}\,\frac{1}{x^2}(-\mathbf{i}) = Gm\frac{M}{L}\int_h^{h+L}\frac{dx}{x^2}\,\mathbf{i}$$

$$\mathbf{F}_g = \frac{GmM}{L}\left[-\frac{1}{x}\right]_h^{h+L}\mathbf{i} = \frac{GmM}{h(h+L)}\,\mathbf{i}$$

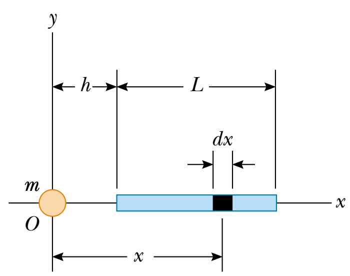


Figure 14.20 The gravitational force exerted by the bar on the particle is directed to the right. Note that the bar is *not* equivalent to a particle of mass M located at the center of mass of the bar.

We see that the force exerted on the particle is in the positive x direction, which is what we expect because the gravitational force is attractive.

Note that in the limit $L \rightarrow 0$, the force varies as $1/h^2$, which is what we expect for the force between two point masses. Furthermore, if $h \gg L$, the force also varies as $1/h^2$. This can be seen by noting that the denominator of the expression for $\mathbf{F}_g$ can be expressed in the form $h^2(1 + L/h)$, which is approximately equal to h^2 when $h \gg L$. Thus, when bodies are separated by distances that are great relative to their characteristic dimensions, they behave like particles.

Optional Section

14.10 THE GRAVITATIONAL FORCE BETWEEN A PARTICLE AND A SPHERICAL MASS

We have already stated that a large sphere attracts a particle outside it as if the total mass of the sphere were concentrated at its center. We now describe the force acting on a particle when the extended object is either a spherical shell or a solid sphere, and then apply these facts to some interesting systems.

Spherical Shell

Case 1. If a particle of mass m is located outside a spherical shell of mass M at, for instance, point P in Figure 14.21a, the shell attracts the particle as though the mass of the shell were concentrated at its center. We can show this, as Newton did, with integral calculus. Thus, as far as the gravitational force acting on a particle outside the shell is concerned, a spherical shell acts no differently from the solid spherical distributions of mass we have seen.

Case 2. If the particle is located inside the shell (at point P in Fig. 14.21b), the gravitational force acting on it can be shown to be zero.

We can express these two important results in the following way:

Force on a particle due to a spherical shell

$$\mathbf{F}_g = -\frac{GMm}{r^2}\,\hat{\mathbf{r}} \qquad \text{for } r \geq R \qquad \textbf{(14.25a)}$$

$$\mathbf{F}_g = 0 \qquad \text{for } r < R \qquad \textbf{(14.25b)}$$

The gravitational force as a function of the distance r is plotted in Figure 14.21c.

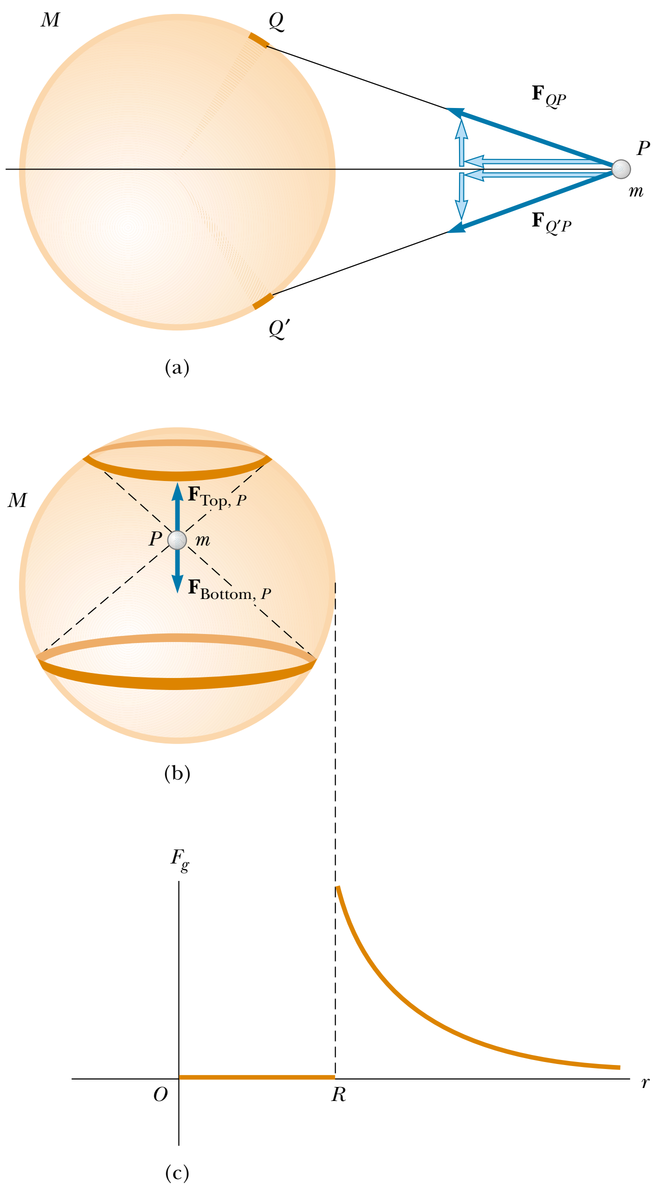


Figure 14.21 (a) The nonradial components of the gravitational forces exerted on a particle of mass m located at point P outside a spherical shell of mass M cancel out. (b) The spherical shell can be broken into rings. Even though point P is closer to the top ring than to the bottom ring, the bottom ring is larger, and the gravitational forces exerted on the particle at P by the matter in the two rings cancel each other. Thus, for a particle located at any point P inside the shell, there is no gravitational force exerted on the particle by the mass M of the shell. (c) The magnitude of the gravitational force versus the radial distance r from the center of the shell.

The shell does not act as a gravitational shield, which means that a particle inside a shell may experience forces exerted by bodies outside the shell.

Solid Sphere

Case 1. If a particle of mass m is located outside a homogeneous solid sphere of mass M (at point P in Fig. 14.22), the sphere attracts the particle as though the mass of the sphere were concentrated at its center. We have used this notion at several places in this chapter already, and we can argue it from Equation 14.25a. A solid sphere can be considered to be a collection of concentric spherical shells. The masses of all of the shells can be interpreted as being concentrated at their common center, and the gravitational force is equivalent to that due to a particle of mass M located at that center.

Case 2. If a particle of mass m is located inside a homogeneous solid sphere of mass M (at point Q in Fig. 14.22), the gravitational force acting on it is due *only* to the mass M' contained within the sphere of radius $r < R$, shown in Figure 14.22. In other words,

Force on a particle due to a solid sphere

$$\mathbf{F}_g = -\frac{GmM}{r^2}\hat{\mathbf{r}} \qquad \text{for } r \geq R \tag{14.26a}$$

$$\mathbf{F}_g = -\frac{GmM'}{r^2}\hat{\mathbf{r}} \qquad \text{for } r < R \tag{14.26b}$$

This also follows from spherical-shell Case 1 because the part of the sphere that is

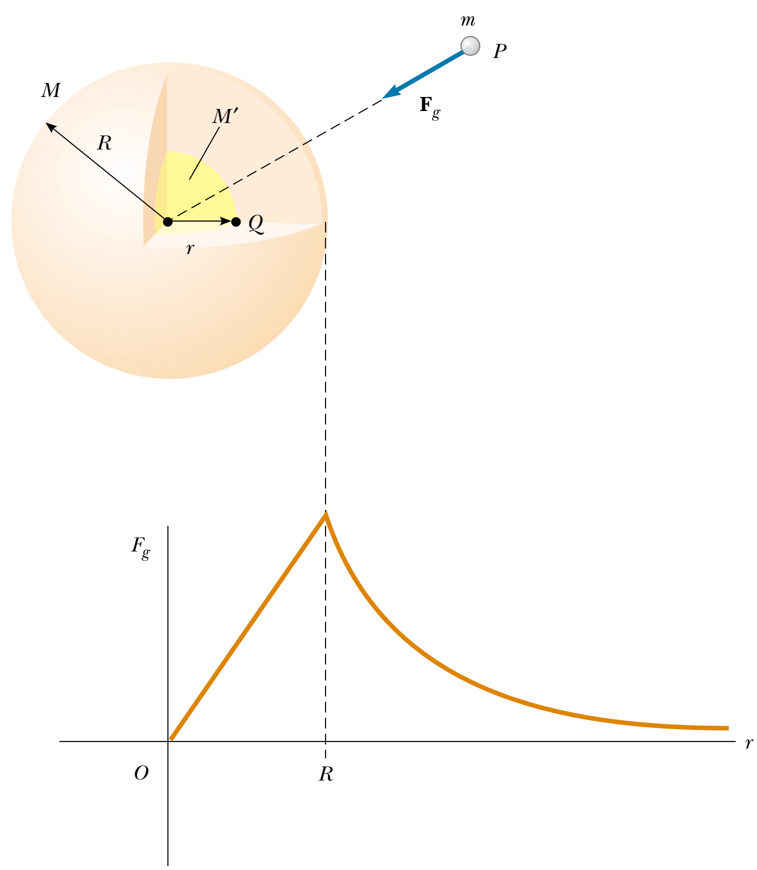


Figure 14.22 The gravitational force acting on a particle when it is outside a uniform solid sphere is GMm/r^2 and is directed toward the center of the sphere. The gravitational force acting on the particle when it is inside such a sphere is proportional to r and goes to zero at the center.

farther from the center than Q can be treated as a series of concentric spherical shells that do not exert a net force on the particle because the particle is inside them. Because the sphere is assumed to have a uniform density, it follows that the ratio of masses M'/M is equal to the ratio of volumes V'/V, where V is the total volume of the sphere and V' is the volume within the sphere of radius r only:

$$\frac{M'}{M} = \frac{V'}{V} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}$$

Solving this equation for M' and substituting the value obtained into Equation 14.26b, we have

$$\mathbf{F}_g = -\frac{GmM}{R^3} r\hat{\mathbf{r}} \qquad \text{for } r < R \tag{14.27}$$

This equation tells us that at the center of the solid sphere, where $r = 0$, the gravitational force goes to zero, as we intuitively expect. The force as a function of r is plotted in Figure 14.22.

Case 3. If a particle is located inside a solid sphere having a density ρ that is spherically symmetric but not uniform, then M' in Equation 14.26b is given by an integral of the form $M' = \int \rho \, dV$, where the integration is taken over the volume contained within the sphere of radius r in Figure 14.22. We can evaluate this integral if the radial variation of ρ is given. In this case, we take the volume element dV as the volume of a spherical shell of radius r and thickness dr, and thus $dV = 4\pi r^2 \, dr$. For example, if $\rho = Ar$, where A is a constant, it is left to a problem (Problem 63) to show that $M' = \pi A r^4$. Hence, we see from Equation 14.26b that F is proportional to r^2 in this case and is zero at the center.

Quick Quiz 14.4

A particle is projected through a small hole into the interior of a spherical shell. Describe

EXAMPLE 14.10 A Free Ride, Thanks to Gravity

An object of mass m moves in a smooth, straight tunnel dug between two points on the Earth's surface (Fig. 14.23). Show that the object moves with simple harmonic motion, and find the period of its motion. Assume that the Earth's density is uniform.

Solution The gravitational force exerted on the object acts toward the Earth's center and is given by Equation 14.27:

$$\mathbf{F}_g = -\frac{GmM}{R^3} r\hat{\mathbf{r}}$$

We receive our first indication that this force should result in simple harmonic motion by comparing it to Hooke's law, first seen in Section 7.3. Because the gravitational force on the object is linearly proportional to the displacement, the object experiences a Hooke's law force.

The y component of the gravitational force on the object is balanced by the normal force exerted by the tunnel wall, and the x component is

$$F_x = -\frac{GmM_E}{R_E^3} r \cos\theta$$

Because the x coordinate of the object is $x = r\cos\theta$, we can write

$$F_x = -\frac{GmM_E}{R_E^3} x$$

Applying Newton's second law to the motion along the x direction gives

$$F_x = -\frac{GmM_E}{R_E^3} x = ma_x$$

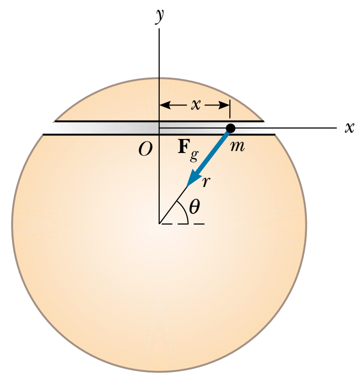


Figure 14.23 An object moves along a tunnel dug through the Earth. The component of the gravitational force $\mathbf{F}_g$ along the x axis is the driving force for the motion. Note that this component always acts toward O.

Solving for a_x, we obtain

$$a_x = -\frac{GM_E}{R_E^{\ 3}}\, x$$

If we use the symbol ω^2 for the coefficient of x—$GM_E/R_E^{\ 3} = \omega^2$— we see that

$$(1) \qquad a_x = -\omega^2 x$$

an expression that matches the mathematical form of Equation 13.9, which gives the acceleration of a particle in simple harmonic motion: $a_x = -\omega^2 x$. Therefore, Equation (1), which we have derived for the acceleration of our object in the tunnel, is the acceleration equation for simple harmonic motion at angular speed ω with

$$\omega = \sqrt{\frac{GM_E}{R_E^{\ 3}}}$$

Thus, the object in the tunnel moves in the same way as a block hanging from a spring! The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^{\ 3}}{GM_E}}$$

$$= 2\pi\sqrt{\frac{(6.37 \times 10^6 \text{ m})^3}{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})}}$$

$$= 5.06 \times 10^3 \text{ s} = \boxed{84.3 \text{ min}}$$

This period is the same as that of a satellite traveling in a circular orbit just above the Earth's surface (ignoring any trees, buildings, or other objects in the way). Note that the result is independent of the length of the tunnel.

A proposal has been made to operate a mass-transit system between any two cities, using the principle described in this example. A one-way trip would take about 42 min. A more precise calculation of the motion must account for the fact that the Earth's density is not uniform. More important, there are many practical problems to consider. For instance, it would be impossible to achieve a frictionless tunnel, and so some auxiliary power source would be required. Can you think of other problems?

the motion of the particle inside the shell.

SUMMARY

Newton's law of universal gravitation states that the gravitational force of attraction between any two particles of masses m_1 and m_2 separated by a distance r has the magnitude

$$F_g = G\frac{m_1 m_2}{r^2} \tag{14.1}$$

where $G = 6.673 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2$ is the universal gravitational constant. This equation enables us to calculate the force of attraction between masses under a wide variety of circumstances.

An object at a distance h above the Earth's surface experiences a gravitational force of magnitude mg', where g' is the free-fall acceleration at that elevation:

$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \tag{14.6}$$

In this expression, M_E is the mass of the Earth and R_E is its radius. Thus, the weight of an object decreases as the object moves away from the Earth's surface.

Kepler's laws of planetary motion state that

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

Kepler's third law can be expressed as

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)r^3 \tag{14.7}$$

where M_S is the mass of the Sun and r is the orbital radius. For elliptical orbits, Equation 14.7 is valid if r is replaced by the semimajor axis a. Most planets have nearly circular orbits around the Sun.

The **gravitational field** at a point in space equals the gravitational force experienced by any test particle located at that point divided by the mass of the test particle:

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} \tag{14.10}$$

The gravitational force is conservative, and therefore a potential energy function can be defined. The **gravitational potential energy** associated with two particles separated by a distance r is

$$U = -\frac{Gm_1m_2}{r} \tag{14.15}$$

where U is taken to be zero as $r \rightarrow \infty$. The total potential energy for a system of particles is the sum of energies for all pairs of particles, with each pair represented by a term of the form given by Equation 14.15.

If an isolated system consists of a particle of mass m moving with a speed v in the vicinity of a massive body of mass M, the total energy E of the system is the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \tag{14.17}$$

The total energy is a constant of the motion. If the particle moves in a circular orbit of radius r around the massive body and if $M \gg m$, the total energy of the system is

$$E = -\frac{GMm}{2r} \tag{14.19}$$

The total energy is negative for any bound system.

The **escape speed** for an object projected from the surface of the Earth is

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}} \tag{14.22}$$

QUESTIONS

1. Use Kepler's second law to convince yourself that the Earth must move faster in its orbit during December, when it is closest to the Sun, than during June, when it is farthest from the Sun.
2. The gravitational force that the Sun exerts on the Moon is about twice as great as the gravitational force that the Earth exerts on the Moon. Why doesn't the Sun pull the Moon away from the Earth during a total eclipse of the Sun?
3. If a system consists of five particles, how many terms appear in the expression for the total potential energy? How many terms appear if the system consists of *N* particles?
4. Is it possible to calculate the potential energy function associated with a particle and an extended body without knowing the geometry or mass distribution of the extended body?
5. Does the escape speed of a rocket depend on its mass? Explain.
6. Compare the energies required to reach the Moon for a 10^5-kg spacecraft and a 10^3-kg satellite.
7. Explain why it takes more fuel for a spacecraft to travel from the Earth to the Moon than for the return trip. Estimate the difference.
8. Why don't we put a geosynchronous weather satellite in orbit around the 45th parallel? Wouldn't this be more useful for the United States than such a satellite in orbit around the equator?
9. Is the potential energy associated with the Earth–Moon system greater than, less than, or equal to the kinetic energy of the Moon relative to the Earth?
10. Explain why no work is done on a planet as it moves in a circular orbit around the Sun, even though a gravitational force is acting on the planet. What is the net work done on a planet during each revolution as it moves around the Sun in an elliptical orbit?
11. Explain why the force exerted on a particle by a uniform sphere must be directed toward the center of the sphere. Would this be the case if the mass distribution of the sphere were not spherically symmetric?
12. Neglecting the density variation of the Earth, what would be the period of a particle moving in a smooth hole dug between opposite points on the Earth's surface, passing through its center?
13. At what position in its elliptical orbit is the speed of a planet a maximum? At what position is the speed a minimum?
14. If you were given the mass and radius of planet X, how would you calculate the free-fall acceleration on the surface of this planet?
15. If a hole could be dug to the center of the Earth, do you think that the force on a mass m would still obey Equation 14.1 there? What do you think the force on m would be at the center of the Earth?
16. In his 1798 experiment, Cavendish was said to have "weighed the Earth." Explain this statement.
17. The gravitational force exerted on the *Voyager* spacecraft by Jupiter accelerated it toward escape speed from the Sun. How is this possible?
18. How would you find the mass of the Moon?
19. The *Apollo 13* spaceship developed trouble in the oxygen system about halfway to the Moon. Why did the spaceship continue on around the Moon and then return home, rather than immediately turn back to Earth?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 14.1 Newton's Law of Universal Gravitation
Section 14.2 Measuring the Gravitational Constant
Section 14.3 Free-Fall Acceleration and the Gravitational Force

1. Determine the order of magnitude of the gravitational force that you exert on another person 2 m away. In your solution, state the quantities that you measure or estimate and their values.
2. A 200-kg mass and a 500-kg mass are separated by 0.400 m. (a) Find the net gravitational force exerted by these masses on a 50.0-kg mass placed midway between them. (b) At what position (other than infinitely remote ones) can the 50.0-kg mass be placed so as to experience a net force of zero?
3. Three equal masses are located at three corners of a square of edge length ℓ, as shown in Figure P14.3. Find the gravitational field $\mathbf{g}$ at the fourth corner due to these masses.
4. Two objects attract each other with a gravitational force of magnitude 1.00×10^{-8} N when separated by 20.0 cm. If the total mass of the two objects is 5.00 kg, what is the mass of each?
5. Three uniform spheres of masses 2.00 kg, 4.00 kg, and 6.00 kg are placed at the corners of a right triangle, as illustrated in Figure P14.5. Calculate the resultant gravi-

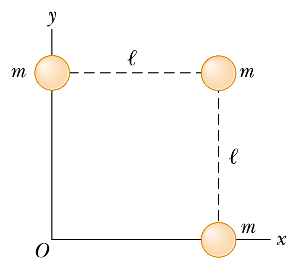


Figure P14.3

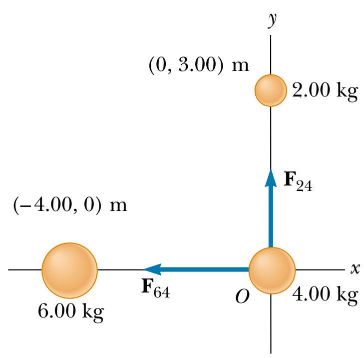


Figure P14.5

tational force on the 4.00-kg mass, assuming that the spheres are isolated from the rest of the Universe.

6. The free-fall acceleration on the surface of the Moon is about one-sixth that on the surface of the Earth. If the radius of the Moon is about $0.250R_E$, find the ratio of their average densities, $\rho_{\text{Moon}}/\rho_{\text{Earth}}$.

7. During a solar eclipse, the Moon, Earth, and Sun all lie on the same line, with the Moon between the Earth and the Sun. (a) What force is exerted by the Sun on the Moon? (b) What force is exerted by the Earth on the Moon? (c) What force is exerted by the Sun on the Earth?

8. The center-to-center distance between the Earth and the Moon is 384 400 km. The Moon completes an orbit in 27.3 days. (a) Determine the Moon's orbital speed. (b) If gravity were switched off, the Moon would move along a straight line tangent to its orbit, as described by Newton's first law. In its actual orbit in 1.00 s, how far does the Moon fall below the tangent line and toward the Earth?

WEB 9. When a falling meteoroid is at a distance above the Earth's surface of 3.00 times the Earth's radius, what is its acceleration due to the Earth's gravity?

10. Two ocean liners, each with a mass of 40 000 metric tons, are moving on parallel courses, 100 m apart. What is the magnitude of the acceleration of one of the liners toward the other due to their mutual gravitational attraction? (Treat the ships as point masses.)

11. A student proposes to measure the gravitational constant G by suspending two spherical masses from the ceiling of a tall cathedral and measuring the deflection of the cables from the vertical. Draw a free-body diagram of one of the masses. If two 100.0-kg masses are suspended at the end of 45.00-m-long cables, and the cables are attached to the ceiling 1.000 m apart, what is the separation of the masses?

12. On the way to the Moon, the Apollo astronauts reached a point where the Moon's gravitational pull became stronger than the Earth's. (a) Determine the distance of this point from the center of the Earth. (b) What is the acceleration due to the Earth's gravity at this point?

Section 14.4 Kepler's Laws

Section 14.5 The Law of Gravity and the Motion of Planets

13. A particle of mass m moves along a straight line with constant speed in the x direction, a distance b from the x axis (Fig. P14.13). Show that Kepler's second law is satisfied by demonstrating that the two shaded triangles in the figure have the same area when $t_4 - t_3 = t_2 - t_1$.

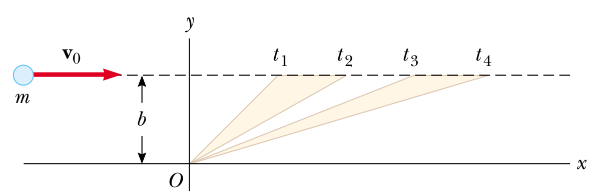


Figure P14.13

14. A communications satellite in geosynchronous orbit remains above a single point on the Earth's equator as the planet rotates on its axis. (a) Calculate the radius of its orbit. (b) The satellite relays a radio signal from a transmitter near the north pole to a receiver, also near the north pole. Traveling at the speed of light, how long is the radio wave in transit?

15. Plaskett's binary system consists of two stars that revolve in a circular orbit about a center of mass midway between them. This means that the masses of the two stars are equal (Fig. P14.15). If the orbital velocity of each star is 220 km/s and the orbital period of each is 14.4 days, find the mass M of each star. (For comparison, the mass of our Sun is 1.99×10^{30} kg.)

16. Plaskett's binary system consists of two stars that revolve in a circular orbit about a center of gravity midway between them. This means that the masses of the two stars are equal (see Fig. P14.15). If the orbital speed of each star is v and the orbital period of each is T, find the mass M of each star.

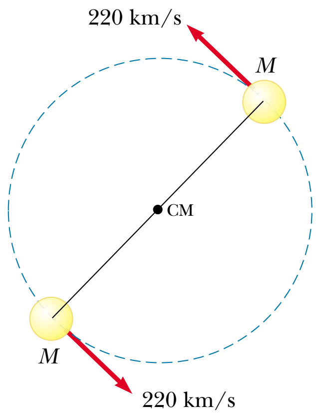


Figure P14.15 Problems 15 and 16.

17. The *Explorer VIII* satellite, placed into orbit November 3, 1960, to investigate the ionosphere, had the following orbit parameters: perigee, 459 km; apogee, 2 289 km (both distances above the Earth's surface); and period, 112.7 min. Find the ratio v_p/v_a of the speed at perigee to that at apogee.

18. Comet Halley (Fig. P14.18) approaches the Sun to within 0.570 AU, and its orbital period is 75.6 years (AU is the symbol for astronomical unit, where 1 AU = 1.50×10^{11} m is the mean Earth–Sun distance). How far from the Sun will Halley's comet travel before it starts its return journey?

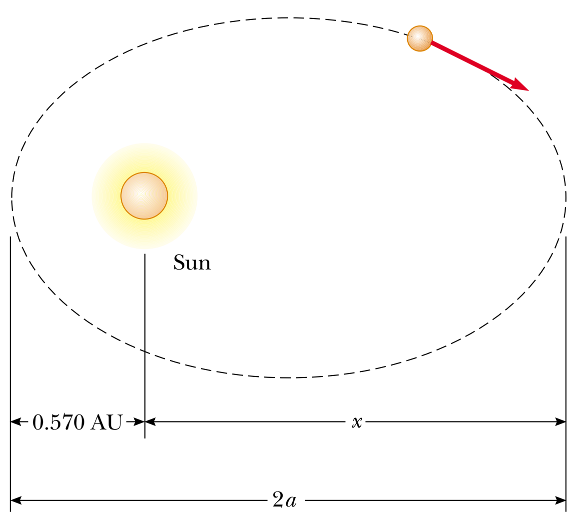


Figure P14.18

WEB **19.** Io, a satellite of Jupiter, has an orbital period of 1.77 days and an orbital radius of 4.22×10^5 km. From these data, determine the mass of Jupiter.

20. Two planets, X and Y, travel counterclockwise in circular orbits about a star, as shown in Figure P14.20. The radii of their orbits are in the ratio 3:1. At some time, they are aligned as in Figure P14.20a, making a straight line with the star. During the next five years, the angular displacement of planet X is 90.0°, as shown in Figure P14.20b. Where is planet Y at this time?

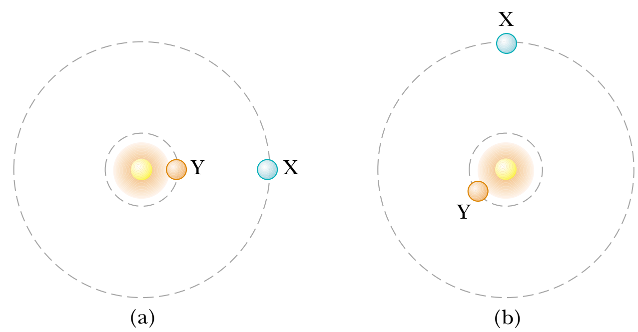


Figure P14.20

21. A synchronous satellite, which always remains above the same point on a planet's equator, is put in orbit around Jupiter so that scientists can study the famous red spot. Jupiter rotates once every 9.84 h. Use the data in Table 14.2 to find the altitude of the satellite.

22. Neutron stars are extremely dense objects that are formed from the remnants of supernova explosions. Many rotate very rapidly. Suppose that the mass of a certain spherical neutron star is twice the mass of the Sun and that its radius is 10.0 km. Determine the greatest possible angular speed it can have for the matter at the surface of the star on its equator to be just held in orbit by the gravitational force.

23. The Solar and Heliospheric Observatory (SOHO) spacecraft has a special orbit, chosen so that its view of the Sun is never eclipsed and it is always close enough to the Earth to transmit data easily. It moves in a near-circle around the Sun that is smaller than the Earth's circular orbit. Its period, however, is not less than 1 yr but is just equal to 1 yr. It is always located between the Earth and the Sun along the line joining them. Both objects exert gravitational forces on the observatory. Show that the spacecraft's distance from the Earth must be between 1.47×10^9 m and 1.48×10^9 m. In 1772 Joseph Louis Lagrange determined theoretically the special location that allows this orbit. The SOHO spacecraft took this position on February 14, 1996. (*Hint:* Use data that are precise to four digits. The mass of the Earth is 5.983×10^{24} kg.)

Section 14.6 The Gravitational Field

24. A spacecraft in the shape of a long cylinder has a length of 100 m, and its mass with occupants is 1 000 kg. It has

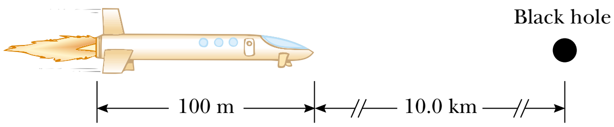


Figure P14.24

strayed too close to a 1.0-m-radius black hole having a mass 100 times that of the Sun (Fig. P14.24). The nose of the spacecraft is pointing toward the center of the black hole, and the distance between the nose and the black hole is 10.0 km. (a) Determine the total force on the spacecraft. (b) What is the difference in the gravitational fields acting on the occupants in the nose of the ship and on those in the rear of the ship, farthest from the black hole?

25. Compute the magnitude and direction of the gravitational field at a point P on the perpendicular bisector of two equal masses separated by a distance $2a$, as shown in Figure P14.25.

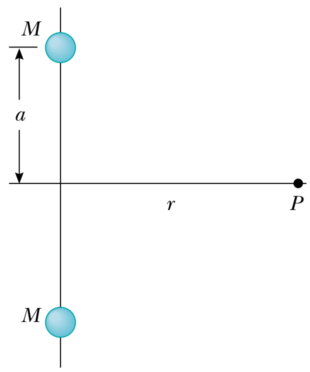


Figure P14.25

26. Find the gravitational field at a distance r along the axis of a thin ring of mass M and radius a.

Section 14.7 Gravitational Potential Energy

Note: Assume that $U = 0$ as $r \to \infty$.

27. A satellite of the Earth has a mass of 100 kg and is at an altitude of 2.00×10^6 m. (a) What is the potential energy of the satellite–Earth system? (b) What is the magnitude of the gravitational force exerted by the Earth on the satellite? (c) What force does the satellite exert on the Earth?
28. How much energy is required to move a 1 000-kg mass from the Earth's surface to an altitude twice the Earth's radius?
29. After our Sun exhausts its nuclear fuel, its ultimate fate may be to collapse to a *white-dwarf* state, in which it has approximately the same mass it has now but a radius equal to the radius of the Earth. Calculate (a) the average density of the white dwarf, (b) the acceleration due to gravity at its surface, and (c) the gravitational potential energy associated with a 1.00-kg object at its surface.
30. At the Earth's surface a projectile is launched straight up at a speed of 10.0 km/s. To what height will it rise? Ignore air resistance.
31. A system consists of three particles, each of mass 5.00 g, located at the corners of an equilateral triangle with sides of 30.0 cm. (a) Calculate the potential energy of the system. (b) If the particles are released simultaneously, where will they collide?
32. How much work is done by the Moon's gravitational field as a 1 000-kg meteor comes in from outer space and impacts the Moon's surface?

Section 14.8 Energy Considerations in Planetary and Satellite Motion

33. A 500-kg satellite is in a circular orbit at an altitude of 500 km above the Earth's surface. Because of air friction, the satellite is eventually brought to the Earth's surface, and it hits the Earth with a speed of 2.00 km/s. How much energy was transformed to internal energy by means of friction?
34. (a) What is the minimum speed, relative to the Sun, that is necessary for a spacecraft to escape the Solar System if it starts at the Earth's orbit? (b) *Voyager 1* achieved a maximum speed of 125 000 km/h on its way to photograph Jupiter. Beyond what distance from the Sun is this speed sufficient for a spacecraft to escape the Solar System?
35. A satellite with a mass of 200 kg is placed in Earth orbit at a height of 200 km above the surface. (a) Assuming a circular orbit, how long does the satellite take to complete one orbit? (b) What is the satellite's speed? (c) What is the minimum energy necessary to place this satellite in orbit (assuming no air friction)?
36. A satellite of mass m is placed in Earth orbit at an altitude h. (a) Assuming a circular orbit, how long does the satellite take to complete one orbit? (b) What is the satellite's speed? (c) What is the minimum energy necessary to place this satellite in orbit (assuming no air friction)?
37. WEB A spaceship is fired from the Earth's surface with an initial speed of 2.00×10^4 m/s. What will its speed be when it is very far from the Earth? (Neglect friction.)
38. A 1 000-kg satellite orbits the Earth at a constant altitude of 100 km. How much energy must be added to the system to move the satellite into a circular orbit at an altitude of 200 km?
39. A "treetop satellite" moves in a circular orbit just above the surface of a planet, which is assumed to offer no air resistance. Show that its orbital speed v and the escape speed from the planet are related by the expression $v_{esc} = \sqrt{2}v$.
40. The planet Uranus has a mass about 14 times the Earth's mass, and its radius is equal to about 3.7 Earth

radii. (a) By setting up ratios with the corresponding Earth values, find the acceleration due to gravity at the cloud tops of Uranus. (b) Ignoring the rotation of the planet, find the minimum escape speed from Uranus.

41. Determine the escape velocity for a rocket on the far side of Ganymede, the largest of Jupiter's moons. The radius of Ganymede is 2.64×10^6 m, and its mass is 1.495×10^{23} kg. The mass of Jupiter is 1.90×10^{27} kg, and the distance between Jupiter and Ganymede is 1.071×10^9 m. Be sure to include the gravitational effect due to Jupiter, but you may ignore the motions of Jupiter and Ganymede as they revolve about their center of mass (Fig. P14.41).

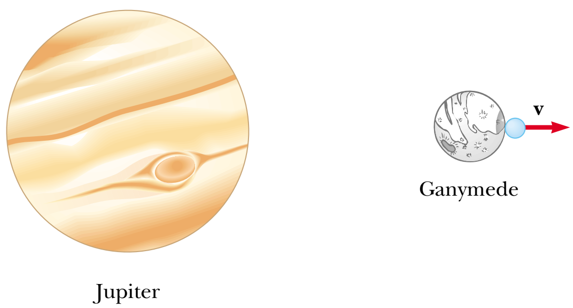


Figure P14.41

42. In Robert Heinlein's *The Moon is a Harsh Mistress,* the colonial inhabitants of the Moon threaten to launch rocks down onto the Earth if they are not given independence (or at least representation). Assuming that a rail gun could launch a rock of mass m at twice the lunar escape speed, calculate the speed of the rock as it enters the Earth's atmosphere. (By *lunar escape speed* we mean the speed required to escape entirely from a stationary Moon alone in the Universe.)

43. Derive an expression for the work required to move an Earth satellite of mass m from a circular orbit of radius $2R_E$ to one of radius $3R_E$.

(Optional)

Section 14.9 The Gravitational Force Between an Extended Object and a Particle

44. Consider two identical uniform rods of length L and mass m lying along the same line and having their closest points separated by a distance d (Fig. P14.44). Show that the mutual gravitational force between these rods has a magnitude

$$F = \frac{Gm^2}{L^2} \ln\left(\frac{(L+d)^2}{d(2L+d)}\right)$$

45. A uniform rod of mass M is in the shape of a semicircle of radius R (Fig. P14.45). Calculate the force on a point mass m placed at the center of the semicircle.

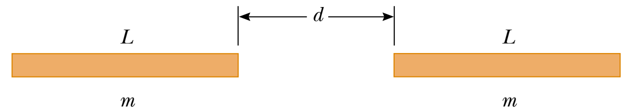


Figure P14.44

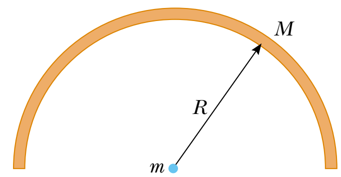


Figure P14.45

(Optional)

Section 14.10 The Gravitational Force Between a Particle and a Spherical Mass

46. (a) Show that the period calculated in Example 14.10 can be written as

$$T = 2\pi\sqrt{\frac{R_E}{g}}$$

where g is the free-fall acceleration on the surface of the Earth. (b) What would this period be if tunnels were made through the Moon? (c) What practical problem regarding these tunnels on Earth would be removed if they were built on the Moon?

47. A 500-kg uniform solid sphere has a radius of 0.400 m. Find the magnitude of the gravitational force exerted by the sphere on a 50.0-g particle located (a) 1.50 m from the center of the sphere, (b) at the surface of the sphere, and (c) 0.200 m from the center of the sphere.

48. A uniform solid sphere of mass m_1 and radius R_1 is inside and concentric with a spherical shell of mass m_2 and radius R_2 (Fig. P14.48). Find the gravitational force exerted by the spheres on a particle of mass m located at (a) $r = a$, (b) $r = b$, and (c) $r = c$, where r is measured from the center of the spheres.

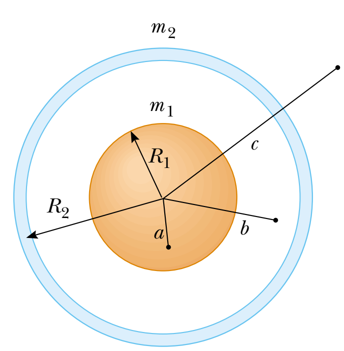


Figure P14.48

ADDITIONAL PROBLEMS

49. Let Δg_M represent the difference in the gravitational fields produced by the Moon at the points on the Earth's surface nearest to and farthest from the Moon. Find the fraction $\Delta g_M/g$, where g is the Earth's gravitational field. (This difference is responsible for the occurrence of the *lunar tides* on the Earth.)

50. Two spheres having masses M and $2M$ and radii R and $3R$, respectively, are released from rest when the distance between their centers is $12R$. How fast will each sphere be moving when they collide? Assume that the two spheres interact only with each other.

51. In Larry Niven's science-fiction novel *Ringworld,* a rigid ring of material rotates about a star (Fig. P14.51). The rotational speed of the ring is 1.25×10^6 m/s, and its radius is 1.53×10^{11} m. (a) Show that the centripetal acceleration of the inhabitants is 10.2 m/s^2. (b) The inhabitants of this ring world experience a normal contact force **n**. Acting alone, this normal force would produce an inward acceleration of 9.90 m/s^2. Additionally, the star at the center of the ring exerts a gravitational force on the ring and its inhabitants. The difference between the total acceleration and the acceleration provided by the normal force is due to the gravitational attraction of the central star. Show that the mass of the star is approximately 10^{32} kg.

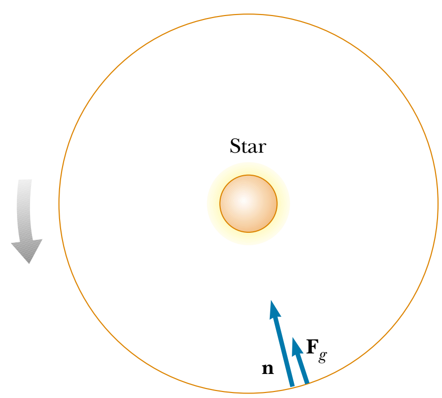


Figure P14.51

52. (a) Show that the rate of change of the free-fall acceleration with distance above the Earth's surface is

$$\frac{dg}{dr} = -\frac{2GM_E}{R_E^{\,3}}$$

This rate of change over distance is called a *gradient.* (b) If h is small compared to the radius of the Earth, show that the difference in free-fall acceleration between two points separated by vertical distance h is

$$|\Delta g| = \frac{2GM_E h}{R_E^{\,3}}$$

(c) Evaluate this difference for $h = 6.00$ m, a typical height for a two-story building.

53. A particle of mass m is located inside a uniform solid sphere of radius R and mass M, at a distance r from its center. (a) Show that the gravitational potential energy of the system is $U = (GmM/2R^3)r^2 - 3GmM/2R$. (b) Write an expression for the amount of work done by the gravitational force in bringing the particle from the surface of the sphere to its center.

54. *Voyagers 1* and *2* surveyed the surface of Jupiter's moon Io and photographed active volcanoes spewing liquid sulfur to heights of 70 km above the surface of this moon. Find the speed with which the liquid sulfur left the volcano. Io's mass is 8.9×10^{22} kg, and its radius is 1 820 km.

55. As an astronaut, you observe a small planet to be spherical. After landing on the planet, you set off, walking always straight ahead, and find yourself returning to your spacecraft from the opposite side after completing a lap of 25.0 km. You hold a hammer and a falcon feather at a height of 1.40 m, release them, and observe that they fall together to the surface in 29.2 s. Determine the mass of the planet.

56. A cylindrical habitat in space, 6.00 km in diameter and 30 km long, was proposed by G. K. O'Neill in 1974. Such a habitat would have cities, land, and lakes on the inside surface and air and clouds in the center. All of these would be held in place by the rotation of the cylinder about its long axis. How fast would the cylinder have to rotate to imitate the Earth's gravitational field at the walls of the cylinder?

WEB **57.** In introductory physics laboratories, a typical Cavendish balance for measuring the gravitational constant G uses lead spheres with masses of 1.50 kg and 15.0 g whose centers are separated by about 4.50 cm. Calculate the gravitational force between these spheres, treating each as a point mass located at the center of the sphere.

58. Newton's law of universal gravitation is valid for distances covering an enormous range, but it is thought to fail for very small distances, where the structure of space itself is uncertain. The crossover distance, far less than the diameter of an atomic nucleus, is called the *Planck length.* It is determined by a combination of the constants G, c, and h, where c is the speed of light in vacuum and h is Planck's constant (introduced briefly in Chapter 11 and discussed in greater detail in Chapter 40) with units of angular momentum. (a) Use dimensional analysis to find a combination of these three universal constants that has units of length. (b) Determine the order of magnitude of the Planck length. (*Hint:* You will need to consider noninteger powers of the constants.)

59. Show that the escape speed from the surface of a planet of uniform density is directly proportional to the radius of the planet.

60. (a) Suppose that the Earth (or another object) has density $\rho(r)$, which can vary with radius but is spherically symmetric. Show that at any particular radius r inside the Earth, the gravitational field strength $g(r)$ will increase as r increases, if and only if the density there exceeds 2/3 the average density of the portion of the Earth inside the radius r. (b) The Earth as a whole has an average density of 5.5 g/cm^3, while the density at the surface is 1.0 g/cm^3 on the oceans and about 3 g/cm^3 on land. What can you infer from this?

WEB **61.** Two hypothetical planets of masses m_1 and m_2 and radii r_1 and r_2, respectively, are nearly at rest when they are an infinite distance apart. Because of their gravitational attraction, they head toward each other on a collision course. (a) When their center-to-center separation is d, find expressions for the speed of each planet and their *relative* velocity. (b) Find the kinetic energy of each planet *just* before they collide, if $m_1 = 2.00 \times 10^{24}$ kg, $m_2 = 8.00 \times 10^{24}$ kg, $r_1 = 3.00 \times 10^6$ m, and $r_2 = 5.00 \times 10^6$ m. (*Hint:* Both energy and momentum are conserved.)

62. The maximum distance from the Earth to the Sun (at our aphelion) is 1.521×10^{11} m, and the distance of closest approach (at perihelion) is 1.471×10^{11} m. If the Earth's orbital speed at perihelion is 30.27 km/s, determine (a) the Earth's orbital speed at aphelion, (b) the kinetic and potential energies at perihelion, and (c) the kinetic and potential energies at aphelion. Is the total energy constant? (Neglect the effect of the Moon and other planets.)

63. A sphere of mass M and radius R has a nonuniform density that varies with r, the distance from its center, according to the expression $\rho = Ar$, for $0 \le r \le R$. (a) What is the constant A in terms of M and R? (b) Determine an expression for the force exerted on a particle of mass m placed outside the sphere. (c) Determine an expression for the force exerted on the particle if it is inside the sphere. (*Hint:* See Section 14.10 and note that the distribution is spherically symmetric.)

64. (a) Determine the amount of work (in joules) that must be done on a 100-kg payload to elevate it to a height of 1 000 km above the Earth's surface. (b) Determine the amount of additional work that is required to put the payload into circular orbit at this elevation.

65. X-ray pulses from Cygnus X-1, a celestial x-ray source, have been recorded during high-altitude rocket flights. The signals can be interpreted as originating when a blob of ionized matter orbits a black hole with a period of 5.0 ms. If the blob is in a circular orbit about a black hole whose mass is $20M_{\text{Sun}}$, what is the orbital radius?

66. Studies of the relationship of the Sun to its galaxy—the Milky Way—have revealed that the Sun is located near the outer edge of the galactic disk, about 30 000 lightyears from the center. Furthermore, it has been found that the Sun has an orbital speed of approximately 250 km/s around the galactic center. (a) What is the period of the Sun's galactic motion? (b) What is the order of magnitude of the mass of the Milky Way galaxy? Suppose that the galaxy is made mostly of stars, of which the Sun is typical. What is the order of magnitude of the number of stars in the Milky Way?

67. The oldest artificial satellite in orbit is *Vanguard I,* launched March 3, 1958. Its mass is 1.60 kg. In its initial orbit, its minimum distance from the center of the Earth was 7.02 Mm, and its speed at this perigee point was 8.23 km/s. (a) Find its total energy. (b) Find the magnitude of its angular momentum. (c) Find its speed at apogee and its maximum (apogee) distance from the center of the Earth. (d) Find the semimajor axis of its orbit. (e) Determine its period.

68. A rocket is given an initial speed vertically upward of $v_i = 2\sqrt{Rg}$ at the surface of the Earth, which has radius R and surface free-fall acceleration g. The rocket motors are quickly cut off, and thereafter the rocket coasts under the action of gravitational forces only. (Ignore atmospheric friction and the Earth's rotation.) Derive an expression for the subsequent speed v as a function of the distance r from the center of the Earth in terms of g, R, and r.

69. Two stars of masses M and m, separated by a distance d, revolve in circular orbits about their center of mass (Fig. P14.69). Show that each star has a period given by

$$T^2 = \frac{4\pi^2 d^3}{G(M + m)}$$

(*Hint:* Apply Newton's second law to each star, and note that the center-of-mass condition requires that $Mr_2 = mr_1$, where $r_1 + r_2 = d$.)

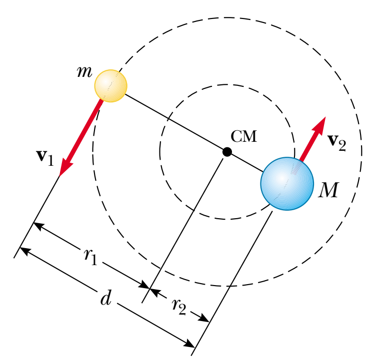


Figure P14.69

70. (a) A 5.00-kg mass is released 1.20×10^7 m from the center of the Earth. It moves with what acceleration relative to the Earth? (b) A 2.00×10^{24} kg mass is released 1.20×10^7 m from the center of the Earth. It moves with what acceleration relative to the Earth? Assume that the objects behave as pairs of particles, isolated from the rest of the Universe.

71. The acceleration of an object moving in the gravitational field of the Earth is

$$\mathbf{a} = -\frac{GM_E}{r^3}\mathbf{r}$$

where **r** is the position vector directed from the center of the Earth to the object. Choosing the origin at the center of the Earth and assuming that the small object is moving in the xy plane, we find that the rectangular (cartesian) components of its acceleration are

$$a_x = -\frac{GM_E x}{(x^2 + y^2)^{3/2}} \qquad a_y = -\frac{GM_E y}{(x^2 + y^2)^{3/2}}$$

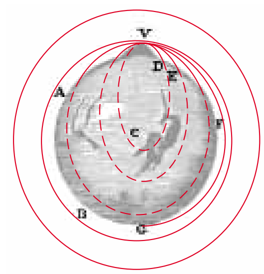
Use a computer to set up and carry out a numerical prediction of the motion of the object, according to Euler's method. Assume that the initial position of the object is $x = 0$ and $y = 2R_E$, where R_E is the radius of the Earth. Give the object an initial velocity of 5 000 m/s in the x direction. The time increment should be made as small as practical. Try 5 s. Plot the x and y coordinates of the object as time goes on. Does the object hit the Earth? Vary the initial velocity until you find a circular orbit.

ANSWERS TO QUICK QUIZZES

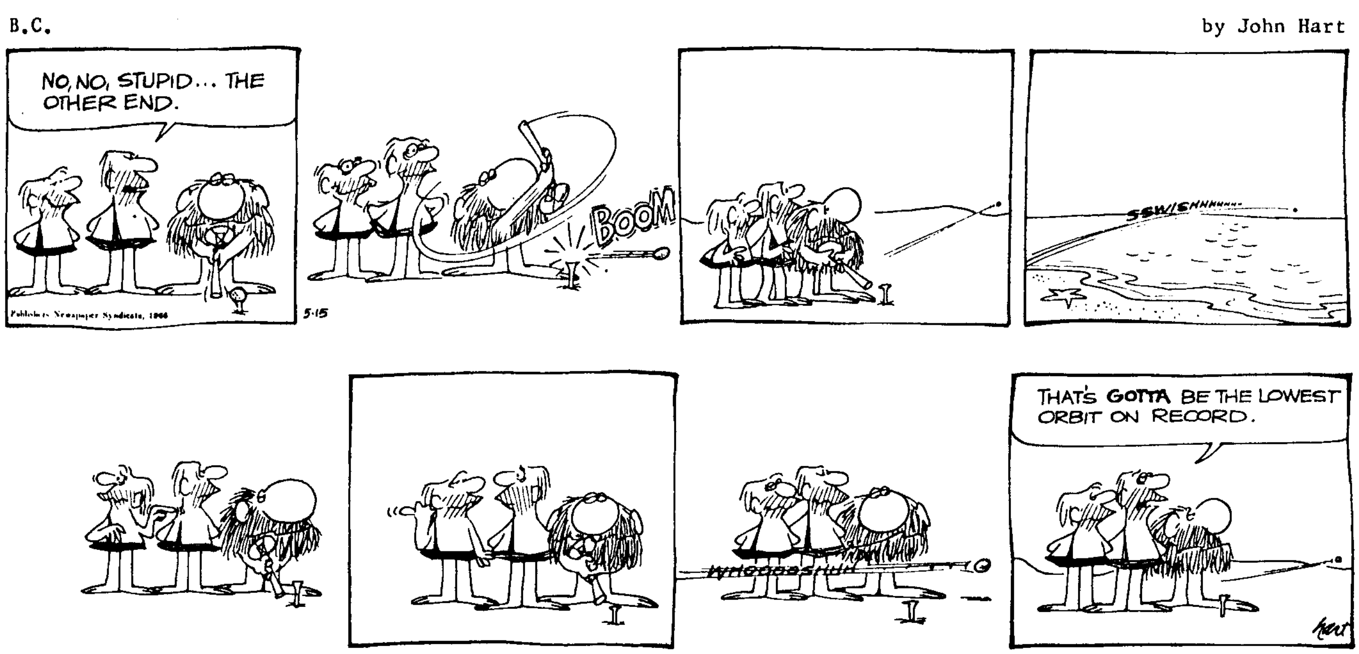
14.1 Kepler's third law (Eq. 14.7), which applies to all the planets, tells us that the period of a planet is proportional to $r^{3/2}$. Because Saturn and Jupiter are farther from the Sun than the Earth is, they have longer periods. The Sun's gravitational field is much weaker at Saturn and Jupiter than it is at the Earth. Thus, these planets experience much less centripetal acceleration than the Earth does, and they have correspondingly longer periods.

14.2 The mass of the asteroid might be so small that you would be able to exceed escape velocity by leg power alone. You would jump up, but you would never come back down!

14.3 Kepler's first law applies not only to planets orbiting the Sun but also to any relatively small object orbiting another under the influence of gravity. Any elliptical path that does not touch the Earth before reaching point G will continue around the other side to point V in a complete orbit (see figure in next column).



14.4 The gravitational force is zero inside the shell (Eq. 14.25b). Because the force on it is zero, the particle moves with constant velocity in the direction of its original motion outside the shell until it hits the wall opposite the entry hole. Its path thereafter depends on the nature of the collision and on the particle's original direction.



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APPENDIX A • Tables

TABLE A.1 Conversion Factors

Length

	m	cm	km	in.	ft	mi
1 meter	1	10^2	10^{-3}	39.37	3.281	6.214×10^{-4}
1 centimeter	10^{-2}	1	10^{-5}	0.393 7	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	10^3	10^5	1	3.937×10^4	3.281×10^3	0.621 4
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	0.304 8	30.48	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1 609	1.609×10^5	1.609	6.336×10^4	5 280	1

Mass

	kg	g	slug	u
1 kilogram	1	10^3	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^4	1	8.789×10^{27}
1 atomic mass unit	1.660×10^{-27}	1.660×10^{-24}	1.137×10^{-28}	1

Note: 1 metric ton = 1 000 kg.

Time

	s	min	h	day	yr
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.994×10^{-4}	1.901×10^{-6}
1 hour	3 600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^4	1 440	24	1	2.738×10^{-5}
1 year	3.156×10^7	5.259×10^5	8.766×10^3	365.2	1

Speed

	m/s	cm/s	ft/s	mi/h
1 meter per second	1	10^2	3.281	2.237
1 centimeter per second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}
1 foot per second	0.304 8	30.48	1	0.681 8
1 mile per hour	0.447 0	44.70	1.467	1

Note: 1 mi/min = 60 mi/h = 88 ft/s.

continued

TABLE A.1 Continued

Force

	N	lb
1 newton	1	0.224 8
1 pound	4.448	1

Work, Energy, Heat

	J	ft · lb	eV
1 joule	1	0.737 6	6.242×10^{18}
1 ft · lb	1.356	1	8.464×10^{18}
1 eV	1.602×10^{-19}	1.182×10^{-19}	1
1 cal	4.186	3.087	2.613×10^{19}
1 Btu	1.055×10^{3}	7.779×10^{2}	6.585×10^{21}
1 kWh	3.600×10^{6}	2.655×10^{6}	2.247×10^{25}

	cal	Btu	kWh
1 joule	0.238 9	9.481×10^{-4}	2.778×10^{-7}
1 ft · lb	0.323 9	1.285×10^{-3}	3.766×10^{-7}
1 eV	3.827×10^{-20}	1.519×10^{-22}	4.450×10^{-26}
1 cal	1	3.968×10^{-3}	1.163×10^{-6}
1 Btu	2.520×10^{2}	1	2.930×10^{-4}
1 kWh	8.601×10^{5}	3.413×10^{2}	1

Pressure

	Pa	atm
1 pascal	1	9.869×10^{-6}
1 atmosphere	1.013×10^{5}	1
1 centimeter mercury[a]	1.333×10^{3}	1.316×10^{-2}
1 pound per inch^2	6.895×10^{3}	6.805×10^{-2}
1 pound per foot^2	47.88	4.725×10^{-4}

	cm Hg	lb/in.^{2}	lb/ft^2
1 newton per meter^2	7.501×10^{-4}	1.450×10^{-4}	2.089×10^{-2}
1 atmosphere	76	14.70	2.116×10^{3}
1 centimeter mercury[a]	1	0.194 3	27.85
1 pound per inch^2	5.171	1	144
1 pound per foot^2	3.591×10^{-2}	6.944×10^{-3}	1

[a] At 0°C and at a location where the acceleration due to gravity has its "standard" value, 9.806 65 m/s^2.

TABLE A.2 Symbols, Dimensions, and Units of Physical Quantities

Quantity	Common Symbol	Unit[a]	Dimensions[b]	Unit in Terms of Base SI Units
Acceleration	**a**	m/s^2	L/T^2	m/s^2
Amount of substance	n	mole		mol
Angle	θ, ϕ	radian (rad)	1	
Angular acceleration	$\boldsymbol{\alpha}$	rad/s^2	T^{-2}	s^{-2}
Angular frequency	ω	rad/s	T^{-1}	s^{-1}
Angular momentum	**L**	$kg \cdot m^2/s$	ML^2/T	$kg \cdot m^2/s$
Angular velocity	$\boldsymbol{\omega}$	rad/s	T^{-1}	s^{-1}
Area	A	m^2	L^2	m^2
Atomic number	Z			
Capacitance	C	farad (F)	Q^2T^2/ML^2	$A^2 \cdot s^4/kg \cdot m^2$
Charge	q, Q, e	coulomb (C)	Q	$A \cdot s$
Charge density				
Line	λ	C/m	Q/L	$A \cdot s/m$
Surface	σ	C/m^2	Q/L^2	$A \cdot s/m^2$
Volume	ρ	C/m^3	Q/L^3	$A \cdot s/m^3$
Conductivity	σ	$1/\Omega \cdot m$	Q^2T/ML^3	$A^2 \cdot s^3/kg \cdot m^3$
Current	I	AMPERE	Q/T	A
Current density	**J**	A/m^2	Q/T^2	A/m^2
Density	ρ	kg/m^3	M/L^3	kg/m^3
Dielectric constant	κ			
Displacement	**r, s**	METER	L	m
Distance	d, h			
Length	ℓ, L			
Electric dipole moment	**p**	$C \cdot m$	QL	$A \cdot s \cdot m$
Electric field	**E**	V/m	ML/QT^2	$kg \cdot m/A \cdot s^3$
Electric flux	Φ_E	$V \cdot m$	ML^3/QT^2	$kg \cdot m^3/A \cdot s^3$
Electromotive force	$\mathcal{E}$	volt (V)	ML^2/QT^2	$kg \cdot m^2/A \cdot s^3$
Energy	E, U, K	joule (J)	ML^2/T^2	$kg \cdot m^2/s^2$
Entropy	S	J/K	$ML^2/T^2 \cdot K$	$kg \cdot m^2/s^2 \cdot K$
Force	**F**	newton (N)	ML/T^2	$kg \cdot m/s^2$
Frequency	f	hertz (Hz)	T^{-1}	s^{-1}
Heat	Q	joule (J)	ML^2/T^2	$kg \cdot m^2/s^2$
Inductance	L	henry (H)	ML^2/Q^2	$kg \cdot m^2/A^2 \cdot s^2$
Magnetic dipole moment	$\boldsymbol{\mu}$	$N \cdot m/T$	QL^2/T	$A \cdot m^2$
Magnetic field	**B**	tesla (T) $(= Wb/m^2)$	M/QT	$kg/A \cdot s^2$
Magnetic flux	Φ_B	weber (Wb)	ML^2/QT	$kg \cdot m^2/A \cdot s^2$
Mass	m, M	KILOGRAM	M	kg
Molar specific heat	C	$J/mol \cdot K$		$kg \cdot m^2/s^2 \cdot mol \cdot K$
Moment of inertia	I	$kg \cdot m^2$	ML^2	$kg \cdot m^2$
Momentum	**p**	$kg \cdot m/s$	ML/T	$kg \cdot m/s$
Period	T	s	T	s
Permeability of space	μ_0	$N/A^2 (= H/m)$	ML/Q^2T	$kg \cdot m/A^2 \cdot s^2$
Permittivity of space	ϵ_0	$C^2/N \cdot m^2 (= F/m)$	Q^2T^2/ML^3	$A^2 \cdot s^4/kg \cdot m^3$
Potential	V	volt (V) $(= J/C)$	ML^2/QT^2	$kg \cdot m^2/A \cdot s^3$
Power	$\mathcal{P}$	watt (W) $(= J/s)$	ML^2/T^3	$kg \cdot m^2/s^3$

continued

TABLE A.2 Continued

Quantity	Common Symbol	Unit[a]	Dimensions[b]	Unit in Terms of Base SI Units
Pressure	P	pascal (Pa) = (N/m^2)	M/LT^2	kg/m · s^2
Resistance	R	ohm (Ω) (=V/A)	ML^2/Q^{2}T	kg · m^2/A^2 · s^3
Specific heat	c	J/kg · K	L^2/T^2 · K	m^2/s^2 · K
Speed	v	m/s	L/T	m/s
Temperature	T	KELVIN	K	K
Time	t	SECOND	T	s
Torque	$\boldsymbol{\tau}$	N · m	ML^2/T^2	kg · m^2/s^2
Volume	V	m^3	L^3	m^3
Wavelength	λ	m	L	m
Work	W	joule (J) (=N · m)	ML^2/T^2	kg · m^2/s^2

[a] The base SI units are given in uppercase letters.

[b] The symbols M, L, T, and Q denote mass, length, time, and charge, respectively.

TABLE A.3 Table of Atomic Masses[a]

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
0	(Neutron)	n		1*	1.008 665		10.4 min
1	Hydrogen	H	1.007 9	1	1.007 825	99.985	
	Deuterium	D		2	2.014 102	0.015	
	Tritium	T		3*	3.016 049		12.33 yr
2	Helium	He	4.002 60	3	3.016 029	0.000 14	
				4	4.002 602	99.999 86	
				6*	6.018 886		0.81 s
3	Lithium	Li	6.941	6	6.015 121	7.5	
				7	7.016 003	92.5	
				8*	8.022 486		0.84 s
4	Beryllium	Be	9.012 2	7*	7.016 928		53.3 days
				9	9.012 174	100	
				10*	10.013 534		1.5×10^6 yr
5	Boron	B	10.81	10	10.012 936	19.9	
				11	11.009 305	80.1	
				12*	12.014 352		0.020 2 s
6	Carbon	C	12.011	10*	10.016 854		19.3 s
				11*	11.011 433		20.4 min
				12	12.000 000	98.90	
				13	13.003 355	1.10	
				14*	14.003 242		5 730 yr
				15*	15.010 599		2.45 s
7	Nitrogen	N	14.006 7	12*	12.018 613		0.011 0 s
				13*	13.005 738		9.96 min
				14	14.003 074	99.63	
				15	15.000 108	0.37	
				16*	16.006 100		7.13 s
				17*	17.008 450		4.17 s

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
8	Oxygen	O	15.999 4	14*	14.008 595		70.6 s
				15*	15.003 065		122 s
				16	15.994 915	99.761	
				17	16.999 132	0.039	
				18	17.999 160	0.20	
				19*	19.003 577		26.9 s
9	Fluorine	F	18.998 40	17*	17.002 094		64.5 s
				18*	18.000 937		109.8 min
				19	18.998 404	100	
				20*	19.999 982		11.0 s
				21*	20.999.950		4.2 s
10	Neon	Ne	20.180	18*	18.005 710		1.67 s
				19*	19.001 880		17.2 s
				20	19.992 435	90.48	
				21	20.993 841	0.27	
				22	21.991 383	9.25	
				23*	22.994 465		37.2 s
11	Sodium	Na	22.989 87	21*	20.997 650		22.5 s
				22*	21.994 434		2.61 yr
				23	22.989 770	100	
				24*	23.990 961		14.96 h
12	Magnesium	Mg	24.305	23*	22.994 124		11.3 s
				24	23.985 042	78.99	
				25	24.985 838	10.00	
				26	25.982 594	11.01	
				27*	26.984 341		9.46 min
13	Aluminum	Al	26.981 54	26*	25.986 892		7.4×10^5 yr
				27	26.981 538	100	
				28*	27.981 910		2.24 min
14	Silicon	Si	28.086	28	27.976 927	92.23	
				29	28.976 495	4.67	
				30	29.973 770	3.10	
				31*	30.975 362		2.62 h
				32*	31.974 148		172 yr
15	Phosphorus	P	30.973 76	30*	29.978 307		2.50 min
				31	30.973 762	100	
				32*	31.973 908		14.26 days
				33*	32.971 725		25.3 days
16	Sulfur	S	32.066	32	31.972 071	95.02	
				33	32.971 459	0.75	
				34	33.967 867	4.21	
				35*	34.969 033		87.5 days
				36	35.967 081	0.02	
17	Chlorine	Cl	35.453	35	34.968 853	75.77	
				36*	35.968 307		3.0×10^5 yr
				37	36.965 903	24.23	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
18	Argon	Ar	39.948	36	35.967 547	0.337	
				37*	36.966 776		35.04 days
				38	37.962 732	0.063	
				39*	38.964 314		269 yr
				40	39.962 384	99.600	
				42*	41.963 049		33 yr
19	Potassium	K	39.098 3	39	38.963 708	93.258 1	
				40*	39.964 000	0.011 7	1.28×10^9 yr
				41	40.961 827	6.730 2	
20	Calcium	Ca	40.08	40	39.962 591	96.941	
				41*	40.962 279		1.0×10^5 yr
				42	41.958 618	0.647	
				43	42.958 767	0.135	
				44	43.955 481	2.086	
				46	45.953 687	0.004	
				48	47.952 534	0.187	
21	Scandium	Sc	44.955 9	41*	40.969 250		0.596 s
				45	44.955 911	100	
22	Titanium	Ti	47.88	44*	43.959 691		49 yr
				46	45.952 630	8.0	
				47	46.951 765	7.3	
				48	47.947 947	73.8	
				49	48.947 871	5.5	
				50	49.944 792	5.4	
23	Vanadium	V	50.941 5	48*	47.952 255		15.97 days
				50*	49.947 161	0.25	1.5×10^{17} yr
				51	50.943 962	99.75	
24	Chromium	Cr	51.996	48*	47.954 033		21.6 h
				50	49.946 047	4.345	
				52	51.940 511	83.79	
				53	52.940 652	9.50	
				54	53.938 883	2.365	
25	Manganese	Mn	54.938 05	54*	53.940 361		312.1 days
				55	54.938 048	100	
26	Iron	Fe	55.847	54	53.939 613	5.9	
				55*	54.938 297		2.7 yr
				56	55.934 940	91.72	
				57	56.935 396	2.1	
				58	57.933 278	0.28	
				60*	59.934 078		1.5×10^6 yr
27	Cobalt	Co	58.933 20	59	58.933 198	100	
				60*	59.933 820		5.27 yr
28	Nickel	Ni	58.693	58	57.935 346	68.077	
				59*	58.934 350		7.5×10^4 yr
				60	59.930 789	26.223	
				61	60.931 058	1.140	
				62	61.928 346	3.634	
				63*	62.929 670		100 yr
				64	63.927 967	0.926	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
29	Copper	Cu	63.54	63	62.929 599	69.17	
				65	64.927 791	30.83	
30	Zinc	Zn	65.39	64	63.929 144	48.6	
				66	65.926 035	27.9	
				67	66.927 129	4.1	
				68	67.924 845	18.8	
				70	69.925 323	0.6	
31	Gallium	Ga	69.723	69	68.925 580	60.108	
				71	70.924 703	39.892	
32	Germanium	Ge	72.61	70	69.924 250	21.23	
				72	71.922 079	27.66	
				73	72.923 462	7.73	
				74	73.921 177	35.94	
				76	75.921 402	7.44	
33	Arsenic	As	74.921 6	75	74.921 594	100	
34	Selenium	Se	78.96	74	73.922 474	0.89	
				76	75.919 212	9.36	
				77	76.919 913	7.63	
				78	77.917 307	23.78	
				79*	78.918 497		$\leqslant 6.5 \times 10^{4}$ yr
				80	79.916 519	49.61	
				82*	81.916 697	8.73	1.4×10^{20} yr
35	Bromine	Br	79.904	79	78.918 336	50.69	
				81	80.916 287	49.31	
36	Krypton	Kr	83.80	78	77.920 400	0.35	
				80	79.916 377	2.25	
				81*	80.916 589		2.1×10^{5} yr
				82	81.913 481	11.6	
				83	82.914 136	11.5	
				84	83.911 508	57.0	
				85*	84.912 531		10.76 yr
				86	85.910 615	17.3	
37	Rubidium	Rb	85.468	85	84.911 793	72.17	
				87*	86.909 186	27.83	4.75×10^{10} yr
38	Strontium	Sr	87.62	84	83.913 428	0.56	
				86	85.909 266	9.86	
				87	86.908 883	7.00	
				88	87.905 618	82.58	
				90*	89.907 737		29.1 yr
39	Yttrium	Y	88.905 8	89	88.905 847	100	
40	Zirconium	Zr	91.224	90	89.904 702	51.45	
				91	90.905 643	11.22	
				92	91.905 038	17.15	
				93*	92.906 473		1.5×10^{6} yr
				94	93.906 314	17.38	
				96	95.908 274	2.80	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
41	Niobium	Nb	92.906 4	91*	90.906 988		6.8×10^2 yr
				92*	91.907 191		3.5×10^7 yr
				93	92.906 376	100	
				94*	93.907 280		2×10^4 yr
42	Molybdenum	Mo	95.94	92	91.906 807	14.84	
				93*	92.906 811		3.5×10^3 yr
				94	93.905 085	9.25	
				95	94.905 841	15.92	
				96	95.904 678	16.68	
				97	96.906 020	9.55	
				98	97.905 407	24.13	
				100	99.907 476	9.63	
43	Technetium	Tc		97*	96.906 363		2.6×10^6 yr
				98*	97.907 215		4.2×10^6 yr
				99*	98.906 254		2.1×10^5 yr
44	Ruthenium	Ru	101.07	96	95.907 597	5.54	
				98	97.905 287	1.86	
				99	98.905 939	12.7	
				100	99.904 219	12.6	
				101	100.905 558	17.1	
				102	101.904 348	31.6	
				104	103.905 428	18.6	
45	Rhodium	Rh	102.905 5	103	102.905 502	100	
46	Palladium	Pd	106.42	102	101.905 616	1.02	
				104	103.904 033	11.14	
				105	104.905 082	22.33	
				106	105.903 481	27.33	
				107*	106.905 126		6.5×10^6 yr
				108	107.903 893	26.46	
				110	109.905 158	11.72	
47	Silver	Ag	107.868	107	106.905 091	51.84	
				109	108.904 754	48.16	
48	Cadmium	Cd	112.41	106	105.906 457	1.25	
				108	107.904 183	0.89	
				109*	108.904 984		462 days
				110	109.903 004	12.49	
				111	110.904 182	12.80	
				112	111.902 760	24.13	
				113*	112.904 401	12.22	9.3×10^{15} yr
				114	113.903 359	28.73	
				116	115.904 755	7.49	
49	Indium	In	114.82	113	112.904 060	4.3	
				115*	114.903 876	95.7	4.4×10^{14} yr
50	Tin	Sn	118.71	112	111.904 822	0.97	
				114	113.902 780	0.65	
				115	114.903 345	0.36	
				116	115.901 743	14.53	
				117	116.902 953	7.68	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
(50)	(Tin)			118	117.901 605	24.22	
				119	118.903 308	8.58	
				120	119.902 197	32.59	
				121*	120.904 237		55 yr
				122	121.903 439	4.63	
				124	123.905 274	5.79	
51	Antimony	Sb	121.76	121	120.903 820	57.36	
				123	122.904 215	42.64	
				125*	124.905 251		2.7 yr
52	Tellurium	Te	127.60	120	119.904 040	0.095	
				122	121.903 052	2.59	
				123*	122.904 271	0.905	1.3×10^{13} yr
				124	123.902 817	4.79	
				125	124.904 429	7.12	
				126	125.903 309	18.93	
				128*	127.904 463	31.70	$>8 \times 10^{24}$ yr
				130*	129.906 228	33.87	$\leqslant 1.25 \times 10^{21}$ yr
53	Iodine	I	126.904 5	127	126.904 474	100	
				129*	128.904 984		1.6×10^{7} yr
54	Xenon	Xe	131.29	124	123.905 894	0.10	
				126	125.904 268	0.09	
				128	127.903 531	1.91	
				129	128.904 779	26.4	
				130	129.903 509	4.1	
				131	130.905 069	21.2	
				132	131.904 141	26.9	
				134	133.905 394	10.4	
				136*	135.907 215	8.9	$\geqslant 2.36 \times 10^{21}$ yr
55	Cesium	Cs	132.905 4	133	132.905 436	100	
				134*	133.906 703		2.1 yr
				135*	134.905 891		2×10^{6} yr
				137*	136.907 078		30 yr
56	Barium	Ba	137.33	130	129.906 289	0.106	
				132	131.905 048	0.101	
				133*	132.905 990		10.5 yr
				134	133.904 492	2.42	
				135	134.905 671	6.593	
				136	135.904 559	7.85	
				137	136.905 816	11.23	
				138	137.905 236	71.70	
57	Lanthanum	La	138.905	137*	136.906 462		6×10^{4} yr
				138*	137.907 105	0.090 2	1.05×10^{11} yr
				139	138.906 346	99.909 8	
58	Cerium	Ce	140.12	136	135.907 139	0.19	
				138	137.905 986	0.25	
				140	139.905 434	88.43	
				142*	141.909 241	11.13	$>5 \times 10^{16}$ yr
59	Praseodymium	Pr	140.907 6	141	140.907 647	100	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
60	Neodymium	Nd	144.24	142	141.907 718	27.13	
				143	142.909 809	12.18	
				144*	143.910 082	23.80	2.3×10^{15} yr
				145	144.912 568	8.30	
				146	145.913 113	17.19	
				148	147.916 888	5.76	
				150*	149.920 887	5.64	$>1 \times 10^{18}$ yr
61	Promethium	Pm		143*	142.910 928		265 days
				145*	144.912 745		17.7 yr
				146*	145.914 698		5.5 yr
				147*	146.915 134		2.623 yr
62	Samarium	Sm	150.36	144	143.911 996	3.1	
				146*	145.913 043		1.0×10^{8} yr
				147*	146.914 894	15.0	1.06×10^{11} yr
				148*	147.914 819	11.3	7×10^{15} yr
				149*	148.917 180	13.8	$>2 \times 10^{15}$ yr
				150	149.917 273	7.4	
				151*	150.919 928		90 yr
				152	151.919 728	26.7	
				154	153.922 206	22.7	
63	Europium	Eu	151.96	151	150.919 846	47.8	
				152*	151.921 740		13.5 yr
				153	152.921 226	52.2	
				154*	153.922 975		8.59 yr
				155*	154.922 888		4.7 yr
64	Gadolinium	Gd	157.25	148*	147.918 112		75 yr
				150*	149.918 657		1.8×10^{6} yr
				152*	151.919 787	0.20	1.1×10^{14} yr
				154	153.920 862	2.18	
				155	154.922 618	14.80	
				156	155.922 119	20.47	
				157	156.923 957	15.65	
				158	157.924 099	24.84	
				160	159.927 050	21.86	
65	Terbium	Tb	158.925 3	159	158.925 345	100	
66	Dysprosium	Dy	162.50	156	155.924 277	0.06	
				158	157.924 403	0.10	
				160	159.925 193	2.34	
				161	160.926 930	18.9	
				162	161.926 796	25.5	
				163	162.928 729	24.9	
				164	163.929 172	28.2	
67	Holmium	Ho	164.930 3	165	164.930 316	100	
				166*	165.932 282		1.2×10^{3} yr
68	Erbium	Er	167.26	162	161.928 775	0.14	
				164	163.929 198	1.61	
				166	165.930 292	33.6	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
(68)	(Erbium)			167	166.932 047	22.95	
				168	167.932 369	27.8	
				170	169.935 462	14.9	
69	Thulium	Tm	168.934 2	169	168.934 213	100	
				171*	170.936 428		1.92 yr
70	Ytterbium	Yb	173.04	168	167.933 897	0.13	
				170	169.934 761	3.05	
				171	170.936 324	14.3	
				172	171.936 380	21.9	
				173	172.938 209	16.12	
				174	173.938 861	31.8	
				176	175.942 564	12.7	
71	Lutecium	Lu	174.967	173*	172.938 930		1.37 yr
				175	174.940 772	97.41	
				176*	175.942 679	2.59	3.78×10^{10} yr
72	Hafnium	Hf	178.49	174*	173.940 042	0.162	2.0×10^{15} yr
				176	175.941 404	5.206	
				177	176.943 218	18.606	
				178	177.943 697	27.297	
				179	178.945 813	13.629	
				180	179.946 547	35.100	
73	Tantalum	Ta	180.947 9	180	179.947 542	0.012	
				181	180.947 993	99.988	
74	Tungsten (Wolfram)	W	183.85	180	179.946 702	0.12	
				182	181.948 202	26.3	
				183	182.950 221	14.28	
				184	183.950 929	30.7	
				186	185.954 358	28.6	
75	Rhenium	Re	186.207	185	184.952 951	37.40	
				187*	186.955 746	62.60	4.4×10^{10} yr
76	Osmium	Os	190.2	184	183.952 486	0.02	
				186*	185.953 834	1.58	2.0×10^{15} yr
				187	186.955 744	1.6	
				188	187.955 832	13.3	
				189	188.958 139	16.1	
				190	189.958 439	26.4	
				192	191.961 468	41.0	
				194*	193.965 172		6.0 yr
77	Iridium	Ir	192.2	191	190.960 585	37.3	
				193	192.962 916	62.7	
78	Platinum	Pt	195.08	190*	189.959 926	0.01	6.5×10^{11} yr
				192	191.961 027	0.79	
				194	193.962 655	32.9	
				195	194.964 765	33.8	
				196	195.964 926	25.3	
				198	197.967 867	7.2	
79	Gold	Au	196.966 5	197	196.966 543	100	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
80	Mercury	Hg	200.59	196	195.965 806	0.15	
				198	197.966 743	9.97	
				199	198.968 253	16.87	
				200	199.968 299	23.10	
				201	200.970 276	13.10	
				202	201.970 617	29.86	
				204	203.973 466	6.87	
81	Thallium	Tl	204.383	203	202.972 320	29.524	
				204*	203.973 839		3.78 yr
				205	204.974 400	70.476	
		(Ra E″)		206*	205.976 084		4.2 min
		(Ac C″)		207*	206.977 403		4.77 min
		(Th C″)		208*	207.981 992		3.053 min
		(Ra C″)		210*	209.990 057		1.30 min
82	Lead	Pb	207.2	202*	201.972 134		5×10^4 yr
				204*	203.973 020	1.4	$\geqslant 1.4 \times 10^{17}$ yr
				205*	204.974 457		1.5×10^7 yr
				206	205.974 440	24.1	
				207	206.975 871	22.1	
				208	207.976 627	52.4	
		(Ra D)		210*	209.984 163		22.3 yr
		(Ac B)		211*	210.988 734		36.1 min
		(Th B)		212*	211.991 872		10.64 h
		(Ra B)		214*	213.999 798		26.8 min
83	Bismuth	Bi	208.980 3	207*	206.978 444		32.2 yr
				208*	207.979 717		3.7×10^5 yr
				209	208.980 374	100	
		(Ra E)		210*	209.984 096		5.01 days
		(Th C)		211*	210.987 254		2.14 min
				212*	211.991 259		60.6 min
		(Ra C)		214*	213.998 692		19.9 min
				215*	215.001 836		7.4 min
84	Polonium	Po		209*	208.982 405		102 yr
		(Ra F)		210*	209.982 848		138.38 days
		(Ac C′)		211*	210.986 627		0.52 s
		(Th C′)		212*	211.988 842		0.30 μs
		(Ra C′)		214*	213.995 177		164 μs
		(Ac A)		215*	214.999 418		0.001 8 s
		(Th A)		216*	216.001 889		0.145 s
		(Ra A)		218*	218.008 965		3.10 min
85	Astatine	At		215*	214.998 638		≈ 100 μs
				218*	218.008 685		1.6 s
				219*	219.011 294		0.9 min
86	Radon	Rn					
		(An)		219*	219.009 477		3.96 s
		(Tn)		220*	220.011 369		55.6 s
		(Rn)		222*	222.017 571		3.823 days
87	Francium	Fr					
		(Ac K)		223*	223.019 733		22 min

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
88	Radium	Ra					
		(Ac X)		223*	223.018 499		11.43 days
		(Th X)		224*	224.020 187		3.66 days
		(Ra)		226*	226.025 402		1 600 yr
		(Ms Th_1)		228*	228.031 064		5.75 yr
89	Actinium	Ac		227*	227.027 749		21.77 yr
		(Ms Th_2)		228*	228.031 015		6.15 h
90	Thorium	Th	232.038 1				
		(Rd Ac)		227*	227.027 701		18.72 days
		(Rd Th)		228*	228.028 716		1.913 yr
				229*	229.031 757		7 300 yr
		(Io)		230*	230.033 127		75.000 yr
		(UY)		231*	231.036 299		25.52 h
		(Th)		232*	232.038 051	100	1.40×10^{10} yr
		(UX_1)		234*	234.043 593		24.1 days
91	Protactinium	Pa		231*	231.035 880		32.760 yr
		(Uz)		234*	234.043 300		6.7 h
92	Uranium	U	238.028 9	232*	232.037 131		69 yr
				233*	233.039 630		1.59×10^5 yr
				234*	234.040 946	0.005 5	2.45×10^5 yr
		(Ac U)		235*	235.043 924	0.720	7.04×10^8 yr
				236*	236.045 562		2.34×10^7 yr
		(UI)		238*	238.050 784	99.274 5	4.47×10^9 yr
93	Neptunium	Np		235*	235.044 057		396 days
				236*	236.046 560		1.15×10^5 yr
				237*	237.048 168		2.14×10^6 yr
94	Plutonium	Pu		236*	236.046 033		2.87 yr
				238*	238.049 555		87.7 yr
				239*	239.052 157		2.412×10^4 yr
				240*	240.053 808		6 560 yr
				241*	241.056 846		14.4 yr
				242*	242.058 737		3.73×10^6 yr
				244*	244.064 200		8.1×10^7 yr

[a] The masses in the sixth column are atomic masses, which include the mass of Z electrons. Data are from the National Nuclear Data Center, Brookhaven National Laboratory, prepared by Jagdish K. Tuli, July 1990. The data are based on experimental results reported in *Nuclear Data Sheets* and *Nuclear Physics* and also from *Chart of the Nuclides,* 14th ed. Atomic masses are based on those by A. H. Wapstra, G. Audi, and R. Hoekstra. Isotopic abundances are based on those by N. E. Holden.

APPENDIX B • *Mathematics Review*

These appendices in mathematics are intended as a brief review of operations and methods. Early in this course, you should be totally familiar with basic algebraic techniques, analytic geometry, and trigonometry. The appendices on differential and integral calculus are more detailed and are intended for those students who have difficulty applying calculus concepts to physical situations.

B.1 SCIENTIFIC NOTATION

Many quantities that scientists deal with often have very large or very small values. For example, the speed of light is about 300 000 000 m/s, and the ink required to make the dot over an *i* in this textbook has a mass of about 0.000 000 001 kg. Obviously, it is very cumbersome to read, write, and keep track of numbers such as these. We avoid this problem by using a method dealing with powers of the number 10:

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 10 \times 10 = 100$$

$$10^3 = 10 \times 10 \times 10 = 1000$$

$$10^4 = 10 \times 10 \times 10 \times 10 = 10\,000$$

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$$

and so on. The number of zeros corresponds to the power to which 10 is raised, called the **exponent** of 10. For example, the speed of light, 300 000 000 m/s, can be expressed as 3×10^8 m/s.

In this method, some representative numbers smaller than unity are

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-2} = \frac{1}{10 \times 10} = 0.01$$

$$10^{-3} = \frac{1}{10 \times 10 \times 10} = 0.001$$

$$10^{-4} = \frac{1}{10 \times 10 \times 10 \times 10} = 0.000\,1$$

$$10^{-5} = \frac{1}{10 \times 10 \times 10 \times 10 \times 10} = 0.000\,01$$

In these cases, the number of places the decimal point is to the left of the digit 1 equals the value of the (negative) exponent. Numbers expressed as some power of 10 multiplied by another number between 1 and 10 are said to be in **scientific notation.** For example, the scientific notation for 5 943 000 000 is 5.943×10^9 and that for 0.000 083 2 is 8.32×10^{-5}.

When numbers expressed in scientific notation are being multiplied, the following general rule is very useful:

$$10^n \times 10^m = 10^{n+m} \quad \textbf{(B.1)}$$

where n and m can be *any* numbers (not necessarily integers). For example, $10^2 \times 10^5 = 10^7$. The rule also applies if one of the exponents is negative: $10^3 \times 10^{-8} = 10^{-5}$.

When dividing numbers expressed in scientific notation, note that

$$\frac{10^n}{10^m} = 10^n \times 10^{-m} = 10^{n-m} \quad \textbf{(B.2)}$$

Exercises

With help from the above rules, verify the answers to the following:

1. $86\,400 = 8.64 \times 10^4$
2. $9\,816\,762.5 = 9.816\,762\,5 \times 10^6$
3. $0.000\,000\,039\,8 = 3.98 \times 10^{-8}$
4. $(4 \times 10^8)(9 \times 10^9) = 3.6 \times 10^{18}$
5. $(3 \times 10^7)(6 \times 10^{-12}) = 1.8 \times 10^{-4}$
6. $\dfrac{75 \times 10^{-11}}{5 \times 10^{-3}} = 1.5 \times 10^{-7}$
7. $\dfrac{(3 \times 10^6)(8 \times 10^{-2})}{(2 \times 10^{17})(6 \times 10^5)} = 2 \times 10^{-18}$

B.2 ALGEBRA

Some Basic Rules

When algebraic operations are performed, the laws of arithmetic apply. Symbols such as x, y, and z are usually used to represent quantities that are not specified, what are called the **unknowns.**

First, consider the equation

$$8x = 32$$

If we wish to solve for x, we can divide (or multiply) each side of the equation by the same factor without destroying the equality. In this case, if we divide both sides by 8, we have

$$\frac{8x}{8} = \frac{32}{8}$$

$$x = 4$$

Next consider the equation

$$x + 2 = 8$$

In this type of expression, we can add or subtract the same quantity from each side. If we subtract 2 from each side, we get

$$x + 2 - 2 = 8 - 2$$

$$x = 6$$

In general, if $x + a = b$, then $x = b - a$.

Now consider the equation

$$\frac{x}{5} = 9$$

If we multiply each side by 5, we are left with x on the left by itself and 45 on the right:

$$\left(\frac{x}{5}\right)(5) = 9 \times 5$$

$$x = 45$$

In all cases, *whatever operation is performed on the left side of the equality must also be performed on the right side.*

The following rules for multiplying, dividing, adding, and subtracting fractions should be recalled, where a, b, and c are three numbers:

	Rule	Example
Multiplying	$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$	$\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}$
Dividing	$\frac{(a/b)}{(c/d)} = \frac{ad}{bc}$	$\frac{2/3}{4/5} = \frac{(2)(5)}{(4)(3)} = \frac{10}{12}$
Adding	$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$	$\frac{2}{3} - \frac{4}{5} = \frac{(2)(5) - (4)(3)}{(3)(5)} = -\frac{2}{15}$

Exercises

In the following exercises, solve for x:

	Answers
1. $a = \frac{1}{1 + x}$	$x = \frac{1 - a}{a}$
2. $3x - 5 = 13$	$x = 6$
3. $ax - 5 = bx + 2$	$x = \frac{7}{a - b}$
4. $\frac{5}{2x + 6} = \frac{3}{4x + 8}$	$x = -\frac{11}{7}$

Powers

When powers of a given quantity x are multiplied, the following rule applies:

$$x^n x^m = x^{n+m} \tag{B.3}$$

For example, $x^2x^4 = x^{2+4} = x^6$.

When dividing the powers of a given quantity, the rule is

$$\frac{x^n}{x^m} = x^{n-m} \tag{B.4}$$

For example, $x^8/x^2 = x^{8-2} = x^6$.

A power that is a fraction, such as $\frac{1}{3}$, corresponds to a root as follows:

$$x^{1/n} = \sqrt[n]{x} \tag{B.5}$$

For example, $4^{1/3} = \sqrt[3]{4} = 1.5874$. (A scientific calculator is useful for such calculations.)

Finally, any quantity x^n raised to the mth power is

$$(x^n)^m = x^{nm} \tag{B.6}$$

Table B.1 summarizes the rules of exponents.

TABLE B.1
Rules of Exponents

$x^0 = 1$
$x^1 = x$
$x^nx^m = x^{n+m}$
$x^n/x^m = x^{n-m}$
$x^{1/n} = \sqrt[n]{x}$
$(x^n)^m = x^{nm}$

Exercises

Verify the following:

1. $3^2 \times 3^3 = 243$
2. $x^5x^{-8} = x^{-3}$
3. $x^{10}/x^{-5} = x^{15}$
4. $5^{1/3} = 1.709\,975$ (Use your calculator.)
5. $60^{1/4} = 2.783\,158$ (Use your calculator.)
6. $(x^4)^3 = x^{12}$

Factoring

Some useful formulas for factoring an equation are

$ax + ay + az = a(x + y + x)$ common factor

$a^2 + 2ab + b^2 = (a + b)^2$ perfect square

$a^2 - b^2 = (a + b)(a - b)$ differences of squares

Quadratic Equations

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{B.7}$$

where x is the unknown quantity and a, b, and c are numerical factors referred to as **coefficients** of the equation. This equation has two roots, given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{B.8}$$

If $b^2 \geq 4ac$, the roots are real.

EXAMPLE 1

The equation $x^2 + 5x + 4 = 0$ has the following roots corresponding to the two signs of the square-root term:

$$x = \frac{-5 \pm \sqrt{5^2 - (4)(1)(4)}}{2(1)} = \frac{-5 \pm \sqrt{9}}{2} = \frac{-5 \pm 3}{2}$$

$$x_+ = \frac{-5+3}{2} = \boxed{-1} \qquad x_- = \frac{-5-3}{2} = \boxed{-4}$$

where x_+ refers to the root corresponding to the positive sign and x_- refers to the root corresponding to the negative sign.

EXERCISES

Solve the following quadratic equations:

		Answers
1. $x^2 + 2x - 3 = 0$	$x_+ = 1$	$x_- = -3$
2. $2x^2 - 5x + 2 = 0$	$x_+ = 2$	$x_- = \frac{1}{2}$
3. $2x^2 - 4x - 9 = 0$	$x_+ = 1 + \sqrt{22}/2$	$x_- = 1 - \sqrt{22}/2$

Linear Equations

A linear equation has the general form

$$y = mx + b \tag{B.9}$$

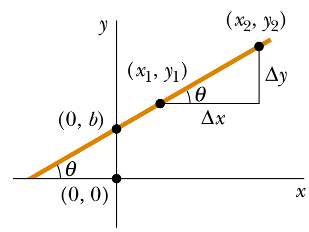


Figure B.1

where m and b are constants. This equation is referred to as being linear because the graph of y versus x is a straight line, as shown in Figure B.1. The constant b, called the ***y*-intercept,** represents the value of y at which the straight line intersects the y axis. The constant m is equal to the **slope** of the straight line and is also equal to the tangent of the angle that the line makes with the x axis. If any two points on the straight line are specified by the coordinates (x_1, y_1) and (x_2, y_2), as in Figure B.1, then the slope of the straight line can be expressed as

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \tan\theta \tag{B.10}$$

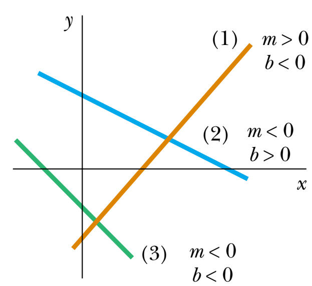


Figure B.2

Note that m and b can have either positive or negative values. If $m > 0$, the straight line has a *positive* slope, as in Figure B1. If $m < 0$, the straight line has a *negative* slope. In Figure B.1, both m and b are positive. Three other possible situations are shown in Figure B.2.

EXERCISES

1. Draw graphs of the following straight lines:
   (a) $y = 5x + 3$ (b) $y = -2x + 4$ (c) $y = -3x - 6$
2. Find the slopes of the straight lines described in Exercise 1.

Answers (a) 5 (b) -2 (c) -3

3. Find the slopes of the straight lines that pass through the following sets of points:
   (a) $(0, -4)$ and $(4, 2)$, (b) $(0, 0)$ and $(2, -5)$, and (c) $(-5, 2)$ and $(4, -2)$

Answers (a) $3/2$ (b) $-5/2$ (c) $-4/9$

Solving Simultaneous Linear Equations

Consider the equation $3x + 5y = 15$, which has two unknowns, x and y. Such an equation does not have a unique solution. For example, note that $(x = 0, y = 3)$, $(x = 5, y = 0)$, and $(x = 2, y = 9/5)$ are all solutions to this equation.

If a problem has two unknowns, a unique solution is possible only if we have *two* equations. In general, if a problem has n unknowns, its solution requires n equations. In order to solve two simultaneous equations involving two unknowns, x and y, we solve one of the equations for x in terms of y and substitute this expression into the other equation.

EXAMPLE 2

Solve the following two simultaneous equations:

$$(1) \quad 5x + y = -8$$

$$(2) \quad 2x - 2y = 4$$

Solution From (2), $x = y + 2$. Substitution of this into (1) gives

$$5(y + 2) + y = -8$$

$$6y = -18$$

$$y = -3$$

$$x = y + 2 = \boxed{-1}$$

Alternate Solution Multiply each term in (1) by the factor 2 and add the result to (2):

$$10x + 2y = -16$$

$$\underline{2x - 2y = 4}$$

$$12x = -12$$

$$x = -1$$

$$y = x - 2 = \boxed{-3}$$

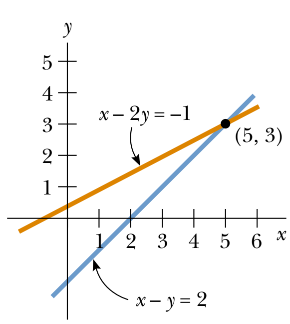


Figure B.3

Two linear equations containing two unknowns can also be solved by a graphical method. If the straight lines corresponding to the two equations are plotted in a conventional coordinate system, the intersection of the two lines represents the solution. For example, consider the two equations

$$x - y = 2$$

$$x - 2y = -1$$

These are plotted in Figure B.3. The intersection of the two lines has the coordinates $x = 5$, $y = 3$. This represents the solution to the equations. You should check this solution by the analytical technique discussed above.

EXERCISES

Solve the following pairs of simultaneous equations involving two unknowns:

Answers

1. $x + y = 8$ $\qquad$ $x = 5, y = 3$
   $x - y = 2$

2. $98 - T = 10a$ $\quad T = 65,\ a = 3.27$
   $T - 49 = 5a$
3. $6x + 2y = 6$ $\quad x = 2,\ y = -3$
   $8x - 4y = 28$

Logarithms

Suppose that a quantity x is expressed as a power of some quantity a:

$$x = a^y \tag{B.11}$$

The number a is called the **base** number. The **logarithm** of x with respect to the base a is equal to the exponent to which the base must be raised in order to satisfy the expression $x = a^y$:

$$y = \log_a x \tag{B.12}$$

Conversely, the **antilogarithm** of y is the number x:

$$x = \text{antilog}_a\, y \tag{B.13}$$

In practice, the two bases most often used are base 10, called the *common* logarithm base, and base $e = 2.718\ldots$, called Euler's constant or the *natural* logarithm base. When common logarithms are used,

$$y = \log_{10} x \qquad (\text{or } x = 10^y) \tag{B.14}$$

When natural logarithms are used,

$$y = \ln_e x \qquad (\text{or } x = e^y) \tag{B.15}$$

For example, $\log_{10} 52 = 1.716$, so that $\text{antilog}_{10} 1.716 = 10^{1.716} = 52$. Likewise, $\ln_e 52 = 3.951$, so $\text{antiln}_e 3.951 = e^{3.951} = 52$.

In general, note that you can convert between base 10 and base e with the equality

$$\ln_e x = (2.302\,585) \log_{10} x \tag{B.16}$$

Finally, some useful properties of logarithms are

$$\log(ab) = \log a + \log b$$
$$\log(a/b) = \log a - \log b$$
$$\log(a^n) = n \log a$$
$$\ln e = 1$$
$$\ln e^a = a$$
$$\ln\left(\frac{1}{a}\right) = -\ln a$$

B.3 GEOMETRY

The **distance** d between two points having coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.17}$$

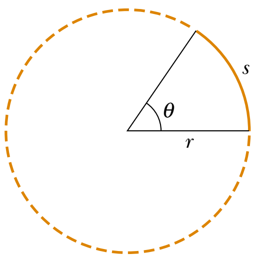


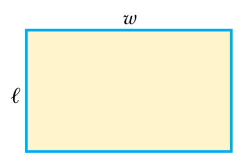
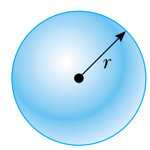
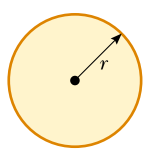
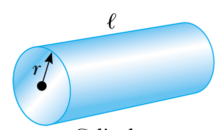
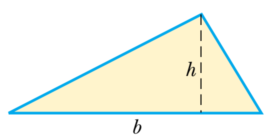
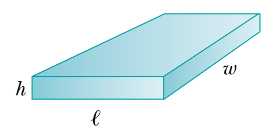
Figure B.4

Radian measure: The arc length s of a circular arc (Fig. B.4) is proportional to the radius r for a fixed value of θ (in radians):

$$s = r\theta$$
$$\theta = \frac{s}{r} \qquad \textbf{(B.18)}$$

Table B.2 gives the areas and volumes for several geometric shapes used throughout this text:

TABLE B.2 Useful Information for Geometry

Shape	Area or Volume	Shape	Area or Volume
 Rectangle	Area = ℓw	 Sphere	Surface area = $4\pi r^2$ Volume = $\frac{4\pi r^3}{3}$
 Circle	Area = πr^2 (Circumference = $2\pi r$)	 Cylinder	Lateral surface area = $2\pi r\ell$ Volume = $\pi r^2\ell$
 Triangle	Area = $\frac{1}{2}bh$	 Rectangular box	Area = $2(\ell h + \ell w + hw)$ Volume = ℓwh

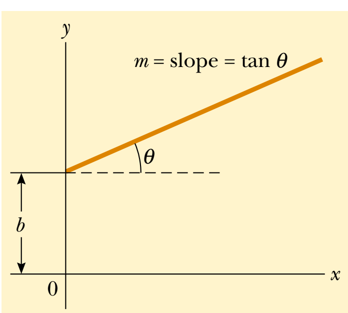


Figure B.5

The equation of a **straight line** (Fig. B.5) is

$$y = mx + b \qquad \textbf{(B.19)}$$

where b is the y-intercept and m is the slope of the line.

The equation of a **circle** of radius R centered at the origin is

$$x^2 + y^2 = R^2 \qquad \textbf{(B.20)}$$

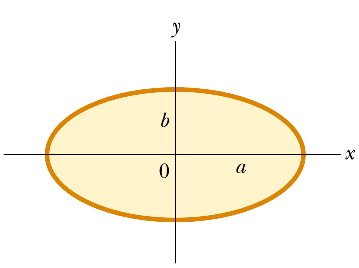


Figure B.6

The equation of an **ellipse** having the origin at its center (Fig. B.6) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \textbf{(B.21)}$$

where a is the length of the semi-major axis (the longer one) and b is the length of the semi-minor axis (the shorter one).

The equation of a **parabola** the vertex of which is at $y = b$ (Fig. B.7) is

$$y = ax^2 + b \tag{B.22}$$

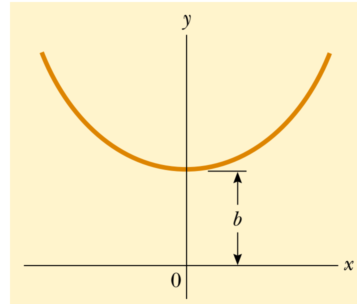


Figure B.7

The equation of a **rectangular hyperbola** (Fig. B.8) is

$$xy = \text{constant} \tag{B.23}$$

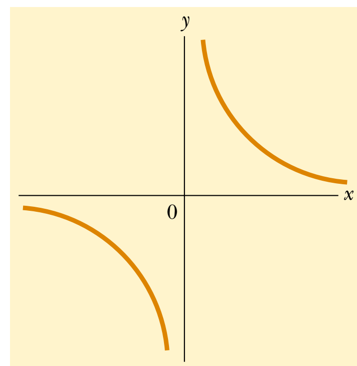


Figure B.8

B.4 TRIGONOMETRY

That portion of mathematics based on the special properties of the right triangle is called trigonometry. By definition, a right triangle is one containing a 90° angle. Consider the right triangle shown in Figure B.9, where side a is opposite the angle θ, side b is adjacent to the angle θ, and side c is the hypotenuse of the triangle. The three basic trigonometric functions defined by such a triangle are the sine (sin), cosine (cos), and tangent (tan) functions. In terms of the angle θ, these functions are defined by

$$\sin\theta \equiv \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{a}{c} \tag{B.24}$$

$$\cos\theta \equiv \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{b}{c} \tag{B.25}$$

$$\tan\theta \equiv \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{a}{b} \tag{B.26}$$

The Pythagorean theorem provides the following relationship between the sides of a right triangle:

$$c^2 = a^2 + b^2 \tag{B.27}$$

From the above definitions and the Pythagorean theorem, it follows that

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The cosecant, secant, and cotangent functions are defined by

$$\csc\theta \equiv \frac{1}{\sin\theta} \qquad \sec\theta \equiv \frac{1}{\cos\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}$$

The relationships below follow directly from the right triangle shown in Figure B.9:

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\cot\theta = \tan(90° - \theta)$$

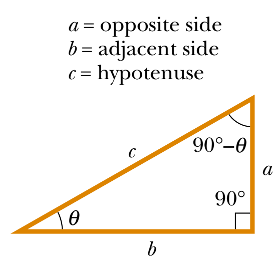


Figure B.9

Some properties of trigonometric functions are

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

The following relationships apply to *any* triangle, as shown in Figure B.10:

$$\alpha + \beta + \gamma = 180°$$

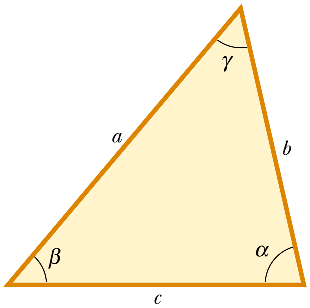


Figure B.10

Law of cosines
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Law of sines
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Table B.3 lists a number of useful trigonometric identities.

TABLE B.3 Some Trigonometric Identities

$\sin^2\theta + \cos^2\theta = 1$	$\csc^2\theta = 1 + \cot^2\theta$
$\sec^2\theta = 1 + \tan^2\theta$	$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$
$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\tan\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$
$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	

EXAMPLE 3

Consider the right triangle in Figure B.11, in which $a = 2$, $b = 5$, and c is unknown. From the Pythagorean theorem, we have

$$c^2 = a^2 + b^2 = 2^2 + 5^2 = 4 + 25 = 29$$

$$c = \sqrt{29} = 5.39$$

To find the angle θ, note that

$$\tan\theta = \frac{a}{b} = \frac{2}{5} = 0.400$$

From a table of functions or from a calculator, we have

$$\theta = \tan^{-1}(0.400) = 21.8°$$

where $\tan^{-1}(0.400)$ is the notation for "angle whose tangent is 0.400," sometimes written as arctan (0.400).

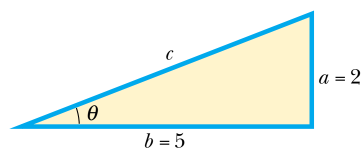


Figure B.11

EXERCISES

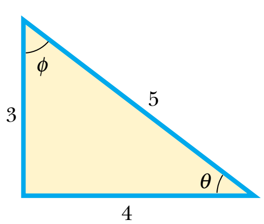


Figure B.12

1. In Figure B.12, identify (a) the side opposite θ and (b) the side adjacent to ϕ and then find (c) $\cos\theta$, (d) $\sin\phi$, and (e) $\tan\phi$.

Answers (a) 3, (b) 3, (c) $\frac{4}{5}$, (d) $\frac{4}{5}$, and (e) $\frac{4}{3}$

2. In a certain right triangle, the two sides that are perpendicular to each other are 5 m and 7 m long. What is the length of the third side?

Answer 8.60 m

3. A right triangle has a hypotenuse of length 3 m, and one of its angles is 30°. What is the length of (a) the side opposite the 30° angle and (b) the side adjacent to the 30° angle?

Answers (a) 1.5 m, (b) 2.60 m

B.5 SERIES EXPANSIONS

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1 \pm x) = \pm x - \tfrac{1}{2}x^2 \pm \tfrac{1}{3}x^3 - \cdots$$

$$\left.\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \qquad |x| < \pi/2 \end{aligned}\right\} x \text{ in radians}$$

For $x \ll 1$, the following approximations can be used[1]:

$$(1+x)^n \approx 1 + nx \qquad \sin x \approx x$$

$$e^x \approx 1 + x \qquad \cos x \approx 1$$

$$\ln(1 \pm x) \approx \pm x \qquad \tan x \approx x$$

B.6 DIFFERENTIAL CALCULUS

In various branches of science, it is sometimes necessary to use the basic tools of calculus, invented by Newton, to describe physical phenomena. The use of calculus is fundamental in the treatment of various problems in Newtonian mechanics, electricity, and magnetism. In this section, we simply state some basic properties and "rules of thumb" that should be a useful review to the student.

First, a **function** must be specified that relates one variable to another (such as a coordinate as a function of time). Suppose one of the variables is called y (the dependent variable), the other x (the independent variable). We might have a function relationship such as

$$y(x) = ax^3 + bx^2 + cx + d$$

If a, b, c, and d are specified constants, then y can be calculated for any value of x. We usually deal with continuous functions, that is, those for which y varies "smoothly" with x.

[1] The approximations for the functions $\sin x$, $\cos x$, and $\tan x$ are for $x \leq 0.1$ rad.

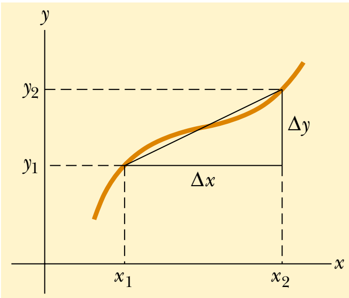


Figure B.13

The **derivative** of y with respect to x is defined as the limit, as Δx approaches zero, of the slopes of chords drawn between two points on the y versus x curve. Mathematically, we write this definition as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \tag{B.28}$$

where Δy and Δx are defined as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (Fig. B.13). It is important to note that dy/dx *does not* mean dy divided by dx, but is simply a notation of the limiting process of the derivative as defined by Equation B.28.

A useful expression to remember when $y(x) = ax^n$, where a is a *constant* and n is *any* positive or negative number (integer or fraction), is

$$\frac{dy}{dx} = nax^{n-1} \tag{B.29}$$

If $y(x)$ is a polynomial or algebraic function of x, we apply Equation B.29 to *each* term in the polynomial and take $d[\text{constant}]/dx = 0$. In Examples 4 through 7, we evaluate the derivatives of several functions.

EXAMPLE 4

Suppose $y(x)$ (that is, y as a function of x) is given by

$$y(x) = ax^3 + bx + c$$

where a and b are constants. Then it follows that

$$y(x + \Delta x) = a(x + \Delta x)^3 + b(x + \Delta x) + c$$

$$y(x + \Delta x) = a(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + b(x + \Delta x) + c$$

so

$$\Delta y = y(x + \Delta x) - y(x) = a(3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + b\Delta x$$

Substituting this into Equation B.28 gives

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [3ax^2 + 3x\Delta x + \Delta x^2] + b$$

$$\frac{dy}{dx} = 3ax^2 + b$$

EXAMPLE 5

$$y(x) = 8x^5 + 4x^3 + 2x + 7$$

Solution Applying Equation B.29 to each term independently, and remembering that d/dx (constant) $= 0$, we have

$$\frac{dy}{dx} = 8(5)x^4 + 4(3)x^2 + 2(1)x^0 + 0$$

$$\frac{dy}{dx} = 40x^4 + 12x^2 + 2$$

Special Properties of the Derivative

A. Derivative of the product of two functions If a function *f(x)* is given by the product of two functions, say, *g(x)* and *h(x)*, then the derivative of *f(x)* is defined as

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x)h(x)] = g\frac{dh}{dx} + h\frac{dg}{dx} \tag{B.30}$$

B. Derivative of the sum of two functions If a function $f(x)$ is equal to the sum of two functions, then the derivative of the sum is equal to the sum of the derivatives:

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x) + h(x)] = \frac{dg}{dx} + \frac{dh}{dx} \tag{B.31}$$

C. Chain rule of differential calculus If $y = f(x)$ and $x = g(z)$, then dy/dz can be written as the product of two derivatives:

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} \tag{B.32}$$

D. The second derivative The second derivative of y with respect to x is defined as the derivative of the function dy/dx (the derivative of the derivative). It is usually written

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \tag{B.33}$$

EXAMPLE 6

Find the derivative of $y(x) = x^3/(x+1)^2$ with respect to x.

Solution We can rewrite this function as $y(x) = x^3(x+1)^{-2}$ and apply Equation B.30:

$$\frac{dy}{dx} = (x+1)^{-2}\frac{d}{dx}(x^3) + x^3\frac{d}{dx}(x+1)^{-2}$$

$$= (x+1)^{-2}3x^2 + x^3(-2)(x+1)^{-3}$$

$$\frac{dy}{dx} = \frac{3x^2}{(x+1)^2} - \frac{2x^3}{(x+1)^3}$$

EXAMPLE 7

A useful formula that follows from Equation B.30 is the derivative of the quotient of two functions. Show that

$$\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{h\dfrac{dg}{dx} - g\dfrac{dh}{dx}}{h^2}$$

Solution We can write the quotient as gh^{-1} and then apply Equations B.29 and B.30:

$$\frac{d}{dx}\left(\frac{g}{h}\right) = \frac{d}{dx}(gh^{-1}) = g\frac{d}{dx}(h^{-1}) + h^{-1}\frac{d}{dx}(g)$$

$$= -gh^{-2}\frac{dh}{dx} + h^{-1}\frac{dg}{dx}$$

$$= \frac{h\dfrac{dg}{dx} - g\dfrac{dh}{dx}}{h^2}$$

Some of the more commonly used derivatives of functions are listed in Table B.4.

B.7 INTEGRAL CALCULUS

We think of integration as the inverse of differentiation. As an example, consider the expression

$$f(x) = \frac{dy}{dx} = 3ax^2 + b \tag{B.34}$$

which was the result of differentiating the function

$$y(x) = ax^3 + bx + c$$

TABLE B.4
Derivatives for Several Functions

$$\frac{d}{dx}(a) = 0$$
$$\frac{d}{dx}(ax^n) = nax^{n-1}$$
$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$
$$\frac{d}{dx}(\sin ax) = a\cos ax$$
$$\frac{d}{dx}(\cos ax) = -a\sin ax$$
$$\frac{d}{dx}(\tan ax) = a\sec^2 ax$$
$$\frac{d}{dx}(\cot ax) = -a\csc^2 ax$$
$$\frac{d}{dx}(\sec x) = \tan x \sec x$$
$$\frac{d}{dx}(\csc x) = -\cot x \csc x$$
$$\frac{d}{dx}(\ln ax) = \frac{1}{x}$$

Note: The letters a and n are constants.

in Example 4. We can write Equation B.34 as $dy = f(x)\,dx = (3ax^2 + b)\,dx$ and obtain $y(x)$ by "summing" over all values of x. Mathematically, we write this inverse operation

$$y(x) = \int f(x)\,dx$$

For the function $f(x)$ given by Equation B.34, we have

$$y(x) = \int (3ax^2 + b)\,dx = ax^3 + bx + c$$

where c is a constant of the integration. This type of integral is called an *indefinite integral* because its value depends on the choice of c.

A general **indefinite integral** $I(x)$ is defined as

$$I(x) = \int f(x)\,dx \qquad \textbf{(B.35)}$$

where $f(x)$ is called the *integrand* and $f(x) = \frac{dI(x)}{dx}$.

For a *general continuous* function $f(x)$, the integral can be described as the area under the curve bounded by $f(x)$ and the x axis, between two specified values of x, say, x_1 and x_2, as in Figure B.14.

The area of the blue element is approximately $f(x_i)\Delta x_i$. If we sum all these area elements from x_1 and x_2 and take the limit of this sum as $\Delta x_i \to 0$, we obtain the *true* area under the curve bounded by $f(x)$ and x, between the limits x_1 and x_2:

$$\text{Area} = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\,\Delta x_i = \int_{x_1}^{x_2} f(x)\,dx \qquad \textbf{(B.36)}$$

Integrals of the type defined by Equation B.36 are called **definite integrals.**

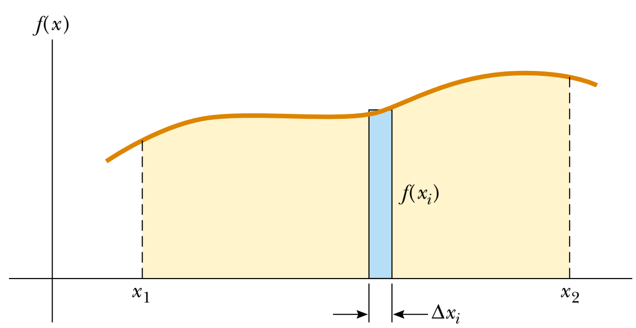


Figure B.14

One common integral that arises in practical situations has the form

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1) \qquad \textbf{(B.37)}$$

This result is obvious, being that differentiation of the right-hand side with respect to x gives $f(x) = x^n$ directly. If the limits of the integration are known, this integral becomes a *definite integral* and is written

$$\int_{x_1}^{x_2} x^n\,dx = \frac{x_2^{\,n+1} - x_1^{\,n+1}}{n+1} \qquad (n \neq -1) \qquad \textbf{(B.38)}$$

Examples

1. $\int_0^a x^2\,dx = \frac{x^3}{3}\Big]_0^a = \frac{a^3}{3}$

2. $\int_0^b x^{3/2}\,dx = \frac{x^{5/2}}{5/2}\Big]_0^b = \frac{2}{5}\,b^{5/2}$

3. $\int_3^5 x\,dx = \frac{x^2}{2}\Big]_3^5 = \frac{5^2 - 3^2}{2} = 8$

Partial Integration

Sometimes it is useful to apply the method of *partial integration* (also called "integrating by parts") to evaluate certain integrals. The method uses the property that

$$\int u\,dv = uv - \int v\,du \qquad \textbf{(B.39)}$$

where u and v are *carefully* chosen so as to reduce a complex integral to a simpler one. In many cases, several reductions have to be made. Consider the function

$$I(x) = \int x^2 e^x\,dx$$

This can be evaluated by integrating by parts twice. First, if we choose $u = x^2$, $v = e^x$, we get

$$\int x^2 e^x\,dx = \int x^2\,d(e^x) = x^2 e^x - 2\int e^x x\,dx + c_1$$

Now, in the second term, choose $u = x$, $v = e^x$, which gives

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2\int e^x\,dx + c_1$$

or

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + c_2$$

The Perfect Differential

Another useful method to remember is the use of the *perfect differential,* in which we look for a change of variable such that the differential of the function is the differential of the independent variable appearing in the integrand. For example, consider the integral

$$I(x) = \int \cos^2 x \sin x\,dx$$

This becomes easy to evaluate if we rewrite the differential as $d(\cos x) = -\sin x\,dx$. The integral then becomes

$$\int \cos^2 x \sin x\,dx = -\int \cos^2 x\,d(\cos x)$$

If we now change variables, letting $y = \cos x$, we obtain

$$\int \cos^2 x \sin x\,dx = -\int y^2 dy = -\frac{y^3}{3} + c = -\frac{\cos^3 x}{3} + c$$

Table B.5 lists some useful indefinite integrals. Table B.6 gives Gauss's probability integral and other definite integrals. A more complete list can be found in various handbooks, such as *The Handbook of Chemistry and Physics,* CRC Press.

TABLE B.5 Some Indefinite Integrals (An arbitrary constant should be added to each of these integrals.)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (\text{provided } n \neq -1)$$

$$\int \frac{dx}{x} = \int x^{-1}\,dx = \ln x$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$$

$$\int \frac{x\,dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$$

$$\int \frac{dx}{x(x+a)} = -\frac{1}{a}\ln\frac{x+a}{x}$$

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} \qquad (a^2 - x^2 > 0)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} \qquad (x^2 - a^2 > 0)$$

$$\int \frac{x\,dx}{a^2 \pm x^2} = \pm\tfrac{1}{2}\ln(a^2 \pm x^2)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a} \qquad (a^2 - x^2 > 0)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2})$$

$$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$$

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right)$$

$$\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2\ln(x + \sqrt{x^2 \pm a^2})]$$

$$\int x(\sqrt{x^2 \pm a^2})\,dx = \tfrac{1}{3}(x^2 \pm a^2)^{3/2}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int \ln ax\,dx = (x\ln ax) - x$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int \frac{dx}{a+be^{cx}} = \frac{x}{a} - \frac{1}{ac}\ln(a+be^{cx})$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \tan ax\,dx = \frac{1}{a}\ln(\cos ax) = \frac{1}{a}\ln(\sec ax)$$

$$\int \cot ax\,dx = \frac{1}{a}\ln(\sin ax)$$

$$\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\left[\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right]$$

$$\int \csc ax\,dx = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\left(\tan\frac{ax}{2}\right)$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

$$\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax$$

$$\int \tan^2 ax\,dx = \frac{1}{a}(\tan ax) - x$$

$$\int \cot^2 ax\,dx = -\frac{1}{a}(\cot ax) - x$$

$$\int \sin^{-1} ax\,dx = x(\sin^{-1} ax) + \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \cos^{-1} ax\,dx = x(\cos^{-1} ax) - \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

TABLE B.6 Gauss's Probability Integral and Other Definite Integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$I_0 = \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \text{(Gauss's probability integral)}$$

$$I_1 = \int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a}$$

$$I_2 = \int_0^\infty x^2 e^{-ax^2}\,dx = -\frac{dI_0}{da} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$I_3 = \int_0^\infty x^3 e^{-ax^2}\,dx = -\frac{dI_1}{da} = \frac{1}{2a^2}$$

$$I_4 = \int_0^\infty x^4 e^{-ax^2}\,dx = \frac{d^2I_0}{da^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

$$I_5 = \int_0^\infty x^5 e^{-ax^2}\,dx = \frac{d^2I_1}{da^2} = \frac{1}{a^3}$$

$$\vdots$$

$$I_{2n} = (-1)^n \frac{d^n}{da^n} I_0$$

$$I_{2n+1} = (-1)^n \frac{d^n}{da^n} I_1$$

APPENDIX C • Periodic Table of the Elements

Symbol — Ca 20 — Atomic number
Atomic mass † — 40.08
4s^2 — Electron configuration

Group I	Group II	Transition elements						
H 1 1.008 0 $1s^1$								
Li 3 6.94 $2s^1$	Be 4 9.012 $2s^2$							
Na 11 22.99 $3s^1$	Mg 12 24.31 $3s^2$							
K 19 39.102 $4s^1$	Ca 20 40.08 $4s^2$	Sc 21 44.96 $3d^14s^2$	Ti 22 47.90 $3d^24s^2$	V 23 50.94 $3d^34s^2$	Cr 24 51.996 $3d^54s^1$	Mn 25 54.94 $3d^54s^2$	Fe 26 55.85 $3d^64s^2$	Co 27 58.93 $3d^74s^2$
Rb 37 85.47 $5s^1$	Sr 38 87.62 $5s^2$	Y 39 88.906 $4d^15s^2$	Zr 40 91.22 $4d^25s^2$	Nb 41 92.91 $4d^45s^1$	Mo 42 95.94 $4d^55s^1$	Tc 43 (99) $4d^55s^2$	Ru 44 101.1 $4d^75s^1$	Rh 45 102.91 $4d^85s^1$
Cs 55 132.91 $6s^1$	Ba 56 137.34 $6s^2$	57-71*	Hf 72 178.49 $5d^26s^2$	Ta 73 180.95 $5d^36s^2$	W 74 183.85 $5d^46s^2$	Re 75 186.2 $5d^56s^2$	Os 76 190.2 $5d^66s^2$	Ir 77 192.2 $5d^76s^2$
Fr 87 (223) $7s^1$	Ra 88 (226) $7s^2$	89-103**	Rf 104 (261) $6d^27s^2$	Db 105 (262) $6d^37s^2$	Sg 106 (263)	Bh 107 (262)	Hs 108 (265)	Mt 109 (266)

*Lanthanide series	La 57 138.91 $5d^16s^2$	Ce 58 140.12 $5d^14f^16s^2$	Pr 59 140.91 $4f^36s^2$	Nd 60 144.24 $4f^46s^2$	Pm 61 (147) $4f^56s^2$	Sm 62 150.4 $4f^66s^2$
**Actinide series	Ac 89 (227) $6d^17s^2$	Th 90 (232) $6d^27s^2$	Pa 91 (231) $5f^26d^17s^2$	U 92 (238) $5f^36d^17s^2$	Np 93 (239) $5f^46d^17s^2$	Pu 94 (239) $5f^66d^07s^2$

Atomic mass values given are averaged over isotopes in the percentages in which they exist in nature.
† For an unstable element, mass number of the most stable known isotope is given in parentheses.
†† Elements 110, 111, 112, and 114 have not yet been named.
††† For a description of the atomic data, visit **physics.nist.gov/atomic**

			Group III	Group IV	Group V	Group VI	Group VII	Group 0
							H 1 1.008 0 $1s^1$	**He** 2 4.002 6 $1s^2$
			B 5 10.81 $2p^1$	**C** 6 12.011 $2p^2$	**N** 7 14.007 $2p^3$	**O** 8 15.999 $2p^4$	**F** 9 18.998 $2p^5$	**Ne** 10 20.18 $2p^6$
			Al 13 26.98 $3p^1$	**Si** 14 28.09 $3p^2$	**P** 15 30.97 $3p^3$	**S** 16 32.06 $3p^4$	**Cl** 17 35.453 $3p^5$	**Ar** 18 39.948 $3p^6$
Ni 28 58.71 $3d^8 4s^2$	**Cu** 29 63.54 $3d^{10} 4s^1$	**Zn** 30 65.37 $3d^{10} 4s^2$	**Ga** 31 69.72 $4p^1$	**Ge** 32 72.59 $4p^2$	**As** 33 74.92 $4p^3$	**Se** 34 78.96 $4p^4$	**Br** 35 79.91 $4p^5$	**Kr** 36 83.80 $4p^6$
Pd 46 106.4 $4d^{10}$	**Ag** 47 107.87 $4d^{10} 5s^1$	**Cd** 48 112.40 $4d^{10} 5s^2$	**In** 49 114.82 $5p^1$	**Sn** 50 118.69 $5p^2$	**Sb** 51 121.75 $5p^3$	**Te** 52 127.60 $5p^4$	**I** 53 126.90 $5p^5$	**Xe** 54 131.30 $5p^6$
Pt 78 195.09 $5d^9 6s^1$	**Au** 79 196.97 $5d^{10} 6s^1$	**Hg** 80 200.59 $5d^{10} 6s^2$	**Tl** 81 204.37 $6p^1$	**Pb** 82 207.2 $6p^2$	**Bi** 83 208.98 $6p^3$	**Po** 84 (210) $6p^4$	**At** 85 (218) $6p^5$	**Rn** 86 (222) $6p^6$
110†† (269)	111†† (272)	112†† (277)		114†† (289)				

Eu 63 152.0 $4f^7 6s^2$	**Gd** 64 157.25 $5d^1 4f^7 6s^2$	**Tb** 65 158.92 $5d^1 4f^8 6s^2$	**Dy** 66 162.50 $4f^{10} 6s^2$	**Ho** 67 164.93 $4f^{11} 6s^2$	**Er** 68 167.26 $4f^{12} 6s^2$	**Tm** 69 168.93 $4f^{13} 6s^2$	**Yb** 70 173.04 $4f^{14} 6s^2$	**Lu** 71 174.97 $5d^1 4f^{14} 6s^2$
Am 95 (243) $5f^7 6d^0 7s^2$	**Cm** 96 (245) $5f^7 6d^1 7s^2$	**Bk** 97 (247) $5f^8 6d^1 7s^2$	**Cf** 98 (249) $5f^{10} 6d^0 7s^2$	**Es** 99 (254) $5f^{11} 6d^0 7s^2$	**Fm** 100 (253) $5f^{12} 6d^0 7s^2$	**Md** 101 (255) $5f^{13} 6d^0 7s^2$	**No** 102 (255) $6d^0 7s^2$	**Lr** 103 (257) $6d^1 7s^2$

APPENDIX D • SI Units

TABLE D.1 SI Units

	SI Base Unit	
Base Quantity	**Name**	**Symbol**
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

TABLE D.2 Some Derived SI Units

Quantity	Name	Symbol	Expression in Terms of Base Units	Expression in Terms of Other SI Units
Plane angle	radian	rad	m/m	
Frequency	hertz	Hz	s^{-1}	
Force	newton	N	$kg \cdot m/s^2$	J/m
Pressure	pascal	Pa	$kg/m \cdot s^2$	N/m^2
Energy; work	joule	J	$kg \cdot m^2/s^2$	$N \cdot m$
Power	watt	W	$kg \cdot m^2/s^3$	J/s
Electric charge	coulomb	C	$A \cdot s$	
Electric potential	volt	V	$kg \cdot m^2/A \cdot s^3$	W/A
Capacitance	farad	F	$A^2 \cdot s^4/kg \cdot m^2$	C/V
Electric resistance	ohm	Ω	$kg \cdot m^2/A^2 \cdot s^3$	V/A
Magnetic flux	weber	Wb	$kg \cdot m^2/A \cdot s^2$	$V \cdot s$
Magnetic field intensity	tesla	T	$kg/A \cdot s^2$	
Inductance	henry	H	$kg \cdot m^2/A^2 \cdot s^2$	$T \cdot m^2/A$

APPENDIX E • *Nobel Prizes*

All Nobel Prizes in physics are listed (and marked with a P), as well as relevant Nobel Prizes in Chemistry (C). The key dates for some of the scientific work are supplied; they often antedate the prize considerably.

1901 (P) *Wilhelm Roentgen* for discovering x-rays (1895).
1902 (P) *Hendrik A. Lorentz* for predicting the Zeeman effect and *Pieter Zeeman* for discovering the Zeeman effect, the splitting of spectral lines in magnetic fields.
1903 (P) *Antoine-Henri Becquerel* for discovering radioactivity (1896) and *Pierre* and *Marie Curie* for studying radioactivity.
1904 (P) *Lord Rayleigh* for studying the density of gases and discovering argon.
(C) *William Ramsay* for discovering the inert gas elements helium, neon, xenon, and krypton, and placing them in the periodic table.
1905 (P) *Philipp Lenard* for studying cathode rays, electrons (1898–1899).
1906 (P) *J. J. Thomson* for studying electrical discharge through gases and discovering the electron (1897).
1907 (P) *Albert A. Michelson* for inventing optical instruments and measuring the speed of light (1880s).
1908 (P) *Gabriel Lippmann* for making the first color photographic plate, using interference methods (1891).
(C) *Ernest Rutherford* for discovering that atoms can be broken apart by alpha rays and for studying radioactivity.
1909 (P) *Guglielmo Marconi* and *Carl Ferdinand Braun* for developing wireless telegraphy.
1910 (P) *Johannes D. van der Waals* for studying the equation of state for gases and liquids (1881).
1911 (P) *Wilhelm Wien* for discovering Wien's law giving the peak of a blackbody spectrum (1893).
(C) *Marie Curie* for discovering radium and polonium (1898) and isolating radium.
1912 (P) *Nils Dalén* for inventing automatic gas regulators for lighthouses.
1913 (P) *Heike Kamerlingh Onnes* for the discovery of superconductivity and liquefying helium (1908).
1914 (P) *Max T. F. von Laue* for studying x-rays from their diffraction by crystals, showing that x-rays are electromagnetic waves (1912).
(C) *Theodore W. Richards* for determining the atomic weights of sixty elements, indicating the existence of isotopes.
1915 (P) *William Henry Bragg* and *William Lawrence Bragg,* his son, for studying the diffraction of x-rays in crystals.
1917 (P) *Charles Barkla* for studying atoms by x-ray scattering (1906).
1918 (P) *Max Planck* for discovering energy quanta (1900).
1919 (P) *Johannes Stark,* for discovering the Stark effect, the splitting of spectral lines in electric fields (1913).

1920 (P) *Charles-Édouard Guillaume* for discovering invar, a nickel-steel alloy with low coefficient of expansion.
(C) *Walther Nernst* for studying heat changes in chemical reactions and formulating the third law of thermodynamics (1918).

1921 (P) *Albert Einstein* for explaining the photoelectric effect and for his services to theoretical physics (1905).
(C) *Frederick Soddy* for studying the chemistry of radioactive substances and discovering isotopes (1912).

1922 (P) *Niels Bohr* for his model of the atom and its radiation (1913).
(C) *Francis W. Aston* for using the mass spectrograph to study atomic weights, thus discovering 212 of the 287 naturally occurring isotopes.

1923 (P) *Robert A. Millikan* for measuring the charge on an electron (1911) and for studying the photoelectric effect experimentally (1914).

1924 (P) *Karl M. G. Siegbahn* for his work in x-ray spectroscopy.

1925 (P) *James Franck* and *Gustav Hertz* for discovering the Franck-Hertz effect in electron-atom collisions.

1926 (P) *Jean-Baptiste Perrin* for studying Brownian motion to validate the discontinuous structure of matter and measure the size of atoms.

1927 (P) *Arthur Holly Compton* for discovering the Compton effect on x-rays, their change in wavelength when they collide with matter (1922), and *Charles T. R. Wilson* for inventing the cloud chamber, used to study charged particles (1906).

1928 (P) *Owen W. Richardson* for studying the thermionic effect and electrons emitted by hot metals (1911).

1929 (P) *Louis Victor de Broglie* for discovering the wave nature of electrons (1923).

1930 (P) *Chandrasekhara Venkata Raman* for studying Raman scattering, the scattering of light by atoms and molecules with a change in wavelength (1928).

1932 (P) *Werner Heisenberg* for creating quantum mechanics (1925).

1933 (P) *Erwin Schrödinger* and *Paul A. M. Dirac* for developing wave mechanics (1925) and relativistic quantum mechanics (1927).
(C) *Harold Urey* for discovering heavy hydrogen, deuterium (1931).

1935 (P) *James Chadwick* for discovering the neutron (1932).
(C) *Irène* and *Frédéric Joliot-Curie* for synthesizing new radioactive elements.

1936 (P) *Carl D. Anderson* for discovering the positron in particular and antimatter in general (1932) and *Victor F. Hess* for discovering cosmic rays.
(C) *Peter J. W. Debye* for studying dipole moments and diffraction of x-rays and electrons in gases.

1937 (P) *Clinton Davisson* and *George Thomson* for discovering the diffraction of electrons by crystals, confirming de Broglie's hypothesis (1927).

1938 (P) *Enrico Fermi* for producing the transuranic radioactive elements by neutron irradiation (1934–1937).

1939 (P) *Ernest O. Lawrence* for inventing the cyclotron.

1943 (P) *Otto Stern* for developing molecular-beam studies (1923), and using them to discover the magnetic moment of the proton (1933).

1944 (P) *Isidor I. Rabi* for discovering nuclear magnetic resonance in atomic and molecular beams.
(C) *Otto Hahn* for discovering nuclear fission (1938).

1945 (P) *Wolfgang Pauli* for discovering the exclusion principle (1924).

1946 (P) *Percy W. Bridgman* for studying physics at high pressures.

1947 (P) *Edward V. Appleton* for studying the ionosphere.

1948 (P) *Patrick M. S. Blackett* for studying nuclear physics with cloud-chamber photographs of cosmic-ray interactions.

1949 (P) *Hideki Yukawa* for predicting the existence of mesons (1935).

1950 (P) *Cecil F. Powell* for developing the method of studying cosmic rays with photographic emulsions and discovering new mesons.

1951 (P) *John D. Cockcroft* and *Ernest T. S. Walton* for transmuting nuclei in an accelerator (1932).

(C) *Edwin M. McMillan* for producing neptunium (1940) and *Glenn T. Seaborg* for producing plutonium (1941) and further transuranic elements.

1952 (P) *Felix Bloch* and *Edward Mills Purcell* for discovering nuclear magnetic resonance in liquids and gases (1946).

1953 (P) *Frits Zernike* for inventing the phase-contrast microscope, which uses interference to provide high contrast.

1954 (P) *Max Born* for interpreting the wave function as a probability (1926) and other quantum-mechanical discoveries and *Walther Bothe* for developing the coincidence method to study subatomic particles (1930–1931), producing, in particular, the particle interpreted by Chadwick as the neutron.

1955 (P) *Willis E. Lamb, Jr.,* for discovering the Lamb shift in the hydrogen spectrum (1947) and *Polykarp Kusch* for determining the magnetic moment of the electron (1947).

1956 (P) *John Bardeen, Walter H. Brattain,* and *William Shockley* for inventing the transistor (1956).

1957 (P) *T.-D. Lee* and *C.-N. Yang* for predicting that parity is not conserved in beta decay (1956).

1958 (P) *Pavel A. Čerenkov* for discovering Čerenkov radiation (1935) and *Ilya M. Frank* and *Igor Tamm* for interpreting it (1937).

1959 (P) *Emilio G. Segrè* and *Owen Chamberlain* for discovering the antiproton (1955).

1960 (P) *Donald A. Glaser* for inventing the bubble chamber to study elementary particles (1952).

(C) *Willard Libby* for developing radiocarbon dating (1947).

1961 (P) *Robert Hofstadter* for discovering internal structure in protons and neutrons and *Rudolf L. Mössbauer* for discovering the Mössbauer effect of recoilless gamma-ray emission (1957).

1962 (P) *Lev Davidovich Landau* for studying liquid helium and other condensed matter theoretically.

1963 (P) *Eugene P. Wigner* for applying symmetry principles to elementary-particle theory and *Maria Goeppert Mayer* and *J. Hans D. Jensen* for studying the shell model of nuclei (1947).

1964 (P) *Charles H. Townes, Nikolai G. Basov,* and *Alexandr M. Prokhorov* for developing masers (1951–1952) and lasers.

1965 (P) *Sin-itiro Tomonaga, Julian S. Schwinger,* and *Richard P. Feynman* for developing quantum electrodynamics (1948).

1966 (P) *Alfred Kastler* for his optical methods of studying atomic energy levels.

1967 (P) *Hans Albrecht Bethe* for discovering the routes of energy production in stars (1939).

1968 (P) *Luis W. Alvarez* for discovering resonance states of elementary particles.

1969 (P) *Murray Gell-Mann* for classifying elementary particles (1963).

1970 (P) *Hannes Alfvén* for developing magnetohydrodynamic theory and *Louis Eugène Félix Néel* for discovering antiferromagnetism and ferrimagnetism (1930s).

1971 (P) *Dennis Gabor* for developing holography (1947).
(C) *Gerhard Herzberg* for studying the structure of molecules spectroscopically.

1972 (P) *John Bardeen, Leon N. Cooper,* and *John Robert Schrieffer* for explaining superconductivity (1957).

1973 (P) *Leo Esaki* for discovering tunneling in semiconductors, *Ivar Giaever* for discovering tunneling in superconductors, and *Brian D. Josephson* for predicting the Josephson effect, which involves tunneling of paired electrons (1958–1962).

1974 (P) *Anthony Hewish* for discovering pulsars and *Martin Ryle* for developing radio interferometry.

1975 (P) *Aage N. Bohr, Ben R. Mottelson,* and *James Rainwater* for discovering why some nuclei take asymmetric shapes.

1976 (P) *Burton Richter* and *Samuel C. C. Ting* for discovering the J/psi particle, the first charmed particle (1974).

1977 (P) *John H. Van Vleck, Nevill F. Mott,* and *Philip W. Anderson* for studying solids quantum-mechanically.
(C) *Ilya Prigogine* for extending thermodynamics to show how life could arise in the face of the second law.

1978 (P) *Arno A. Penzias* and *Robert W. Wilson* for discovering the cosmic background radiation (1965) and *Pyotr Kapitsa* for his studies of liquid helium.

1979 (P) *Sheldon L. Glashow, Abdus Salam,* and *Steven Weinberg* for developing the theory that unified the weak and electromagnetic forces (1958–1971).

1980 (P) *Val Fitch* and *James W. Cronin* for discovering CP (charge-parity) violation (1964), which possibly explains the cosmological dominance of matter over antimatter.

1981 (P) *Nicolaas Bloembergen* and *Arthur L. Schawlow* for developing laser spectroscopy and *Kai M. Siegbahn* for developing high-resolution electron spectroscopy (1958).

1982 (P) *Kenneth G. Wilson* for developing a method of constructing theories of phase transitions to analyze critical phenomena.

1983 (P) *William A. Fowler* for theoretical studies of astrophysical nucleosynthesis and *Subramanyan Chandrasekhar* for studying physical processes of importance to stellar structure and evolution, including the prediction of white dwarf stars (1930).

1984 (P) *Carlo Rubbia* for discovering the W and Z particles, verifying the electroweak unification, and *Simon van der Meer,* for developing the method of stochastic cooling of the CERN beam that allowed the discovery (1982–1983).

1985 (P) *Klaus von Klitzing* for the quantized Hall effect, relating to conductivity in the presence of a magnetic field (1980).

1986 (P) *Ernst Ruska* for inventing the electron microscope (1931), and *Gerd Binnig* and *Heinrich Rohrer* for inventing the scanning-tunneling electron microscope (1981).

1987 (P) *J. Georg Bednorz* and *Karl Alex Müller* for the discovery of high temperature superconductivity (1986).

1988 (P) *Leon M. Lederman, Melvin Schwartz,* and *Jack Steinberger* for a collaborative experiment that led to the development of a new tool for studying the weak nuclear force, which affects the radioactive decay of atoms.

1989 (P) *Norman Ramsay* (U.S.) for various techniques in atomic physics; and *Hans Dehmelt* (U.S.) and *Wolfgang Paul* (Germany) for the development of techniques for trapping single charge particles.

1990 (P) *Jerome Friedman, Henry Kendall* (both U.S.), and *Richard Taylor* (Canada) for experiments important to the development of the quark model.

1991 (P) *Pierre-Gilles de Gennes* for discovering that methods developed for studying order phenomena in simple systems can be generalized to more complex forms of matter, in particular to liquid crystals and polymers.

1992 (P) *George Charpak* for developing detectors that trace the paths of evanescent subatomic particles produced in particle accelerators.

1993 (P) *Russell Hulse* and *Joseph Taylor* for discovering evidence of gravitational waves.

1994 (P) *Bertram N. Brockhouse* and *Clifford G. Shull* for pioneering work in neutron scattering.

1995 (P) *Martin L. Perl* and *Frederick Reines* for discovering the tau particle and the neutrino, respectively.

1996 (P) *David M. Lee, Douglas C. Osheroff,* and *Robert C. Richardson* for developing a superfluid using helium-3.

1997 (P) *Steven Chu, Claude Cohen-Tannoudji,* and *William D. Phillips* for developing methods to cool and trap atoms with laser light.

1998 (P) *Robert B. Laughlin, Horst L. Störmer,* and *Daniel C. Tsui* for discovering a new form of quantum fluid with fractionally charged excitations.